Probabilities for Successive Memorable Hands

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Abstract

We provide details of the computations for a variety of successive hold'em hands.

This file provides the details for the probabilities given in the article "Memorable Back-To-Back Hands" that appeared in *Canadian Poker Player*, Vol. 1, No. 11, November, 2004.

The total number of hold'em hands a player may be dealt is $\binom{52}{2} = 1,326$. Thus, the probability that a player will be dealt the identical two cards she was dealt on the previous hand is precisely 1/1,326 which is approximately .00075.

A more interesting question is the following. Given that a player plays a hold'em session over which she is dealt n hands, what is the probability that at least once over the course of the session she is dealt identical hands back-to-back? This is equivalent to asking how many sequences of length n are there from an alphabet of 1,326 letters so that no two successive entries are identical.

The total number of sequences of length n using an alphabet of 1,326 letters is $1,326^n$. This follows because there are 1,326 choices for each entry of the sequence. It is now easy to count the number of sequences without two successive entries that are the same. The first entry may be anything, that is, there are 1,326 choices for the first entry. Once the first entry is chosen, the second entry may be any of 1,325. Once the second entry is chosen, the third entry may be any of 1,325.

We continue in this way and see that the number of sequences of length n with no two successive entries the same is

$$1,326(1,325)^{n-1}.$$

Thus, the probability that out of n hands no two successive hands are exactly the same is given by

$$(\frac{1,325}{1,326})^{n-1}.$$

To get the probability that a player is dealt the same hand back-to-back at least once in a session of n hands, we subtract the preceding number from 1. The following table contains the probabilities for a few values of n.

n	2	10	50	150	200
probability	.00075	.00677	.03629	.10632	.1394

Back-to-back Identical Hold'Em Hands

Back-to-back hands consisting of the same two cards is something a player notices. Another successive occurrence players notice is being dealt back-to-back special hands. For example, back-to-back pocket aces usually draws comments when it happens. When this happens, sometimes you will hear a player ask, "What are the chances of being dealt successive hands of pocket aces?"

The problem with the question as asked is that it makes no sense because it may be interpreted in several ways. The probability of being dealt A-A on a particular deal is 1/221. Thus, if you interpret the preceding question to be asking for the probability of being dealt A-A on two particular successive deals, the answer is

$$(\frac{1}{221})^2 = \frac{1}{48,841}.$$

If, on the other hand, you interpret the preceding question to be asking for the probability of being dealt A-A on the next hand given that you have just been dealt A-A, then the answer is 1/221. This interpretation is not terribly interesting.

The most interesting interpretation is to ask for the probability of being dealt back-to-back pocket aces sometime during a hold'em session of n hands. This is the interpretation we consider for the rest of this paper.

Of the 1,326 hold'em hands, 6 consist of A-A. We shall think of a hold'em session of n hands as a sequence of length n, where each entry may be any of 1,326 objects (the possible hold'em hands). There are $(1,326)^n$ such sequences and we are interested in the number of such sequences with no successive occurrences of any of the 6 objects corresponding to A-A.

Define c_i, $i \geq 0$, to be the number of sequences of length $i+1$ that do not contain successive objects corresponding to A-A hands. It is easy to see that

$$c_0 = 1,326 \tag{1}$$

because a session of one hand cannot possibly produce back-to-back pocket aces.

There are $1,326^2 = 1,758,276$ sequences of length 2 and 36 of them consist of two successive hands of A-A. Hence,

$$c_1 = 1,758,240 \tag{2}$$

as we must remove the 36 sequences.

We now are going to derive a recurrence relation for c_i, $i \geq 2$. Let $c_i(a)$ denote the sequences of length $i+1$ whose last entry is an object corresponding to an A-A hand, but the sequence itself contains no successive objects corresponding to A-A hands. Similarly, let $c_i(b)$ denote the sequences of length $i+1$

whose last entry is an object not corresponding to an A-A hand, and the sequence itself contains no successive objects corresponding to A-A hands. It is then apparent that

$$c_i = c_i(a) + c_i(b) \tag{3}$$

because every sequence counted in c_i must end with one of the two types of objects.

If a sequence of length $i+1$ counted in c_i ends with an object corresponding to an A-A hand, then the first i coordinates of the sequence must be a sequence of length i being counted in $c_{i-1}(b)$. This tells us that

$$c_i(a) = 6c_{i-1}(b), \tag{4}$$

for $i \geq 1$.

If the last element of a sequence counted in c_i corresponds to a hand that is not A-A, then the rest of the sequence is unrestricted as long as it has no successive objects corresponding to A-A hands. Thus,

$$c_i(b) = 1,320c_{i-1}. \tag{5}$$

Combining (3), (4) and (5), we obtain

$$c_i = 1,320c_{i-1} + 7,920c_{i-2}, \tag{6}$$

for all $i \geq 2$. This is a linear recurrence of order 2.

Now let $f(x)$ be the generating function for the sequence $\{c_i\}_{i=0}^{\infty}$, that is,

$$f(x) = \sum_{i=0}^{\infty} c_i x^i. \tag{7}$$

Because of the recurrence relation (6), we have that

$$f(x) - 1,320xf(x) - 7,920x^2 f(x) = 1,326 + 7,920x$$

or

$$f(x) = \frac{1,326 + 7,920x}{1 - 1,320x - 7,920x^2}. \tag{8}$$

The polynomial in the denominator of (8) factors into

$$(1 - (660 + 24\sqrt{770})x)(1 - (660 - 24\sqrt{770})x)$$

. We use the method of partial fractions in order to write $f(x)$ in the form

$$f(x) = \frac{A}{1 - (660 + 24\sqrt{770})x} + \frac{B}{1 - (660 - 24\sqrt{770})x}, \tag{9}$$

where A and B are constants. We then use (9) to derive an expression for directly calculating any particular coefficient of $f(x)$.

Performing the calculations to put the expression in (9) over a common denominator, we obtain the following two equations in A and B from (8).

$$\begin{aligned} A + B &= 1,326 \\ (24\sqrt{770} - 660)A - (24\sqrt{770} + 660)B &= 7,920. \end{aligned}$$

Solving these two linear equations produces the solutions

$$A = 663 + \frac{669\sqrt{770}}{28}$$

and

$$B = 663 - \frac{669\sqrt{770}}{28}.$$

The key to using the preceding results to obtain a direct expression for the coefficients is based on the fact that that

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots + y^n + \cdots \qquad (10)$$

in the context of the algebra of generating functions. If $y = Cx$, where C is a constant, then we see that the term involving x^n is just $C^n x^n$. In other words, the coefficient of x^n is just C^n. Therefore, using (9) and the solutions for A and B, we have that

$$c_i = (663 + \frac{669\sqrt{770}}{28})(660 + 24\sqrt{770})^i + (663 - \frac{669\sqrt{770}}{28})(660 - 24\sqrt{770})^i,$$

for all integers $i \geq 0$.

Since c_i gives us the number of hold'em sessions consisting of $i+1$ deals such that a given player is not dealt back-to-back pocket aces, the probability of a player being dealt back-to-back aces sometime during a session of $i+1$ hold'em hands is given by subtracting from 1 the value of c_i divided by $1,326^{i+1}$. The next table has a few sample values.

n	2	10	50	150	200
probability	.00002	.00018	.000998	.00303	.00405

Back-to-back Pocket Aces