Chances of Making Omaha Low Hands

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7 July 2009

Abstract

We determine the probabilities for a player making lows and nut lows with various Omaha starting hands.

1 Introduction

We are going to provide details for determining the probabilities of making a low hand and a nut low hand with certain starting hands in Omaha. We are using the standard rule that a player must make an 8-low or better in order to have a low hand.

For cards of ranks A through 8 we simply shall designate the rank by the rank itself. For cards of ranks 9, 10, J, Q or K, we shall designate the rank by B.

The basic methodology we use is to first count the possible number of boards that a fixed hand may encounter. We then count the number of the boards that give the player a low and the number of boards that give the player a nut low. We then divide by the total number of boards to obtain the appropriate probability.

An Omaha hand has four cards so that the total number of boards for such a hand is given by

$$\binom{48}{5} = 1,712,304. \tag{1}$$

2 Low Boards

When one-to-one correspondences work they are splendid to use because they provide such an elegant way to prove two (or more) sets have the same cardinality. We use them consistently to establish the following result.

2.1 Theorem. *The probability of an Omaha hand making an 8-low or better is completely determined by the number of distinct ranks from* $\{A,2,3,4,5,6,7,8\}$ *appearing in the hand.*

PROOF. Let $R = \{A, 2, 3, 4, 5, 6, 7, 8\}$. Of course, an Omaha hand with no ranks or just one rank from R cannot make a low. Thus, the probability of making a low is 0 for all such hands. So the conclusion holds for such hands.

Consider hands having cards with precisely two ranks chosen from R. Letting x and y represent the two small ranks, such a hand may have any of the forms xxxy, xxyy, xxyB, or xyBB.

Consider a hand H_1 with the form xxxy. Without loss of generality we may let the cards of rank x be clubs, diamonds and hearts, and let the other card be the y of clubs. Making these assumptions on the suits has no effect on the number of boards giving the hand a low.

Now consider a hand H_2 of the form xxxz, where z is different from y, the suits of the cards of rank x are the same as in H_1, and z is a club. If a board for H_1 has any cards of ranks z or y , then change any card of rank z to the card of rank y in the same suit and change any card of rank y to the card of rank z in the same suit to get the corresponding board for H_2. If a board for H_1 contains no cards of ranks y or z, then let the corresponding board for H_2 be the same.

It is easy to see that the preceding is a one-to-one correspondence between the boards for H_1 and the boards for H_2. If a board for H_1 makes a low and there are three low ranks making the low without using a card of rank z, then these same three low ranks are present in the corresponding board for H_2. Hence, H_2 makes a low with the corresponding board. On the other hand, if H_1 needs a card of rank z from the board to make a low, then H_2 has a card of rank y in the corresponding board to use towards a low. Hence, H_2 makes a low with the corresponding board.

If we consider the low boards for H_2, it is easy to see that the corresponding boards for H_1 give H_1 a low. Thus, we have a one-to-one correspondence between low boards for H_1 and low boards for H_2.

Let H_3 be the hand yyyx, where the cards of rank y are clubs, diamonds, hearts, with the x of clubs. So H_1 and H_2 share the x and y of clubs. To go from a board for H_1 to a board for H_3, change any y of diamonds or hearts to x of diamonds or hearts, respectively. This clearly establishes a one-to-one correspondence between boards for H_1 and boards for H_3. Because no small ranks other than x and y are altered, the correspondence clearly is one-to-one between boards giving each hand lows.

Now let H_4 have the form zzzy, where z is different from x and the suits are clubs, diamonds, and hearts for the cards of rank z and y is a club. To go from a board for H_1, replace any card of rank z with the card of rank x in the same suit and replace the x of spades with the z of spades. This establishes a one-to-one correspondence between the boards for H_1 and the boards for H_4.

If a board gives a low for H_1 and there are three small ranks different from z, then these three small ranks do not change for the corresponding board for H_4. Thus, H_4 has a low with the corresponding board. If a board gives H_1 a low and the rank z is needed from a card on board, then this becomes a card of rank x in the corresponding board for H_4. This gives H_4 a low with the corresponding board so that every low board for H_1 produces a low board for H_4 under the one-to-one correspondence. It is easy to see that low boards for

H_4 come from low boards for H_1 under the correspondence.

The above allows us to conclude that all hands of the form xxxy have the same number of boards that give the hand a low. Now let H_5 be the hand xxyy, where the suits are clubs and diamonds for both ranks. Choose the hand H_1 to compare with H_5. If a board for H_1 contains the y of diamonds, change it to the x of hearts to give the corresponding board for H_5, otherwise leave the board as it is. This clearly is a one-to-one correspondence between boards for H_1 and boards for H_5. It has absolutely no effect on low boards so that the latter are in one-to-one correspondence as well. Hence, the number of boards giving H_5 a low is the same as for H_1.

Next look at hands H_6 of the form xxyB. Compare it to H_5 where the suits are the same except the y of diamonds is replaced by the B of clubs. If a board for H_5 has the B of clubs, then replace it with the y of diamonds to get the corresponding board for H_6, otherwise leave the board as it is. This yields a one-to-one correspondence between boards for H_5 and H_6 that has no effect on boards allowing lows. Thus, the boards allowing lows also are in one-to-one correspondence. This implies that hands of the form xxyB have the same number of boards giving the hand a low.

The last form of hand we consider is xyBB which we denote H_7. Because the ranks of the big cards have no effect on how many boards produce a low, we assume the second big card is the B of diamonds. If a board for H_6 has the B of diamonds, replace it with the x of diamonds to get the corresponding board for H_7, otherwise, leave the board as it is. We get the same conclusion we have obtained several times.

The above shows that all Omaha hands with exactly two low ranks have the same probability of making a low. We now turn to hands with three low ranks. They must have one of two forms: xxyz or xyzB.

Let H_8 have the form xyzB with all cards clubs. Then choose H_9 to be xxyz with the second card of rank x being a diamond. If a board for H_9 has the B of clubs, then change this to the x of diamonds. Leave the other boards unaltered. This is a one-to-one correspondence from boards for H_9 to boards for H_8. The correspondence has no effect on lows so that H_8 and H_9 have the same number of boards giving the hands a low.

This implies that if we show all hands of the form xyzB have the same number of boards giving a low, then we are done. We certainly may assume the big card is the same in each hand. If H_{10} has the form xywB, where w is different from z, then for any card of rank w in a board for H_8, replace it with a card of rank z in the same suit to get a board for H_{10}. As in the preceding arguments, this establishes a one-to-one correspondence between the boards for H_8 and H_{10}. It is easy to see that the low boards for each also are in one-to-one correspondence. This means if we change one of the low ranks the number of low boards does not change. Hence, all hands of the form xyzB have the same number of boards making a low. Thus, all hands with exactly three ranks from {A,2,3,4,5,6,7,8} have the same probability of making a low.

Consider H_{11} to have the form xyzw, with all suits clubs, and let H_{12} be xyzv, with all suits clubs and v different from z. If a board for H_{11} has a card of rank v, change each such card to the card of rank w in the same suit to get the corresponding board for H_{12}. It is again easy to see that this establishes a one-to-one correspondence between both boards and low boards for H_{11} and H_{12}.

By changing one rank at a time, we obtain that all hands of the form xyzw have the same probability of making a low. This then completes the proof as we have covered all possible forms. ■

Because of Theorem 1 we need only check the probability of making a low for three types of hands based on the number of distinct low ranks contained in the hand. We are going to go through the details for one hand to illustrate how one does the counting and then place the information for the other types in a table. This way anyone who wishes to check the calculations knows the numbers we are using. The hand we check has the form A-2-B-B.

In order for a player to make a low, the board must have at least three cards with distinct low ranks chosen from {3,4,5,6,7,8}. We consider various subcases in such a way that it becomes easy to determine as well the number of boards giving the player a nut low.

Suppose the board has exactly three cards of low ranks. There are $\binom{6}{3} = 20$ choices for the three ranks, there are 4 choices for a card of each rank, and the other two cards of big rank can be chosen in $\binom{18}{2} = 153$ ways. Multiplying yields 195,840 boards of this type. The player has a nut low for all of these boards.

Now suppose the board has exactly four cards of low ranks. If the four cards of low rank represent only three ranks, then there is a pair of low rank. There are 6 choices for the paired rank, 6 choices for the pair, 10 choices for the other two ranks, 16 choices of cards of these ranks, and 18 choices for the card of big rank. We then get 103,680 boards of this type and all of them give the player a nut low.

If all four ranks are distinct and chosen from 3 through 8, there are $\binom{6}{4} = 15$ choices of ranks, 4 choices for each rank, and 18 choices for a card of big rank. We get 69,120 boards of this type and each gives the player nut low.

If all four ranks are distinct and only three are chosen from 3 through 8, then there are 20 choices for the three ranks, 64 choices for cards of those ranks, 6 choices for the card of rank A or 2, and 18 choices for a card of big rank. Multiplying yields 138,240 boards of this type. Because one of the board cards has rank A or 2, not all of these boards give the player a nut low. It is easy to see that the player has a nut low only if she has a wheel, that is, A-2-3-4-5. Hence, the number of boards giving her a nut low is $6 \cdot 64 \cdot 18 = 6,912$.

Now suppose all five board cards have low rank. Consider xxxyz first. All three ranks must be chosen from {3,4,5,6,7,8}. There are 6 choices for rank x, 4 choices for trips of that rank, 10 choices for the other two ranks, and 4 choices for each card of the other ranks. This produces 3,840 boards and all give the player a nut low.

Next consider xxyyz with all three ranks from {3,4,5,6,7,8}. There are $\binom{6}{2} =$ 15 choices for the ranks that are paired, 6 choices for each pair, and 16 choices for the other card. This produces 8,640 boards all of which give the player a nut low.

Move to xxyzw with all four ranks from {3,4,5,6,7,8}. There are 6 choices for x, 6 choices for the pair, 10 choices for the other three ranks, and 4 choices for a card of each rank. Altogether this gives 23,040 boards and each is a nut low for the player.

Next look at xxyzw with just three ranks from {3,4,5,6,7,8}. If xx is either A or 2, then there are 6 choices for the pair, 20 choices for the other three ranks, and 64 choices for the cards of those three ranks. If x is from {3,4,5,6,7,8}, then there are 6 choices for the counterfeiting card, 6 choices for x, 6 choices for a pair of rank x, 10 choices for the other two ranks, and 16 choices for cards of the latter two ranks. Taking the two products and summing yields 42,240 boards of this type. Again, the only way the player can have a nut low is by holding a wheel. The number of boards for this is given by $(6 \cdot 64) + (6 \cdot 3 \cdot 6 \cdot 16) = 2,112$.

We move to xyzwv last. There are three subcases here. If all five ranks are chosen from {3,4,5,6,7,8}, then the number of such boards is 6,144. All of them give the player a nut low.

If four of the ranks are chosen from {3,4,5,6,7,8}, then the number of boards is given by $6\binom{6}{4}4^4 = 23,040$. The player must have a wheel to have a nut low. There are $6 \cdot 64 \cdot 12 = 4,608$ such boards.

If three of the ranks are chosen from {3,4,5,6,7,8}, then there are $9 \cdot 20 \cdot 64 = 11,520$ boards giving the player a low. Because the player needs a wheel to have a nut low, there are $9 \cdot 64 = 576$ boards of this type giving the player a nut low.

Summing all of the boards that give a player holding A2BB a low produces 625,344. If we divide by (1), we obtain a probability of .3562 that the player makes a low. If we sum the boards that give the player a nut low, we find there are 424,512 such boards. This yields a probability of .2479 that the player makes a nut low.

We go through the same steps for A-2-3-B and A-2-3-4 and put the numbers in the following table. They are derived in the same way we got the numbers for A-2-B-B although there are a few additional cases because there may be more duplicates. Let's say a few words about the notation in the following table. Lower case letters indicate ranks chosen from {A,2,3,4,5,6,7,8}, that is, ranks that may be used for low. The notation (dup) means that one of the low ranks on board duplicates one of the ranks in the hand, (2dup) means that two low ranks on board duplicate ranks in the hand, and so on. The symbol B represents cards whose rank cannot be used for low.

board form	A-2-B-B	A-2-3-B	A-2-3-4
abcBB	195,840	109,640	48,640
abcBB(dup)	-	246,240	218,880
abcBB(2dup)	-	-	164,160
aabcB	103,680	54,720	23,040
aabcB(dup)	-	109,440	92,160
aabcB(2dup)	-	-	60,480
abcdB	69,120	23,420	5,120
abcdB(dup)	138,240	109,440	61,440
abcdB(2dup)	-	82,080	103,680
abcdB(3dup)	-	-	34,560
aaabc	3,840	1,920	768
aaabc(dup)	-	3,360	2,688
aaabc(2dup)	-	-	1,440
aabbc	8,640	4,320	1,728
aabbc(dup)	-	7,560	6,048
aabbc(2dup)	-	-	3,456
aabcd	23,040	7,680	1,536
aabcd(dup)	42,240	31,680	16,896
aabcd(2dup)	-	18,144	10,368
aabcd(3dup)	-	-	5,184
abcde	6,144	1,024	-
abcde(dup)	23,040	11,520	3,072
abcde(2dup)	11,520	17,280	2,016
abcde(3dup)	-	4,320	10,368
abcde(4dup)	-	-	1,296
Totals	625,344	844,48	879,024

NUMBERS OF LOW BOARDS

3 Nut Lows

We have seen that the probability of making a low depends only on the number of distinct low ranks in a player's hand. However, it is clear that not all potential low hands are created equal. For example, it takes little common sense to know that there is something better about the hand A-2-4-K as compared to 2-5-7-K with regards to potential low hands. One huge difference is captured by the notion of a *nut* low, that is, a low hand that cannot be beaten. That is what we examine in this section.

When we were examining boards that gave A-2-B-B a low, we simultaneously counted the number of boards giving the hand a nut low. We saw that there are 424,512 such boards. We are going to go through the details for one other hand with regard to nut lows and then give information in a table for the others.

The other hands with two low ranks we shall consider are A-3-B-B, A-4-B-B, 2-3-B-B, and 2-4-B-B because it is hard to imagine anyone playing many other hands with only two low ranks because of low potential. As far as nut lows are concerned, A-3-B-B and 2-3-B-B are identical because they have one low rank forced to be on board for a nut low. Here is a one-to-one correspondence between their boards that makes it explicit. Assume the low cards are clubs and the two big cards are identical in the two hands. Given a board for A-3-B-B, replace any cards of rank 2 with an ace in the same suit and replace any ace with a deuce in the same suit to get the corresponding board for 2-3-B-B. It is easy to see that the boards are in one-to-one correspondence and that boards giving nut lows also are in one-to-one correspondence. Similarly, A-4-B-B and 2-4-B-B are identical for nut lows because they have two forced ranks for nut lows.

When the player holds A-3-B-B and the low ranks on boards are from {2,4,5,6,7,8}, the player has a nut low if and only if there is a 2 on board. Hence, when there are exactly three low cards on board, there are 4 choices for a 2, 10 choices for the other two ranks, 16 choices for cards of those ranks, and $\binom{18}{2} = 153$ choices for the two big cards. This gives 97,920 boards for the player to have a nut low.

When there are four low cards on board and they have the form xxyz, either the deuces are paired or the deuce is a singleton. This gives

$$18(6\binom{5}{2}16) + 18(4 \cdot 5 \cdot 6 \cdot 16) = 51,840$$

boards with the player having a nut low.

If there are four low cards of the form xyzw on board all chosen from {2,4,5,6,7,8}, then the number of boards giving the player a nut low is given by $4\binom{5}{3}64 \cdot 18 = 46,080$.

If there are four low cards of the form xyzw on board with three chosen from {2,4,5,6,7,8}, then the player has a nut low if and only if she has a wheel. This means there are 6 choices for the card of rank A or 3, and 4 choices for each of the cards of ranks 2, 4 and 5. The number of boards is then $6 \cdot 64 \cdot 18 = 6,912$.

Now consider five low cards on board. If the board has the form xxxyz, she needs a deuce on board to have the nut low. The number of such boards is given by

$$(4 \cdot 10 \cdot 16) + (5 \cdot 4 \cdot 4 \cdot 16) = 1,920.$$

When the board has the form xxyyz, then the number of boards giving the player a nut low is given by

$$(4 \cdot 10 \cdot 36) + (6 \cdot 5 \cdot 6 \cdot 16) = 4,320.$$

If the board has the form xxyzw and none of the cards has rank A or 3, then the number of boards giving the player a nut low is given by

$$(6\binom{5}{3}64) + (4 \cdot 5 \cdot 6 \cdot \binom{4}{2}16) = 15,360.$$

hands	low	nut low boards	nut low
A-2-3-4	.5267	863,932	.5045
A-2-3-5	.5267	773,904	.4520
A-2-3-B	.4932	738,648	.4314
A-2-4-5	.5267	633,638	.3700
A-2-4-B	.4932	561,192	.3277
A-2-5-B	.4932	445,152	.2600
A-3-4-5	.5267	433,376	.2531
2-3-4-5	.5267	433,376	.2531
A-2-B-B	.3562	424,512	.2479
A-3-4-B	.4932	388,112	.2267
2-3-4-B	.4932	388,112	.2267
A-3-B-B	.3562	236,768	.1383
2-3-B-B	.3562	236,768	.1383
A-4-B-B	.3562	117,568	.0687
2-4-B-B	.3562	117,568	.0687

Table 2

If the board has the form xxyzw with a card of rank A or 3, the player must have a wheel for nut low. This means the number of boards giving the player a nut low is

$$(6 \cdot 64) + (6 \cdot 3 \cdot 6 \cdot 16) = 2,112.$$

If the board has five distinct low ranks chosen from {2,4,5,6,7,8}, then the number of boards giving the player a nut low is given by $4 \cdot 5 \cdot 4^4 = 5,120$. If one of the five ranks is A or 3, then the player needs a wheel for a nut low. The number of boards doing it for the player is then $6 \cdot 4^3 \cdot 12 = 4,608$. Finally, if the board has an A and a 3, then the number of boards giving the player a wheel is $9 \cdot 64 = 576$.

Summing the boards giving a player holding A-3-B-B a nut low leads to 236,768 such boards. Dividing by (1) yields a probability of .1383 that the player makes a nut low.

For the hand A-4-B-B, there must be a 2-3 on board in order for the player to have a nut low. This severely limits the nut lows. Going through the same cases we did for A-3-B-B, we get 117,568 boards that give a nut low for a player holding A-4-B-B. Dividing by (1) above yields a probability of .0687 for this happening. This completes the cases where the player has two working low cards.

The hands with three low ranks we examine are A-2-3-B, A-2-4-B, A-3-4-B, A-3-5-B, 2-3-4-B, and 2-3-5-B. The hands with four low ranks we consider are A-2-3-4, A-2-3-5, A-2-4-5, A-3-4-5, and 2-3-4-5. The information for all of these hands is in the Table 2.

Here are a few comments about Table 2. The first column shows the hand, where B denotes an actual card whose rank is chosen from {9,10,J,Q,K}. The reason we have to be fussy is that the probability of making a low with A-2-10-J and A-2-2-J is the same, but the probability of making a nut low for those two hands is not the same. There is no theorem like Theorem 1 for nut lows.

The column headed "low" gives the probability of making a low, the column headed "nut low boards" gives the number of boards producing a nut low, and the column headed "nut low" is the probability of making a nut low.

The hands have been placed in Table 2 in decreasing probability of making a nut low, but this need not be the end of the story. We continue considering these issues in the file entitled *Counterfeiting Low Hands in Omaha*.