NOWHERE-ZERO FLOWS IN REGULAR MATROIDS AND HADWIGER’S CONJECTURE

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To the memory of Reinhard Börger

Abstract. We present a tool that shows, that the existence of a k-nowhere-zero-flow is compatible with 1-, 2- and 3-sums in regular matroids. As application we present a conjecture for regular matroids that is equivalent to Hadwiger’s conjecture for graphs and Tutte’s 4- and 5-flow conjectures.

Keywords: nowhere zero flow, regular matroid, chromatic number, flow number, total unimodularity

1. Introduction

A (real) matrix is totally unimodular (TUM) if each subdeterminant belongs to \{0, \pm 1\}. Totally unimodular matrices enjoy several nice properties which give them a fundamental role in combinatorial optimization and matroid theory. In this note we prove that the TUM possesses an attractive property.

Let \( S \subseteq \mathbb{R} \) and let \( A \) be a real matrix. A column vector \( f \) is a \( S \)-flow of \( A \) if \( Af = 0 \) and every entry of \( f \) is a member of \( \pm S \).

For any additive abelian group \( \Gamma \) use the notation \( \Gamma^+ = \Gamma \setminus \{0\} \). For a TUM \( A \) and a column vector \( f \) with entries in \( \Gamma \), the product \( Af \) is a well defined column vector with entries in \( \Gamma \), by interpreting \((-1)\gamma \) to be the additive inverse of \( \gamma \).

It is convenient to use the language of matroids. A regular oriented matroid \( M \) is an oriented matroid that is representable \( M = M[A] \) by a TUM matrix \( A \). Here the elements \( E(M) \) of \( M \) label the columns of \( A \). Each (signed) cocircuit \( D = (D^+, D^-) \) of \( M \) corresponds to a \{0, ±1\}-valued vector in the row space of \( A \) and having minimal support. The +1-entries in this vector constitute the sets \( D^+ \). It is known [19, Prop. 1.2.5] that two TUMs represent the same oriented matroid if and only if the first TUM can be converted to the second TUM by a succession of the following operations: multiplying a row by \( -1 \), adding one row to another, deleting a row of zeros, and permuting columns (with their labels).

For \( S \subseteq E(M) \) we use the notation \( f(S) = \sum_{e \in S} f(e) \). Let \( M = M[A] \) be the regular oriented matroid represented by the TUM \( A \). Let \( S \subseteq \Gamma \) where \( \Gamma \) is an abelian group. An \( S \)-flow of \( M \) is a function \( f : E(M) \rightarrow S \) for which \( Af = 0 \), where \( f \) is interpreted to be a vector indexed by the column labels of \( A \). For any \( S \subseteq \Gamma \) we say that a regular matroid \( M \) has an \( S \)-flow if any of the TUMs that represent \( M \) has an \( S \)-flow. By the previous paragraph, this property of \( M \) is well defined. Since the rows of a TUM \( A \) generate the cocycle space of \( M = M[A] \), we have that a function \( f : E(M) \rightarrow \Gamma \) is a flow if and only if for every signed cocircuit \( D = (D^+, D^-) \) we have that \( f(D) = 0 \) where \( f(D) \) is defined to equal \( f(D^+) - f(D^-) \).

Let \( \Gamma \) be a finite abelian group. Let \( M \) be a regular oriented matroid, and let \( F \subseteq E(M) \) and let \( f : F \rightarrow \Gamma \). Let \( \tau_(M, f) \) denote the number of \( \Gamma^* \)-flows of \( M \) which are extensions of \( f \).

Theorem 1. Let \( M \) be an regular oriented matroid. Let \( F \subseteq E(M) \) and let \( f, f' : F \rightarrow \Gamma \). Suppose that for every minor \( N \) of \( M \) satisfying \( E(N) = F \), we have that \( f \) is a \( \Gamma \)-flow of \( N \) if and only \( f' \) is a \( \Gamma \)-flow of \( N \). Then \( \tau_((M, f) = \tau_((M, f) \).

Proof. We proceed by induction on \( d = |E \setminus F| \). If \( d = 0 \), then there is nothing to prove. Otherwise let \( e \in E \setminus F \). If \( e \) is a coloop of \( M \), then \( \tau_((M, f) = \tau_((M, f) = 0 \). If \( e \) is a loop of \( M \), then by applying induction to \( M \setminus e \), we have \( \tau_((M, f) = \tau_((M, f') = (|\Gamma| - 1) \tau_((M \setminus e, f) \). Otherwise we apply Tutte’s deletion/contraction formula [3] and induction to get

\( \tau_((M, f') = \tau_((M / e, f') - \tau_((M \setminus e, f') = \tau_((M / e, f) - \tau_((M \setminus e, f) = \tau_((M, f) \). \)
Corollary 2. Let $D$ be a positively oriented cocircuit of a regular oriented matroid $M$. Let $f, f' : D \to \Gamma$. Suppose that for every $S \subseteq D$ we have that $f(S) = 0$ if and only if $f'(S) = 0$. Then $\tau_f(M, f) = \tau_{f'}(M, f')$.

Proof. Let $N$ be a minor of $M$ satisfying $E(N) = D$. Then $E(N)$ is a disjoint union $\bigcup_i D_i$ of positively oriented cocircuits of $N$ [9, Prop. 9.3.1]. Thus $f$ is a $\Gamma^*$-flow of $N$ if and only if $f$ has no zeros, and $f(D_i) = 0$ for each $i$. The result follows from Theorem 1. \qed

Corollary 3. Let $M$ be a regular oriented matroid which has a $\Gamma^*$-flow $f$.

1. Let $e \in E(M)$ and $\gamma \in \Gamma^*$. Then $M$ has a $\Gamma^*$-flow $f'$ with $f'(e) = \gamma$.

2. Let $D$ be a signed cocircuit of $M$ of cardinality three. Let $f' : D \to \Gamma^*$ satisfy $f'(D) = 0$. Then $f'$ extends to a $\Gamma^*$-flow of $M$.

Proof. (1) In any minor $N$ with $E(N) = \{e\}$, both $f'$ and $f_{\{e\}}$ are $\Gamma^*$-flows of $N$ if and only if $N$ is a loop. Thus by Theorem 1 $\tau_f(M, f') = \tau_f(M, f) > 0$.

(2) Let $S \subseteq D$. For any $e \in D$ we have $f'(D \setminus \{e\}) = f'(D) - f'(e) = -f'(e) \neq 0$. Therefore $f'(S) = 0$ if and only if $S = D$. Since $f$ is a $\Gamma$-flow and $D$ is a positively oriented cocircuit of $D$ we have $f(D) = 0$. Since $f(e) \neq 0$ for $e \in D$ we again have that $f(S) = 0$ if and only if $S = D$. It follows from Theorem 1 that $\tau_f(M, f') = \tau_f(M, f) > 0$. \qed

A k-nowhere zero flow (k-NZF) of a regular oriented matroid $M$ is an $S$-flow of $M$ for $S = \{1, 2, \ldots, k-1\} \subseteq \mathbb{R}$. We frequently use the following observation of Tutte [15].

Proposition 4. Let $\Gamma$ be an abelian group of order $k$, and let $S = \{1, 2, \ldots, k-1\} \subseteq \mathbb{R}$. Then $M$ has a k-NZF if and only if $M$ has a $\Gamma^*$-flow. In particular, the existence of a $\Gamma^*$-flow in $M$ depends only on $|\Gamma|$.

A key step in the proof of Proposition 4 is the conversion of a $\Gamma^*$-flow into a k-NZF, where $\Gamma$ is the group of integers modulo $k$. By modifying this argument, one can show that the statement of Corollary 3 remains true if each occurrence of the symbol $\Gamma^*$ is replaced by the set of integers $S = \{\pm 1, \pm 2, \ldots, \pm (k-1)\}$. We omit the proof of this fact, as it is not needed in this paper.

2. Seymour decomposition

We provide here a description of Seymour’s decomposition theorem for regular oriented matroids. We refer the reader to [13] for further details. We first describe three basic types of regular oriented matroids.

A oriented matroid is graphic if it can be represented by the $\{0, \pm 1\}$-valued vertex-edge incidence matrix of a directed graph, where loops and multiple edges are allowed. Any $\{0, \pm 1\}$-valued matrix which whose rows span the nullspace of a network matrix is called a dual network matrix. Dual network matrices are also TUM, and an oriented matroid is cographic if it is representable by a dual network matrix. The third class consists of all the all the orientations of one regular matroid $R_{10}$. Every orientation of $R_{10}$ can be represented by the matrix $[I|D]$ where $B$ is obtained by negating a subset of the columns of the following matrix.

$$
\begin{bmatrix}
+ & 0 & 0 & + & - \\
- & + & 0 & 0 & + \\
+ & - & + & 0 & 0 \\
0 & + & - & + & 0 \\
0 & 0 & + & + & -
\end{bmatrix}
$$

(1)

Here “+” and “−” respectively denote $+1$ and $-1$.

Let $M_1, M_2$ be regular oriented matroids. If $E(M_1)$ and $E(M_2)$ are disjoint, then the 1-sum $M_1 \oplus M_2$ is just the direct sum of $M_1$ and $M_2$. The signed cocircuits of $M_1 \oplus M_2$ are the signed subsets of $E(M_1) \cup E(M_2)$ which are signed cocircuits of either $M_1$ or $M_2$. If $M_1 \cap M_2 = \{e\}$ and $e$ is neither a loop nor a coloop in each $M_i$, then the 2-sum $M_1 \oplus_2 M_2$ has element set $E(M_1) \Delta E(M_2)$, where “$\Delta$” is the symmetric difference operator. A signed cocircuit is a signed subset of $E(M_1 \oplus_2 M_2)$ that is either a signed cocircuit of $M_1$ or $M_2$, or is a signed set of the form

$$
D = (D_1 ^+ \Delta D_2 ^+, D_1 ^- \Delta D_2 ^-)
$$

(2)
where each \((D_1^+, D_1^-)\) is a signed cocircuit of \(M_i\), and \(e \in (D_2^+ \cap D_2^-) \cup (D_2^- \cap D_2^+)\). If \(M_1 \cap M_2 = B\) and \(B = (B^+, B^-)\) is a signed cocircuit of cardinality 3 in each \(M_i\), then the 3-sum \(M_1 \oplus_3 M_2\) has element set \(E(M_1) \cup E(M_2)\). A signed cocircuit is a signed subset of \(E(M_1) \cup E(M_2)\) that is either a signed cocircuit of \(M_1\) or \(M_2\), or a signed subset of the form (2) where each \((D_1^+, D_1^-)\) is a signed cocircuit of \(M_i\), with \(D_1 \cap D_2 = \emptyset\) and \((B^+, B^-)\) equals one of the following ordered pairs:

\[
\begin{align*}
&\{(D_1^+ \cap B^+) \cup (D_2^+ \cap B^+), (D_1^- \cap B^-) \cup (D_2^- \cap B^-)\}, \\
&\{(D_1^+ \cap B^+) \cup (D_2^- \cap B^-), (D_1^- \cap B^+) \cup (D_2^+ \cap B^-)\}.
\end{align*}
\]

The oriented version of Seymour’s decomposition theorem [13] and can be derived from [5, Theorem 6.6].

**Theorem 5.** Every regular oriented matroid \(M\) can be constructed by means of repeated application of \(k\)-sums, \(k = 1, 2, 3\), starting with oriented matroids, each of which is isomorphic to a minor of \(M\) and each of which is either graphic, cographic, or an orientation of \(R_{10}\).

We note that Schriver [12] states an equivalent version of Theorem 5 in terms of TUMs, that requires a second representation of \(R_{10}\) in (1) due to his implicit selection of a basis.

Here is the main tool of this paper, which we employ in two subsequent applications.

**Theorem 6.** Let \(k \geq 2\) be an integer and let \(\mathcal{M}\) be a set of regular oriented matroids that is closed under minors. If every graphic and cographic member of \(\mathcal{M}\) has a \(k\)-NZF, then every matroid in \(\mathcal{M}\) has a \(k\)-NZF.

**Proof.** Let \(M \in \mathcal{M}\). We proceed by induction on \(|E(M)|\). If \(M\) is an orientation of \(R_{10}\), then \(M\) has a 2-NZF since \(R_{10}\) is a disjoint union of circuits, and each circuit is the support of a \(\{0, \pm 1\}\)-flow in \(M\). If \(M\) is graphic or cographic, then we are done by assumption. Otherwise, by Theorem 5, \(M\) has two proper minors \(M_1, M_2 \in \mathcal{M}\), such that \(M = M_1 \oplus M_2\), for some \(i = 1, 2, 3\). By induction, each \(M_i\) has a \(k\)-NZF. Thus by Proposition 4, both minors have a \(\Gamma^*\)-flow where \(\Gamma\) is any fixed group of order \(k\). By Corollary 3, we may assume that these \(\Gamma^*\)-flows coincide on \(M_1 \cap M_2\). Hence the union of these functions is a well defined \(\Gamma^*\)-flow on \(M\) and we are done by another application of Proposition 4. \(\square\)

### 3. Tutte’s Flow Conjectures and Hadwiger’s Conjecture

In this section we will present a conjecture that unifies two of Tutte’s Flow Conjectures and Hadwiger’s Conjecture on graph colorings.

**Conjecture 7 (H(k)[4]).** If a simple graph is not \(k\)-colorable, then it must have a \(K_{k+1}\)-minor.

While H(1) and H(2) are trivial, Hadwiger proved his conjecture for \(k = 3\) and pointed out that Klaus Wagner proved that H(4) is equivalent to the Four Color Theorem [18, 2, 10]. Robertson, Seymour and Thomas [11] reduced H(5) to the Four Color Theorem. The conjecture remains open for \(k \geq 6\).

Tutte [15] pointed out that the Four Color Theorem is equivalent to the statement that every planar graph admits a 4-NZF. Generalizing this to arbitrary graphs he conjectured that

**Conjecture 8 (Tutte’s Flow Conjecture [15]).** There is a finite number \(k \in \mathbb{N}\) such that every bridgeless graph admits a \(k\)-NZF-flow.

and moreover that

**Conjecture 9 (Tutte’s Five Flow Conjecture [15]).** Every bridgeless graph admits a 5-NZF-flow.

Note that the latter is best possible as the Petersen graph does not admit a 4-NZF-flow. Conjecture 8 has been proven independently by Kilpatrick [7] and Jaeger [6] with \(k = 8\) and improved to \(k = 6\) by Seymour [14].

Conjecture 9 has a sibling which is a more direct generalization of the Four Color Theorem.

**Conjecture 10 (Tutte’s Four Flow Conjecture [16, 17]).** Every graph without a Petersen-minor admits a 4-NZF-flow.

In [16, 17] Tutte cited Hadwiger’s conjecture as a motivating theme and pointed out that while “Hadwiger’s conjecture asserts that the only irreducible chain-group which is graphic is the coboundary group of the complete 5-graph”
Conjecture 10 means that

"the only irreducible chain-group which is cographic is the cycle group of the Petersen graph."

The first statement refers to the case where the rows of a totally unimodular matrix $A$ consist of a basis of signed characteristic vectors of cycles of a digraph.

Combining these we derive the following formulation in terms of regular matroids. First let us call any integer combination of the rows of $A$ a coflow. Clearly, by duality resp. orthogonality, flows and coflows yield the same concept in regular matroids. Note that the existence of a $k$-NZ-coflow in a graph is equivalent to

$k$-colorability [16].

Conjecture 11 (Tutte’s Four Flow Conjecture, matroid version). A regular matroid that does not admit a 4-NZ-flow has either a minor isomorphic to the cographic matroid of the $K_5$ or a minor isomorphic to the graphic matroid of the Petersen graph.

Equivalently, we have

Conjecture 12 (Hadwiger’s Conjecture for regular matroids and $k = 4$). A regular matroid that is not 4-colorable, i.e. that does not admit a NZ-4-coflow, has a $K_5$ or a Petersen-dual as a minor.

Some progress concerning this Conjecture was made by Lai, Li and Poon using the Four Color Theorem

Theorem 13 ([8]). A regular matroid that is not 4-colorable has a $K_5$ or a $K_5$-dual as a minor.

Tutte’s Five Flow Conjecture now suggests the following matroid version of Hadwiger’s conjecture:

Conjecture 14 (Hadwiger’s Conjecture for regular matroids and $k \geq 5$). If a regular matroid is not $k$-colorable for $k \geq 5$, then it must have a $K_{k+1}$-minor.

Theorem 15. (1) Conjecture 11 is equivalent to Conjecture 10.

(2) Conjecture 14 for $k = 5$ is equivalent to Conjecture 9.

(3) Conjecture 14 for $k \geq 6$ is equivalent to Conjecture 7.

Proof. (1) By Weiske’s Theorem [4] a graphic matroid has no $K_5^*$-minor. Hence Conjecture 11 clearly implies Conjecture 10. The other implication is proven by induction on $|E(M)|$. Consider a regular matroid $M$, that is not 4-colorable, i.e. that does not admit a NZ-4-coflow. Clearly, $M$ cannot be isomorphic to $R_{10}$. If $M$ is graphic, it must have a $K_5$-minor by the Four Color Theorem [2, 10] and an observation of Klaus Wagner [18]. If $M$ is cographic it must have a Petersen-dual-minor by Conjecture 10. Otherwise, by Theorem 5, $M$ has two proper minors $M_1, M_2 \in M$ such that $M = M_1 \oplus_i M_2$, for some $i = 1, 2, 3$ and at least one of them is not 4-colorable by Theorem 6. Using induction we find either a Petersen-dual-minor or a $K_5$-minor in one of the $M_i$ and hence also in $M$. Thus, Conjecture 10 implies Conjecture 11.

(2) We proceed as in the first case using $H(5)$ for graphs [11] instead of the Four Color Theorem.

(3) We proceed similar to the first case, with only a slight difference in the base case. If $M$ is graphic, it must have a $K_{k+1}$-minor by Conjecture 7. $M$ cannot be cographic by Seymour’s 6-flow-theorem [14].

Remark 16. James Oxley pointed that Theorem 15 could also be proven using splitting formulas for the Tutte polynomial (see e.g. [1]). Seymour’s decomposition and the fact that the flow number as well as the chromatic number are determined by the smallest non-negative integer non-zero of certain evaluations of the Tutte polynomial.

References

NOWHERE-ZERO FLOWS IN REGULAR MATROIDS AND HADWIGER’S CONJECTURE