

# Nullspaces yield new explicit Runge–Kutta pairs

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## Abstract

Sixty years ago Butcher [1] characterized a natural tabulation of the order conditions for Runge–Kutta methods as an isomorphism from the set of rooted trees having up to  $p$  nodes, and provided examples of explicit and implicit methods of several orders. Within a few years. Fehlberg [3] derived *pairs* of explicit methods of successive orders that could be implemented efficiently by using the difference of each pair of estimates to control the local error. Unfortunately, Fehlberg’s pairs were deficient for quadrature problems. Subsequently, this author [5],[6] derived parametric families of explicit Runge–Kutta pairs of increasing orders 6 to 9 that avoided this problem altogether. These, and most known explicit methods, have been derived by exploiting certain ‘simplifying conditions’ suggested by Butcher [1] that imposed constraints on subsets of the coefficients, and thereby simplified the solution of the order conditions for moderate to high order methods.

‘Test 21’, a MAPLE program developed recently by Butcher [2], was applied to derive known 13-stage pairs of orders 7 and 8. Unexpectedly, results of this application revealed the existence of some previously unknown methods - ie. some that satisfied most, *but not all*, of the previously known simplifying conditions. This present study develops formulas for directly computing exact coefficients of these new pairs together with others lying within this new parametric family of (13,7-8) pairs. While the best of these new pairs falls short of the best of pairs already known, the properties discovered might be utilized to precisely characterize recently reported higher order methods found using other approaches by Khashin [4] and Zhang[7], and possibly lead to finding other Runge–Kutta and related yet unknown methods.

## References

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