Math 800, Commutative Algebra, Lecture 4

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1 Monic and Epic

A morphism $f : A \to B$ is monic if for any $g \neq h : C \to A$ then $fg \neq fh$.
A morphism $f : A \to B$ is epic if for any $g \neq h : B \to C$ then $gf \neq hf$.
Also in any category coming from universal algebra, a $1-1$ homomorphism must be monic and an onto homomorphism must be epic.

Property 1. *In $\mathcal{S}et$, $\mathcal{G}rp$, $R-mod$ the converse holds, i.e all monics are $1-1$, all epics are onto.*

Proof. $\mathcal{S}et$: Take $f : A \to B$ monic and take $a,\ b \in A$ with $f(a) = f(b)$. Suppose $a \neq b$, take C any nonempty set, define $g : C \to A$, $g(x) = a, \forall x \in C$ and $h : C \to A$, $h(x) = b, \forall x \in C$. Since C is nonempty we have $g \neq h$ but $fg = fh$, a contradiction.

Take $f : A \to B$ epic, suppose there is $b \in B$ not in the image of f. Choose C to have at least two elements $c,\ c^{'}$ define $g : B \to C$, $g(x) = c, \forall x \in B$, $h : B \to C$, $h(x) = c, \forall x \neq b,\ x \in B$ and $h(b) = c^{'}$. Then $g \neq h$ but $gf = hf$, a contradiction.

$\mathcal{G}$rp: Take $f : A \to B$ monic, take $g, h : Kerf \to A$ with h the canonical embedding and g the zero map. Then $fg = 0 = fh$ implies $g = h$, so $Kerf = 0$.
Note the same argument works in $R-mod$.

Take $f : A \to B$ epic if $f(A) = B$ done so assume not. Let C be the group of permutations of the left coset of $f(A)$ in B. $g : B \to C$, $g(x) = id$, $\forall x \in B$. $h : B \to C$, $g(x) = map_x : bf(A) \mapsto bx^{-1}f(A)$. Then $gf = id = hf$ so $g = h$

which implies $f(A) = B$.

$R - mod$: Suppose $f : A \to B$ epic, define $g, h : B \to B/f(A)$ with g the zero map and h the quotient map. Then $gf = 0 = hf$ infers $g = h$, so $f(A) = B$. $\square$

Property 2. *In Ring every monic is* $1-1$ *but not every epic is onto.*

Proof. take $f : A \to B$ monic, $P = \{(a, b) \in A \times A : \ f(a) = f(b)\}$. Define $g, h : P \to A$ with g the projection onto the first coordinate and h the projection onto the second coordinate. Then $fg = fh$ so $g = h$. The only way this occurs is $P \cong A$.

Consider the natural map $\tau : \mathbb{Z} \to \mathbb{Q}$. τ is monic since it is $1-1$ but it is also epic as if $g, h : \mathbb{Q} \to R$ then g, h are determined by where they send 1 to. So $g \neq h$ implies $g(1) \neq h(1)$ thus $g\tau(1) \neq h\tau(1)$ so epic but not onto. $\square$

2 Initial and terminal objects

Definition. *An object A in a category $\mathcal{C}$ is an* ***initial object*** *(sometimes "universal repelling") provided that for all $B \in Obj(\mathcal{C})$ there is a unique morphism $A \to B$.*

Example. $\mathcal{S}et$*: $\emptyset$ is an initial object.*

$\mathcal{G}rp$*: 1 is an initial object.*

$\mathcal{R}ing$*: $\mathbb{Z}$ is an initial object.*

Property 3. *Initial objects are unique up to unique isomorphism if they exist.*

Proof. Say A, A' are both initial objects, so there's a unique morphism $f : A \to A'$ and a unique morphism $g : A' \to A$. Consider $fg : A' \to A'$ and $gf : A \to A$ but there is a unique morphism $A' \to A'$ so it must be the identity. Thus $fg = 1_{A'}$ and $gf = 1_A$. $\square$

Definition. *An object A in a category $\mathcal{C}$ is a* ***terminal object*** *(or universal attracting) provided that $\forall B \in Obj(\mathcal{C})$ there exists a unique morphism $B \to A$.*

Example. $\mathcal{S}et$*: any singleton set is a terminal object.*

$\mathcal{G}rp$*: 1 is a terminal object.*

Property 4. *Terminal objects are unique up to unique morphism if they exist.*

Proof is the same.

Definition. *An object which is both an intial and a terminal object is called a* ***zero object****.*

3 Products and Coproducts

Definition. *Let* $\{A_i\}_{i\in I}$ *be a family of objects of* $\mathcal{C}$*. The coproduct of the* A_i *is an object of* $\mathcal{C}$ $\coprod\limits_{i\in I} A_i$ *along with morphisms* $\nu_i : A_i \to \coprod\limits_{i\in I} A_i$ *such that* $\forall X \in Obj(\mathcal{C})$ *and every family of morphism* $\alpha_i : A_i \to X$ *there exists a unique* $\Theta : \coprod\limits_{i\in I} A_i \to X$ *such that* $\forall i$*,*

$$\begin{tikzcd} A_i \arrow[r, "\nu_i"] \arrow[d, "\alpha_i"'] & \coprod\limits_{i\in I} A_i \arrow[dl, dotted, "\Theta"] \\ X & \end{tikzcd} \qquad (*)$$

commutes.

Example. $\mathcal{S}et$: *coproduct is disjoint union.*

$R-mod$: coproduct is direct sum $\bigoplus M_i$*.*

Definition. *Let* $\{A_i\}_{i\in I}$ *be a family of objects of* $\mathcal{C}$*. The product of the* A_i *is an object of* $\mathcal{C}$ $\prod\limits_{i\in I} A_i$ *together with morphisms* $\pi_i : \prod\limits_{i\in I} A_i \to A_i$ *such that* $\forall X \in Obj(\mathcal{C})$ *and all morphism* $\beta_i : X \to A_i$ *there exists a unique* $\Theta : X \to \prod\limits_{i\in I} A_i$ *such that* $\forall i$*,*

$$\begin{tikzcd} A_i & \prod\limits_{i\in I} A_i \arrow[l, "\pi_i"'] \\ X \arrow[u, "\beta_i"] \arrow[ur, dotted, "\Theta"'] & \end{tikzcd} \qquad (**)$$

commutes.

Example. *In every category coming from universal algebra, the product is the cartesian product.*

Property 5. *Products and coproducts are unique if they exist.*

Proof. Build the category where objects are pairs of an element of $\mathcal{C}$ and a family of morphisms from the A_i (coproduct case) or to the A_i (product case). The morphism of the category are morphisms of the first elements of the pair such that (*) commutes (coproduct case) or (**) commutes (product case). Then the definition of coproduct says that the coproduct is initial in this category and the definition of product says that the product is terminal in its category. So they are each unique. $\square$

A reference is "Joy of Cats"

4 Kernels

Definition. *A category has zero morphisms if every $Hom(A, B)$ has a morphism 0_{AB} s.t $\forall X, Y, Z \in Obj(\mathcal{C})$ and $\forall f : Y \to Z,\ g : X \to Y$,*

$$\begin{tikzcd} X \arrow[r, "0_{XY}"] \arrow[d, "g"'] \arrow[dr, "0_{XZ}"] & Y \arrow[d, "f"] \\ Y \arrow[r, "0_{YZ}"'] & Z \end{tikzcd}$$

commutes.

Definition. *In a category with zero morphisms (pointed category) let $f : X \to Y$ be a morphism then a kernel of f is a morphism $k : K \to X$ s.t*

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ K \arrow[ur, "k"] \arrow[rr, "0_{KY}"'] & & Y \end{tikzcd}$$

and given any $k^{'} : K^{'} \to X$ s.t the analogue triangle holds and there exists a unique $v : K^{'} \to K$

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ K^{'} \arrow[ur, "k^{'}"] \arrow[r, dotted, "v"] \arrow[rr, bend right, "0_{K^{'}Y}"'] & K \arrow[u, "k"] \arrow[r, "0_{KY}"] & Y \end{tikzcd}$$

Here kernel of f is defined by a universal property so by the same type of argument as for products and coproducts it is unique if it exists. So write $k = kerf$.

Property 6. *kerf is monic.*

Proof. Take $g \neq h : C \to K$ say $(kerf)g = (kerf)h$ then

$$\begin{tikzcd} & & X \arrow[dr, "f"] & \\ K' \arrow[urr, "(kerf)g=(kerf)h"] \arrow[rr, dotted, "v"] \arrow[rrr, bend right, "0_{K'Y}"'] & & K \arrow[u, "k"] \arrow[r, "0_{KY}"] & Y \end{tikzcd}$$

here map v could be either g or h contradicting the uniqueness in the universal property. □

Note

(1) We have cokernel by reversing the arrows in all of the above.

(2) Rowen sets up a partial order on monics and then picks the biggest as the kernel but this will be the terminal one. Also don't need pre-additive just having zero morphism is enough.

(3) Kernel in usual sense for say $R - mod$ is the K.

(4) But this doesn't capture all algebraic examples of kernels. eg., ring doesn't have 0 morphisms so this definition doesn't work. But it has kernels in algebraic sense.