We continue our discussion of the direct sum and direct products of families of modules $\{M_i\}_{i\in I}$.

The co-product of modules $\{M_i\}_{i\in I}$ will be the sub module of $\prod_{i\in I} M_i$ such that all but finitely many of the terms are zero: This will be denoted

$$\oplus_{i\in I} M_i$$

For each $i \in I$ there is a obvious map $v_i : M_i \hookrightarrow \oplus_{i\in I} M_i$ that sends m_i to the element of $\oplus_{i\in I} M_i$ with m_i "in the i^{th}" spot. Formally it is the element $(a_j)_{j\in I}$ where $a_i = m_i$ and $a_j = 0$ for all $j \neq i$.

To see that $\oplus_{i\in I} M_i$ is a direct sum in the category of R modules let $t_i : M_i \to T$ be R module maps. Define, $\theta : \oplus_{i\in I} M_i \to T$ by

$$\theta((m_i)_{i\in I}) = \sum_{i\in I} t_i(m_i)$$

Since $(m_i)_{i\in I}$ has only finitely many non-zero coordinates the above sum is well defined. It is easily verified that θ is a homomorphism of R modules. The proof is omitted. Furthermore, for some fixed $j \in I$ and $m_j \in M_j$ we have $\theta v_j(m_j) = t_j(m_j)$ as all the coordinates of $v_j(m_j)$ are zero except m_j. So $\theta v_j = t_j$ as desired. Furthermore, if $\psi : \oplus_{i\in I} M_i \to T$ such that $\psi v_j = t_j$ then given an element $(m_i)_{i\in I}$ let $\hat{m}_i$ be the element with m_i in the

$$\psi((m_i)_{i\in I}) = \sum_{i\in I} \psi(v_i(m_i)) = \sum_{i\in I} t_i(m_i) = \theta((m_i)_{i\in I})$$

Where we use that ψ is a module map and the fact that the above are finite sums.

Remark and Question, skip if you want: Do we get anything interesting if instead of restricting to finite subsets we restrict up to a specific cardinality. For example in our case we allow sums of less then $\aleph$ elements. What happens if we decide we want to allow sums less then $\mathfrak{c}$ elements? I guess that such things would make us think about the Continuum Hypothesis and related issues. We could get around it by allowing at most countable sums. So we could ask some questions, is the module that allows at most countable sums ever isomorphic to the co-product? It is not always so as we could take $M_i = \mathbb{Z}$ and then $\oplus_{n=1}^{\infty}\mathbb{Z}$ is countable, but if we allow countable sums the resulting module is not. I can think of at least one situation where it may be useful to consider countable sums. In functional analysis there is the notion of a Hamel Basis and a Schroeder Basis. The Hamel basis, is the usual vector space basis. However, when the dimension is infinite it can be cumbersome to work with when the size is large. One often considers a Schroeder Base which is a sequence of elements $\{p_n\}$such that every element of the space v can be written $v = \sum_{i=1}^{\infty} \lambda_i p_i$ for scalars λ_i. Where the aforementioned sum converges with respect to some norm on the space. We might want to try and model this situation algebraically. I can imagine one might have some idea of "non convergence" in our space, which would be perhaps a sub module, we could then take all countable sums and mod out by the "non-convergent" elements. Of course this is all just me talking out loud and perhaps they are totally worthless.

We resume: We now come to a internal description of the direct product

Suppose that $M = \sum_{i \in I} M_i$ and if $\sum_{\text{finite}} a_i = 0$ with $a_i \in M_i$ then each $a_i = 0$ we call M the internal direct sum of the M_i.

For the case of two sub modules, the above we can see that M is a internal direct sum of K, N if $M = K + N$ and $K \cap N = (0)$. It is clear that in this case $M \cong K \oplus N$.

Split Epics: Definition: Given a short exact sequence

$$0 \to K \xrightarrow{f} M \xrightarrow{\pi} N \to 0$$

we say that the sequence splits if there is some homomorphism $g : N \to M$ such that $\pi g = 1_N$. We also say that in general $\pi : M \to N$ splits if there is some map $g : N \to M$ with $\pi g = 1_N$.

Ex: Every short exact sequence of vector spaces splits.

Warning: Not every short exact sequence splits.

Proposition: Let $N \leq M$. N is a direct summand if and only if there is a split epic $\pi : M \to N$.

Proof: Suppose that $M = N \oplus K$ then π_N, the projection unto N is a split epic. Conversely, suppose that $\pi : M \to N$ is a split epic with $g : N \to M$ such that $\pi g = 1_N$. Put

$$\pi' = 1_M - g\pi$$

Now take $K = \pi'(M)$. Note that
$\pi(1_M - g\pi) = \pi - (\pi g)\pi = \pi - 1_N \pi = \pi - \pi = 0$. So $K \subseteq \ker \pi$. Now take $N_1 = g(N_1)$. Note that g must be a monic as $gn = 0$ means $\pi g n = n = 0$ so that $N_1 \cong N$. I claim that $M = K + N_1$. So for each $m \in M$ we have $\pi'(m) + g\pi(m) = m - g\pi(m) + g\pi(m) = m$. Since $\pi'(m) + g\pi(m) \in K + N_1$ we have my claim. I now claim that $N_1 \cap K = 0$ as suppose that for some $b \in N$ we have $g(b) \in K$ so $g(b) = m - g\pi(m)$ for some $m \in M$. So, $b = \pi g(b) = \pi(m) - \pi g \pi(m) = 0$ which means $g(b) = 0$. So we have verified that condition that M is a internal direct sum of N_1 and K. ■

Definition: A module is said to indecomposable if it cannot be written as a direct sum of proper sub-modules.

It is then reasonable to attempt to find all modules over R and then given a module M find out how to decompose M into indecomposable sub modules. According to Rowen but classifying the indecomposable sub modules is apparently difficult.

Bases and Generating Sets: Definition:

A subset S of a module M is said to be a base for is a base if any element of M can be uniquely written as a R linear combination of S elements.

A module with base is called Free.

Given a commutative ring R we define $R^{(n)} = \oplus_{i=1}^{n} R$.

We write $e_k := (0, ..., 1, ...)$ that has 1 at the k^{th} coordinate and zero elsewhere.

The idea is that rings with bases allow us to do some sort of linear algebra.

Lemma:

Let R have a base $\mathcal{B}$. Then morphisms out of R are determined by their action on the base.

Proof: Any element of R can be written as a finite sum. $\sum r_i b_i$. Now let $(a_b)_{b\in\mathcal{B}}$ be elements in a R module A. Define, $\phi(b) = a_b$ and $\phi(\sum r_i b_i) = \sum r_i \phi(b_i)$. It is easily seen that this defines a homomorphism of R modules. On the other hand, if ϕ and ψ are R modules maps that agree on a base into say M then $\phi(\sum r_i b_i) = \sum r_i \phi(b_i) = \sum r_i \psi(b_i) = \psi(\sum r_i b_i)$ so $\phi = \psi$.

■

Prop:

Let M be a finitely generated module. Then $F/K \cong M$ where F is a free module.

Proof: Suppose that M is generated by n elements, say $m_1, ..., m_n$. Define a map $\phi : R^{(n)} \to M$ with $\phi(e_k) = m_k$. Then it is immediate that ϕ is a surjection, so by the Isomorphism theorem

$$M \cong R^{(n)}/\ker\phi \quad ■$$

Prop:

Any module with a finite base, is isomorphic to a free module.

Proof: Suppose $b_1, ..., b_n$ is a base for a R module M. Define $\psi : R^{(n)} \to M$ as above. I.E. $\psi(e_k) = b_k$. We know that ψ is surjective so it remains to check the kernel. Now suppose that

$$0 = \psi(\sum_{k=1}^{n} r_k e_k) = \sum_{k=1}^{n} r_k \psi(e_k) = \sum_{k=1}^{n} r_k b_k$$

Since the b_k are a base we have $r_k = 0$ for each k so $\ker\phi = (0)$ and so $M \cong R^{(n)}$. ■

We keep building up the theory, of free modules.

Lemma: Suppose that $f : M \to N$ is a onto module map, $\ker f = K$. If N has a base of size n and K has a base of size j then M has a base of size $n + j$.

Proof: Take $f(m_1), ..., f(m_n)$ as a base for N (we can always do this by assumption) and $k_1, ..., k_j$ as base for K. I claim that $m_1, ...m_n, k_1, ..., k_j$ is a base of M. Indeed, let $m \in M$. Then we have
$f(m) = \sum_{i=1}^{n} r_i f(m_i) \Rightarrow m - \sum_{i=1}^{n} r_i m_i \in K$. That is,
$m = \sum_{i=1}^{n} r_i m_i + \sum_{p=1}^{j} s_p k_p$ so that $m_1, ...m_n, k_1, ..., k_j$ span M. On the other

hand, suppose that

$$\sum_{i=1}^{n} r_i m_i + \sum_{p=1}^{j} s_p k_p = 0$$

then applying f we have

$$f(\sum_{i=1}^{n} r_i m_i + \sum_{p=1}^{j} s_p k_p) = f(\sum_{i=1}^{n} r_i m_i) \quad = \textstyle\sum_{i=1}^{n} r_i f(m_i) = 0$$

since $\sum_{p=1}^{j} s_p k_p \in \ker f$. Since the the $f(m_i)$ are a base the $r_i = 0$ for each i. But this means that $\sum_{p=1}^{j} s_p k_p = 0$ so the $s_p = 0$ for each p as the k_i are a base as well. So the $m_1, ...m_n, k_1, ..., k_j$ are indeed a base as desired. ■

We now go into a little more linear algebra.

Given a commutative ring R we define $M_{m\times n}(R)$ to be $m \times n$ matrices with coefficients in R. All the usual linear algebra works, we multiply matrices and so forth the usual way.

We get some standard things. The adjoint of of a $n \times n$ matrix A is the matrix with A_{ij} the matrix formed by taking the determinant of the i, j co factor.

The standard proof goes through so that $Aadj(A) = adj(A)A = (\det A)I_n$.

We also have the following.

Prop: The following are equivalent,

(i)A is right invertible

(ii)A is invertible

(iii) $\det A$ is invertible in R.

Proof: Use the adjoint lemma above. (It is explicitly done in the text on page 59)

Continuing he linear algebra theme we have the following.

Fix Bases $\{e_1, ..., e_n\}$ of $R^{(n)}$ and $\{f_1, ..., f_m\}$ on $R^{(m)}$ where n, m are arbitrary. There is a one to one correspondence

$$\Phi : \{\text{maps } \varphi : R^{(n)} \to R^{(m)}\} \to \{n \times m \text{ matrices over } R\}$$

$\Phi(\varphi) = (r_{i,j})$ is defined by $\varphi(e_i) = \sum_{j=1}^{m} r_{i,j} f_j$. Also, $\Phi(\psi\varphi) = \Phi(\psi)\Phi(\varphi)$.

Proof: See the text, page 60.

We finally get to the the proof we have been working towards.

That, if $R^{(n)} \cong R^{(m)}$ then $n = m$. That is, the rank of a finitely generated free module is well defined. There are at least two ways I know how to prove this. There is the proof in the text that uses matrices and the determinant. Here is a non-matrix version. Well, the matrices at least are not in sight. We still use linear algebra.

Fix a commutative ring R.

Sketch of the proof:$\varphi : R^{(n)} \to R^{(m)}$ be a surjection

Choose a maximal ideal $\mathfrak{m}$ of R.(We can always do this with Zorns lemma) Pass to the quotient $R^{(n)}/\mathfrak{m}R^{(n)} \cong (R/\mathfrak{m})^{(n)}$. Then φ induces a surjection Φ :

$R^{(n)}/\mathfrak{m}R^{(n)} \to R^{(m)}/\mathfrak{m}R^{(m)}$. But this is a map of vector spaces so linear algebra tells us that $n \geq m$. (This is a exercise more or less in our text and in Dummit and Foote. In Dummit and Foote they reference $R^{(n)}/\mathfrak{m}R^{(n)} \cong (R/\mathfrak{m})^{(n)}$. With this idea the proof is relatively easy to come upon)

Lemma: It is easy to show that, If $A_1, ..., A_n$ are R modules over a commutative ring and B_i a sub module of A_i for each i. Then,

$$\prod_{i=1}^{n} A_i / \prod_{i=1}^{n} B_i \cong \prod_{i=1}^{n} (A_i/B_i)$$

Proof: If $a_i \in A_i$ let $\bar{a_i} = a_i + B_i$. Then $f(a_1, ..., a_n) = (\bar{a_1}, ..., \bar{a}_n)$ is a module homomorphism and clearly a surjection. Its kernel is precisely $\prod_{i=1}^{n} B_i$ so we can apply the isomorphism theorem. ■

Now let $\varphi : R^{(n)} \to R^{(m)}$ be a surjection. I claim that $n \geq m$. Let $\mathfrak{m}$ be a maximal ideal of R. For any R module M we can define IM to be all finite sums of elements $\sum i_j m_j$ with $m_j \in M$ and $i_j \in I$. Then, IM is a sub module of M. I claim that,

$$R^{(n)}/\mathfrak{m}R^{(n)} \cong \prod_{i=1}^{n} (R/\mathfrak{m}) \quad (*)$$

This follows from the lemma as $\mathfrak{m}R = \mathfrak{m}$ is a sub-module of R and $\mathfrak{m}R^{(n)} = \prod_{i=1}^{n} \mathfrak{m}$. (Because we defined it this way)

Now, we define $\Phi : R^{(n)}/\mathfrak{m}R^{(n)} \to R^{(m)}/\mathfrak{m}R^{(m)}$ by

$$\Phi(x + \mathfrak{m}R^{(n)}) = \varphi(x) + \mathfrak{m}R^{(m)}$$

To see that this is well defined note that if $x + \mathfrak{m}R^{(n)} = y + \mathfrak{m}R^{(n)}$ then there is a element $z \in \mathfrak{m}R^{(n)}$ such that $x + z = y$.

But $z = \sum_{j=1}^{n} i_j a_j$ with $i_j \in \mathfrak{m}$ and $a_j \in R^{(n)}$. So $\varphi(\sum_{j=1}^{n} i_j a_j) = \sum_{j=1}^{n} i_j \varphi(a_j) \in \mathfrak{m}R^{(m)}$. So $\varphi(z) \in \mathfrak{m}R^{(m)}$ and consequently
$\Phi(y + \mathfrak{m}R^{(n)}) = \varphi(y) + \mathfrak{m}R^{(m)} = \varphi(x+z) + \mathfrak{m}R^{(m)} = \varphi(x) + \varphi(z) + \mathfrak{m}R^{(m)} = \varphi(x) + \mathfrak{m}R^{(m)} = \Phi(x)$.

So Φ is well defined, it is a module map and is a surjection because φ is. But by (*) Φ is a map of vector spaces since $R/\mathfrak{m}$ is a field. So we can apply the usual linear algebra to obtain $n \geq m$ as desired.

I'm going to close out this section a short note on Free modules.

We can also define free modules via a universal property. Let A be a set. A free module on a set A is a module F_A and a set map $\iota : A \to F_A$ such that if M is any module and $f : A \to M$ is any set mapping then there is a unique module homomorphism $\theta : F_A \to M$ that makes the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "f"'] & F_A \arrow[dl, "\theta"', "\exists!"] \\ M & \end{tikzcd}$$

One has to do a general construction to show that such modules always exists for a given set. But they in fact do, so one can to some degree forget about the definition and just work with the universal property. For the details see say Dummit and Foote.

The free modules of rank n that we have constructed are the free modules on $\{1, ..., n\}$ with $\iota(k) = e_k$ for $k = 1, ..., n$ into $R^{(n)}$.

Indeed, let $f : \{1, ..., n\} \to M$ be a set map and let $\iota(k) = e_k$ be a map into $R^{(n)}$.

Now set $\theta(e_k) = f(k)$. So θ is indeed a module map, and $\theta\iota = f$. Furthermore, if $\psi\iota = f$ then $\psi(\iota(k)) = \psi(e_k) = f(k)$ so that ψ and θ agree on a basis and are so the same. So $R^{(n)}$ satisfies the universal property of free modules. (This is basically just that maps are determined by their values on the base) We also obtain via the above definition that F_A is unique up to isomorphism by the usual abstract nonsense argument. Namely let F_1, F_2 satisfy the universal property with maps ι_1, ι_2. We have the following diagram via the universal property applied to F_1, F_2.

$$\begin{tikzcd} F_1 \arrow[d, "\phi"'] & A \arrow[l, "\iota_1"'] \arrow[r, "\iota_2"] & F_2 \arrow[d, "\psi"] \\ F_2 & & F_1 \end{tikzcd}$$

Note that

$$\begin{tikzcd} A \arrow[r, "\iota_i"] \arrow[d, "\iota_i"'] & F_i \arrow[dl, "1_{F_i}"', "\exists!"] \\ F_i & \end{tikzcd}$$

Satisfies the universal property so the identity is the unique map F_i to F_i that satisfies the above diagram. Now, $\psi\phi : F_1 \to F_1$ and

$$\psi\phi\iota_1 \quad = \quad \psi(\phi\iota_1) = \psi\iota_2 = \iota_1$$

Since $\psi\phi$ satisfies the second diagram by uniqueness $\psi\phi = 1_{F_1}$. Similarly, $\phi\psi = 1_{F_2}$ so that $F_1 \cong F_2$ as desired.

Given all that, Free modules are somewhat nice.

For example, suppose that $\{F_\alpha\}_{\alpha \in I}$ are free R modules. Then

$$\oplus_\alpha F_\alpha \text{ is Free}$$

Proof: Let B_α be a base or F_α. The set $\sum_\alpha B_\alpha$ clearly span $\oplus_\alpha F_\alpha$ as given $(x_\alpha)_\alpha \in \oplus_\alpha F_\alpha$. Then, $x_\alpha = \sum_{\text{finite}} r_{\alpha,k} b_{\alpha,k}$. Since there are finitely many non-zero x_α we have

$$(x_\alpha)_\alpha = \sum_{x_\alpha \neq 0} \sum_{\text{finite}} r_{\alpha,k_\alpha} b_{\alpha,k_\alpha}$$

Which is a finite linear combination. On the other hand, suppose that there is a finite linear combination of $\{b_{\alpha_1,k}\}_{k=1}^{n_{\alpha_1}}, \ldots, \{b_{\alpha_p,k}\}_{k=1}^{n_{\alpha_p}}$ with scalars $r_{\alpha_i,k}$ such that

$$\sum_{i=1}^{p} \sum_{k=1}^{n_{\alpha_i}} r_{\alpha_i,k} b_{\alpha_i,k} = 0$$

Then, since we are working with a direct sum we have that each of the components are zero, namely $\sum_{k=1}^{n_{\alpha_i}} r_{\alpha_i,k} b_{\alpha_i,k} = 0$ which means the $r_{\alpha_i,k} = 0$ as the b_{α_i} are basis elements. This applies for each i so all the $r_{\alpha_i,k} = 0$ and so $\sum_\alpha B_\alpha$ is a basis.

So direct sums of free modules are free. However, it is NOT the case the arbitrary products of free modules are free. In fact, $\prod_{i=1}^{\infty} \mathbb{Z}$ can be shown to not be a free $\mathbb{Z}$ module. This is in a exercise of Dummit and Foote, page 358. I may add it at some point or someone else can if they want.