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THE DEFINITION OF WEIL COHOMOLOGY

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1. Introduction

In these notes, we outline the definition of a *Weil cohomology* (WC). It should be noted that a WC encodes all the nice properties any cohomology theory should satisfy. In particular, any universal *motivic* cohomology should encode all the properties of a WC.

These notes are highly unoriginal in the sense that they are a very close (and often verbatim) rewrite of parts of section 3 of Kleiman's paper in the AMS *Motives* volume 1. We have merely added some comments and explanations to the same. The origins of WC lies in the Weil conjectures on the ζ-function of a variety over a finite field.

2. Main definition

Let k and K be fields of arbitrary characteristic and zero characteristic respectively. Let **SmProj** be the category of smooth projective schemes over k with smooth scheme maps as morphisms. Let **VectGr** be the category of **Z**-graded anticommutative K-algebras with K-algebra maps that preserve gradings as morphisms.

Suppose V is an object of **VectGr**. Therefore

$$V = \bigoplus_{i=0}^{n} V_i.$$

We set $V_0 = K$.

A WC is a contravariant functor

$$H^* : \mathbf{SmProj} \longrightarrow \mathbf{VectGr},$$

that satisfies the following properties:

2.1. **Finiteness.** Each $H^i(X)$ has finite dimension. The object $H^i(X) = 0$ unless $0 \leq i \leq 2 \dim X$.

2.2. **Poincaré duality.** We present two equivalent versions of the Poincaré duality. The first version goes as follows: Let $r := \dim X$. For each such X, there is an isomorphism

$$H^{2r}(X) \xrightarrow{\simeq} K$$

and a nondegenerate bilinear pairing

$$H^i(X) \times H^{2r-i} \xrightarrow{\simeq} K$$

given by $(x, y) \mapsto \langle x \cdot y \rangle$.

Given some $u \in H^*(X)$, we denote the image of u under the map $H^*(X) \to H^{2r}(X)$ as $\langle u \rangle$.

The second version of Poincaré duality goes as follows:

$$\widehat{H^i(X)} \simeq H^{2r-i}(X),$$

where $\widehat{H^i(X)} := \mathrm{Hom}(H^i(X), K)$. This version will prove useful when we think of *correspondences* later on.

2.3. **Künneth formula.** Let $X \times_k Y$ denote the fiber product between two objects, X and Y, of **SmProj**. Consider the diagram of projections

$$\begin{tikzcd} X \times_k Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & X \\ Y & \end{tikzcd}$$

These projections induce the isomorphism

$$H^*(X) \otimes_K H^*(Y) \xrightarrow{\simeq} H^*(X \times_k Y).$$

Note that since we are working over fields, the "Tor" part of the Künneth formula that we encounter in topology textbooks vanishes.

2.4. **Cycle maps.** Let $C^i(X)$ denote the free abelian group generated by the closed irreducible subschemes of X of codimension i. By definition, each element of $C^i(X)$ is a *cycle* and the group is referred to as a *group of cycles of codimension i on X*. (With additional structure such as gradation and subject to certain equivalence conditions, this group is referred to in the literature as the *Chow group*.) In a WC, there should be a map ("the cycle map") that is a group homomorphism

$$\gamma_X^i : C^i(X) \longrightarrow H^{2i}(X)$$

satisfying the following properties:

(1) [Functoriality] There are two functorial conditions on cycle maps. The first one ("pull back") goes as follows: Let $r := \dim X$. Let $f : X \to Y$ be a morphism of **SmProj**. Let $C^i(X)$ and $C^i(Y)$ denote the group of cycles of codimension i of X and Y respectively. The following diagram commutes:

$$\begin{tikzcd} C^i(Y) \arrow[r, "f^*"] \arrow[d, "\gamma_Y^i"'] & C^i(X) \arrow[d, "\gamma_X^i"] \\ H^{2i}(Y) \arrow[r, "f^*"] & H^{2i}(X) \end{tikzcd}$$

Notice that this digram commutes because H^* is a contravariant functor. The upper map in the diagram expresses the pull back on cycles and we (by abusing notation) denote it as f^* which is in fact the map we obtain by applying the functor H^* to the morphism f.

The second functorial condition ("push forward") goes as follows: Let $s := \dim Y$. Now consider the pull back map defined above- if

$$H^{2i}(Y) \xrightarrow{f^*} H^{2i}(X),$$

then by Poincaré duality, we obtain the push forward map

$$H^{2r-2i}(X) \xrightarrow{f_*} H^{2s-2i}(Y)$$

such that the following diagram commutes

$$\begin{tikzcd} C^{r-i}(X) \arrow[r, "f_*"] \arrow[d, "\gamma_X^{r-i}"'] & C^{s-i}(Y) \arrow[d, "\gamma_Y^{s-i}"] \\ H^{2r-2i}(X) \arrow[r, "f_*"] & H^{2s-2i}(Y) \end{tikzcd}$$

The map on the top of the diagram expresses push forward of cycles.

(2) [Multiplicativity] $\gamma^{i+j}_{X\times_k Y}(Z\times_k W) = \gamma^i_X(Z)\otimes_K \gamma^j_Y(W)$. We see, by applying the definition of the cyle map and the Künneth formula, that

$$Z\times_k W \in C^{i+j}(X\times_k Y) \xrightarrow{\gamma^{i+j}_{X\times_k Y}} H^{2(i+j)}(X\times_k Y) \xrightarrow{\simeq} \bigoplus_l H^{2(i+j)-l}(X)\otimes_K H^l(Y).$$

Now $H^{2i}\otimes_K H^{2j} \subset \bigoplus_l H^{2(i+j)-l}(X)\otimes_K H^l(Y)$ and $\gamma^i_X\otimes_K\gamma^j_Y \in H^{2i}\otimes_K H^{2j}$ so the multiplicativity axiom makes sense.

(3) [Calibration] Let P be a point. Then the cycle map $\gamma_P : C^0(P) \longrightarrow H^0(P)$ is the same as the inclusion of the integers $\mathbf{Z}$ into K.

2.5. **Lefschetz theorems.** There are two versions of the Lefschetz theorems, one "weak" and the other "strong". Let $h : W \to X$ be the inclusion of a smooth hyperplane section W of some smooth projective scheme X of dimension r.

(1) [Weak Lefschetz] The induced map $h^* : H^i(X) \longrightarrow H^i(W)$ is an isomorphism for $i \le r-2$ and an injection for $i = r-1$.

(2) [Strong Lefschetz] Define the *Lefschetz operator*

$$L : H^i(X) \longrightarrow H^{i+2}(X),$$
$$Lx = x\cdot\gamma^i_X(W).$$

Then, for $i \le r$,

$$L^{r-i} : H^i(X) \xrightarrow{\simeq} H^{2r-i}(X).$$

Of course, we do know that $H^i(X) \xrightarrow{\simeq} H^{2r-i}(X)$ by Poincaré duality. It seems that the point behind strong Lefschetz is to prove Poincaré duality by constructing the operator L.

To summarize: A Weil cohomology is a contravariant functor from **SmProj** to **VectGr** that satisfies the finiteness conditions, Poincaré duality, has a Künneth formula, cycle maps that have certain (above mentioned) requisite properties and, finally, where the Lefschetz theorems hold true.

3. Some properties of cycle maps

We conclude by pointing out some properties that cycles satisfy.

3.1. **Intersection pairing and cup products.** Let Z and W be two properly intersecting cycles in X and let $Z \cdot_{\text{int}} W$ denote the intersection pairing between Z and W. Let $\Delta : X \to X \times_k X$ denote the diagonal map. Then $\gamma_X(Z \cdot_{\text{int}} W) = \gamma_{X \times_k X} \circ \Delta^*(Z \times_k W)$ where Δ^* denotes the pull back of $Z \times_k W$. We from the functoriality of cycle maps that this equals $\Delta^* \circ \gamma_{X \times_k X}(Z \times_k W) = \Delta^*(\gamma_X(Z) \otimes_K \gamma_X(W))$ by the multiplicativity of cycle maps. So we get

$$\Delta^*(\gamma_X(Z) \otimes_K \gamma_X(W)) = \gamma_X(Z) \cdot \gamma_X(W)$$

by applying the pull back again.

3.2. **Correspondences as operators.** We know that $H^*(X \times_k Y) = H^*(X) \otimes_K H^*(Y)$ by the Künneth formula. Now $H^*(X) \otimes_K H^*(Y) = \widehat{H^*(X)} \otimes_K H^*(Y)$ by Poincaré duality. This, by definition, means $H^*(X) \otimes_K H^*(Y) = \text{Hom}(H^*(X), K) \otimes_K H^*(Y)$. This equals $\text{Hom}(H^*(X), H^*(Y))$ by elementary linear algebra. Therefore we can view each element of $H^*(X \times_k Y)$ as defining a linear operator from $H^*(X)$ to $H^*(Y)$. Such operators are called *correspondences*.

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