Patterns in Feynman graph denominators

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CMS winter meeting
December 6, 2010

Story

What do I like to do? I like discrete problems which are inspired by and gain richness from other areas.

I will tell a story about **graphs**, **polynomials**, and **quantum field theory**. For those who have seen this story before, there are new and interesting denominator identities today.

Set up

Let G be a 4-regular graph. Remove any vertex v of G and consider

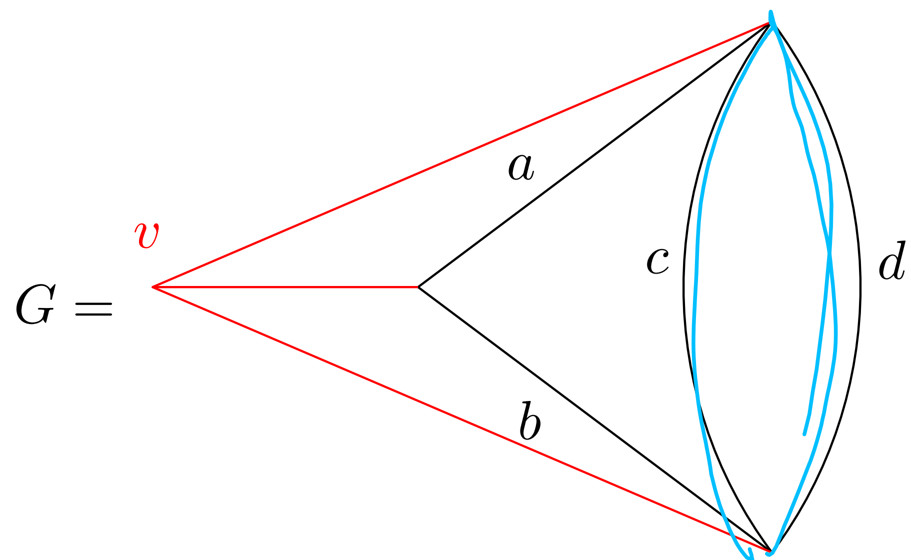
$$\int_{e_i \geq 0} \frac{\delta(e_1 + \cdots + e_n) \prod de_i}{\Psi^2}$$

where Ψ is the Kirchhoff polynomial of $G \smallsetminus v$,

$$\Psi = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G \smallsetminus v}} \prod_{e \notin T} a_e$$

- This will converge provided all proper subgraphs of $G \smallsetminus v$ have more than twice as many edges as independent cycles.
- That this is independent of the choice of removed vertex is a theorem, but it is not known how to see it using this representation of the integral.

Example



Spanning trees of $G \setminus v = \{ac, ad, bc, bd, ab\}$

$$\Psi_G = bd + bc + ad + ac + cd$$

$$\int \frac{\delta(a+b+c+d)}{((c+d)(a+b)+cd)^2}$$

will diverge as c and d get large.

Origin

This is a problem from quantum field theory.

- The graph with a vertex removed is a 4-point graph in massless scalar field theory.
- The integral is its Feynman integral in Schwinger parametric form.
- The convergence criterion is the condition of primitivity in the renormalization Hopf algebra.
- The independence of choice of vertex is conformal invariance.

$$\int_{x=0}^{\infty} \frac{1}{(Ax+B)^2} = \int \frac{1}{AB}$$

A naive approach

Consider

$$\int_{e_i \geq 0} \frac{\prod de_i}{\Psi^2}$$

one edge variable at a time (Francis Brown).

So long as there is always a variable e so that the denominator is a product of two linear polynomials in e,

$$(Ae + B)(Ce + D),$$

then we can do the e integration next, getting explicit, increasingly complex polylogarithms in the numerator and

$$AD - BC$$

in the denominator.

If the denominator is the square of a linear polynomials in e,

$$(Ae + B)^2,$$

then we can again do the e integration. This time the weight of the polylogarithms in the numerator does not increase.

If all is nice we will end up evaluating some polylogarithms at 1. This gives multiple zeta values.

Multiple zeta values

$$\zeta(s_1,\ldots,s_n) = \sum_{a_1>\cdots>a_n\geq 1} \frac{1}{a_1^{s_1}\ldots a_n^{s_n}}$$

The **weight** of $\zeta(s_1,\ldots,s_n)$ is $s_1+\cdots+s_n$.

Multiple zeta values

- generalize special values of the Riemann zeta function
- have an interesting algebra structure and relations
- are the periods of moduli spaces
- $\ldots$

Consequences

Everything is controlled by some combinatorics of polynomials

- We know exactly how things go bad – when the polynomial does not factor. We can understand combinatorial criteria for this to happen, or be avoided.
- We will get a weight drop when the denominator has a factor which is a square or one of the edge variables is missing entirely.
- This works even with $\prod a_e^{\epsilon_e}$ in the numerator – a full epsilon expansion, the numbers physicists really want.

The Dodgson polynomials

The main tool for understanding the denominators are some polynomials

$$\Psi^{I,J}_{K,G}$$

which we can understand **graphically** or via **matrices**. Each viewpoint has its uses.

Matrices

Suppose G has n vertices and m edges. Let $\widehat{E}$ be the incidence matrix with one column removed. Build the matrix

$$M = (-1)^{n+1} \left[\begin{array}{ccc|c} a_1 & & & \\ & \ddots & & \widehat{E} \\ & & a_m & \\ \hline & \widehat{E}^T & & 0 \end{array}\right]$$

Then

$$\Psi_G = \det(M)$$

Let I, J, K be sets of edges of G with $|I| = |J|$. Let $M_G(I,J)_K$ be the matrix obtained from M_G by removing the rows of I, the columns of J, and setting $\alpha_e = 0$ for all $e \in K$. Then

$$\Psi^{I,J}_{G,K} = \det M_G(I,J)_K \ .$$

Spanning forest polynomials

Let $P = P_1 \cup \ldots \cup P_k$ be a set partition of a subset of the vertices of G. Define

$$\Phi_G^P = \sum_F \prod_{e \notin F} \alpha_e$$

where the sum runs over spanning forests $F = T_1 \cup \ldots \cup T_k$ where each tree T_i of F contains the vertices in P_i. Trees consisting of a single vertex are permitted.

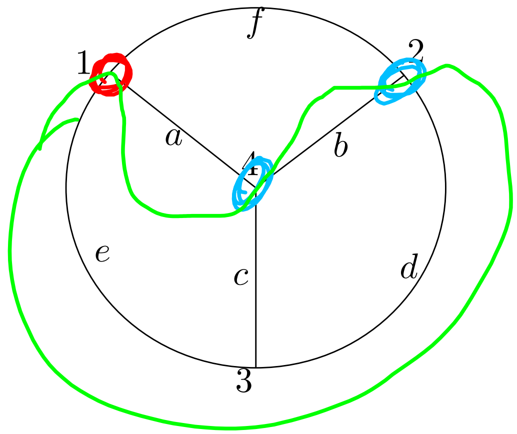
Then

$$\Psi_{G,K}^{I,J} = \sum \pm \Phi^P_{G \setminus I \cup J \cup K}$$

where P runs over partitions of the vertices adjacent to edges in I and J so that the resulting terms are trees after

- cutting I and contracting J **and**
- cutting J and contracting I.

Example



Let $P = \{1\}, \{2, 4\}$. Then

$$\Phi_G^P = af(ec + ed + cd + be)$$

What is $\Psi_G^{a,f}$?

Structure in the denominators

The first few integrations look like:

$$\int \frac{1}{\Psi_G}$$

$$\int \frac{1}{\Psi_G^{1,1}\Psi_{G,1}}$$

$$\int \frac{\text{logs}}{(\Psi_G^{1,2})^2}$$

$$\int \sum \frac{\text{logs}}{\text{stuff}}$$

$$\int \frac{\text{dilogs}}{\Psi_G^{12,34}\Psi_G^{13,24}} + \frac{\text{dilogs}}{\Psi_G^{12,34}\Psi_G^{14,23}} + \frac{\text{dilogs}}{\Psi_G^{13,24}\Psi_G^{14,23}}$$

$$\int \frac{\text{trilogs}}{{}^5\Psi_G(1,2,3,4,5)}$$

The 5-invariant

The denominator after five integrations is given by

$$^{5}\Psi_G(1,2,3,4,5) = \pm \det \begin{pmatrix} \Psi^{12,34}_{G,5} & \Psi^{125,135}_{G} \\ \Psi^{13,24}_{G,5} & \Psi^{135,245}_{G} \end{pmatrix}$$

Up to sign it doesn't depend on order.

4 and 6 and onward

It would be nice to have a 4-invariant too. We have

$$\Psi_G^{12,34}\Psi_G^{13,24}$$
$$\Psi_G^{12,34}\Psi_G^{14,23}$$
$$\Psi_G^{13,24}\Psi_G^{14,23}$$

We'd like to think of **any one** of these as a 4-invariant. This is justified because any one of these gives the 5-invariant at the next integration.

We'd also like to have 6-invariants, 7-invariants, etc. This is not always possible. The 5-invariant may not factor, or it may, but the 6 may not, ...

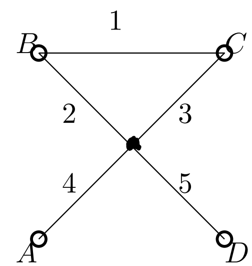
Write

$$D_G^n(i_1,\ldots,i_n)$$

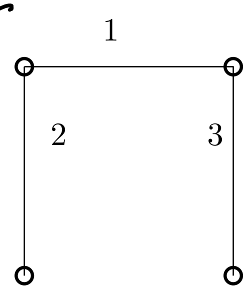
for the nth denominator when it exists.

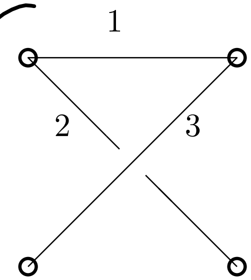
Denominator identities I

Let

$J =$ 

Let

$K_1 =$ 

$K_2 =$ 

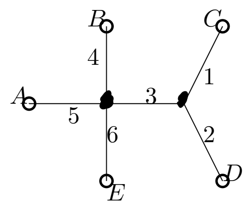
with the same remaining graph connecting at the circled vertices. Pick any 6th edge from among the remaining edges.

Theorem 1

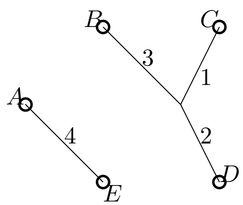
$$D^6_J = \pm D^4_{K_1} \pm D^4_{K_2}$$

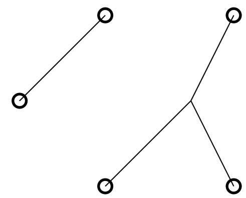
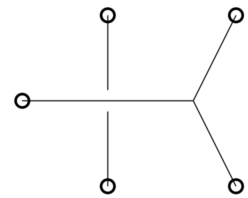
Denominator identities II

Let

$G =$ 

Let

$H_1 =$ 

$H_2 =$ 
$H_3 =$ 

with the same remaining graph connecting at the circled vertices.

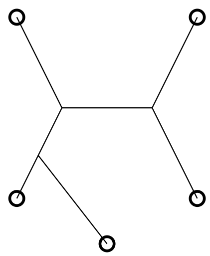
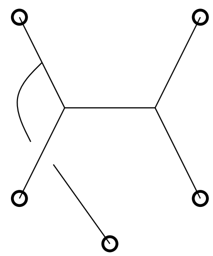
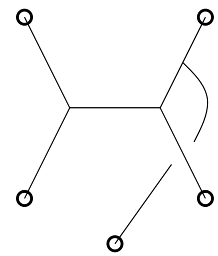
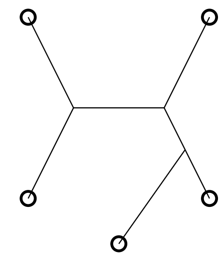
Theorem 2

$$D^6_G = \pm D^4_{H_1} \pm D^4_{H_2} \pm D^4_{H_3}$$

Denominator identities III

We can play the same games even if the graph is not almost 4-regular.

Let

$L_1 =$  $L_2 =$  $L_3 =$  $L_4 =$ 

with the same remaining graph connecting at the circled vertices.

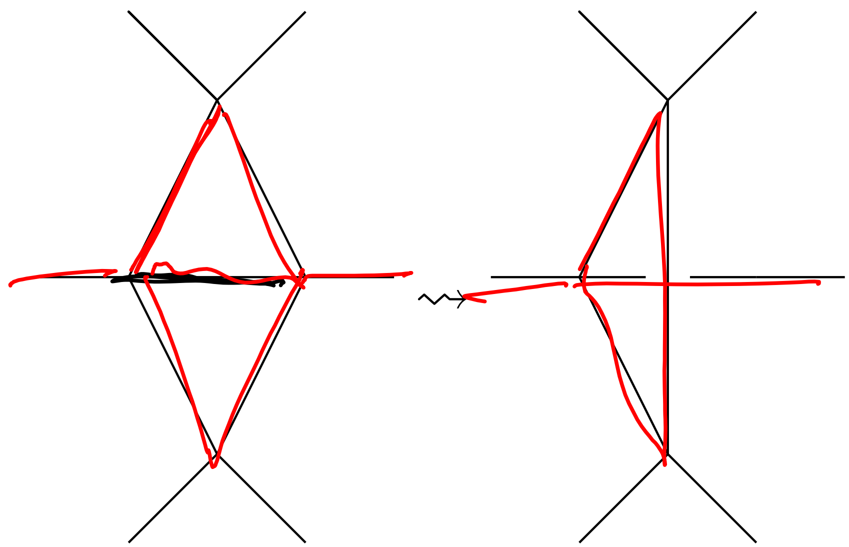
Theorem 3

$$D^7_{L_1} \pm D^7_{L_2} \pm D^7_{L_3} \pm D^7_{L_4} = 0$$

A special case – double triangle

Suppose a graph has two triangles which share an edge.

Contract the shared edge, remove the other edges of the triangles and reconnect the three vertices in a triangle.



These graphs have the same denominator after the implicated edges have been integrated out.

Double triangle is a special case of Theorem 1

Double triangle is special because it says that one denominator is the same as another, and since denominators determine the weight drops, it says that **one graph has weight drop if another, simpler graph does.**

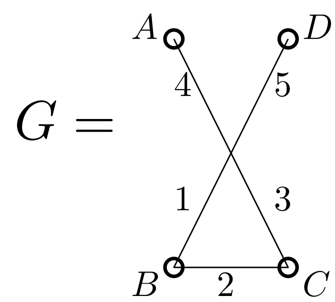
Proofs

All of these theorems are proved by manipulating the Dodgson polynomials and spanning forest polynomials.

Simpler but along the same lines is the direct proof of the double triangle identity. It will be most instructive to show it here.

Proof of double triangle – 1 triangle

Let



with circles to indicate where the rest of the graph is attached. Let K be the rest of the graph.

Calculate

$$^5\Psi_G(1,2,3,4,5) = \pm\Psi_G^{123,245}\Psi_{G,2}^{14,35}.$$

$$\Psi_{G,2}^{14,35} =$$

$$=$$

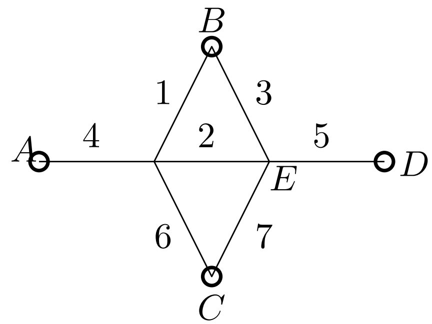
$$\Psi_{G}^{123,245} =$$

$$=$$

So

$$^{5}\Psi_G(1,2,3,4,5) = \pm\Phi_K^{\{A,D\},\{B\},\{C\}}\left(\Phi_K^{\{A,B\},\{C,D\}} - \Phi_K^{\{A,C\},\{B,D\}}\right)$$

Proof of double triangle – 2 triangles

$G' =$ 

Let K again be the rest of the graph.

By the above applied to edges $1, 3, 2, 4, 6$ we know that

$$^5\Psi_{G'}(1,2,3,4,6) = \pm\Phi_{K\cup\{5,7\}}^{\{A,C\},\{B\},\{E\}}\left(\Phi_{K\cup\{5,7\}}^{\{A,B\},\{C,E\}} - \Phi_{K\cup\{5,7\}}^{\{A,E\},\{B,C\}}\right)$$

The two ends of edge 7 are in different parts of $\{A, C\}, \{B\}, \{E\}$ so

$$\Phi_{K\cup\{5,7\}}^{\{A,C\},\{B\},\{E\}} = \alpha_7\Phi_{K\cup 5}^{\{A,C\},\{B\},\{E\}}$$

So we can easily continue the denominator reduction with edge 7.

$$ {}^{6}\Psi_{G'}(1,2,3,4,6,7) = \pm\Phi_{K\cup 5}^{\{A,C\},\{B\},\{E\}}\Phi_{K\cup 5}^{\{A,B\},\{C\}} $$

$$ = $$

From the pictures we can read off the contractions and deletions of edge 5 and deduce that the reduction with respect to edge 5 is

$$ {}^{7}\Psi_{G'}(1,2,3,4,5,6,7) $$

$$ = $$

We have

$$
\begin{aligned}
&{}^{7}\Psi_{G'}(1,2,3,4,5,6,7)\\
&= \pm\left(\Phi_K^{\{A,B\},\{C\},\{D\}}\Phi_K^{\{A,C\},\{B\}} - \Phi_K^{\{A,C\},\{B\},\{D\}}\Phi_K^{\{A,B\},\{C\}}\right)
\end{aligned}
$$

But this is itself a five-invariant, ${}^{5}\Psi_G(1,2,3,4,5)$ so by the previous calculation

$$
D_7(G') = \pm\left(\Phi_K^{\{A,B\},\{C,D\}} - \Phi_K^{\{A,C\},\{B,D\}}\right)\Phi_K^{\{A,D\},\{B\},\{C\}}
$$