Summary: Analytic Nullstellensatz part 2

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Last seminar we saw the (analytic) Nullstellansatz for varieties of complex germs. We also saw a proof for the analytic Nullstellensatz; however we omitted the proof of the key theorems used to and also ended with a floating definition. Now we shall produce some machinery to fill in the gaps of the Nullstellensatz proof from last seminar. First we will give a reminder of what the analytic Nullstellensatz entails.

Theorem 0.1 (Analytic Nullstellensatz). *Let I be an ideal of $n\mathcal{H}_0$ then $id(loc(I)) = \sqrt{I}$.*

Now we shall give some definitions including the definition missing from last seminar.

Definition 0.2. Let V, W be topological spaces. A *proper map* $\pi : V \to W$ is a map that is continuous and if K is compact in W then $\pi^{-1}(K)$ is compact in V.

Definition 0.3. Let V_0 be an open subset of $\mathbb{C}^n$ and W_0 be an open subset of $\mathbb{C}^m$.

Then $\pi : V_0 \to W_0$ is *locally biholopmorphic* if for all $\lambda \in V_0$ there is a neighbourhood, U, of λ such that $\pi : U \to \pi(U)$ is a biholomorphism.

Definition 0.4. Let V be a holomorphic subvariety of $\mathbb{C}^n$ and W be a holomorphic subvariety of $\mathbb{C}^m$.

Let $\pi : V \to W$ be a finite-to-one proper holomorphic map.

Then π is a *finite branched holomorphic cover* if there are dense open subsets $W_0 \subseteq W$, $V_0 \subseteq V$ such that:

- $V_0 = pi^{-1}(W_0)$
- $W \setminus W_0$ is a subvariety of W.
- $\pi : V_0 \to W$ is locally biholomorphic.

We call $\pi : V_0 \to W$ a *dense regular subvariety* of $\pi : V \to W$.

For some examples of a biholomorphic map let $V = \{(z, w)\mathbb{C}^2 : z^2 - w^3 = 0\}$ and $W = \mathbb{C}$. Let

$$\pi_1(z, w) = z \tag{1}$$

and

$$\pi_2(z, w) = w. \tag{2}$$

Let $V_0 = V \setminus \{(0, 0)\}$ and $W_0 = W \setminus \{0\}$. It is left as an exercise that π_1 and π_2 are biholomorphic via W_0 and V_0.

Now we shall show that finite branched holomorphic covers are more than just locally biholomorphic.

Definition 0.5. Let V_0 be an open subset of $\mathbb{C}^n$ and W_0 be an open subset of $\mathbb{C}^m$.

Then $\pi : V \to W$ is a *finite holomorphic covering map* if for all $w \in W_0$ there is a neighbourhood, A, of w such that π^{-1} is a finite disjoint union of open sets which are biholomorphic with A with $\pi : A \to \pi^{-1}$ their biholomorphism.

Prop 0.6 (Proposition 4.4.2)**.** *If $\pi : V \to W$ is a finite branched holomorphic cover with W_0 and V_0 as above then $\pi : V_0 \to W_0$ is a finite holomorphic covering map.*

Proof. Take $w \in W$.

Let $\pi(w)^{-1} = \{\lambda_1, \ldots, \lambda_m\}$.

□

It can be shown that the number of points in $\pi^{-1}(w)$ for $w \in W_0$ is locally constant. If W_0 is connected, $|\pi^{-1}(w)| = r$ is a constant. We say that $\pi : V \to W$ and $\pi : V_0 \to W_0$ are *pure order* r.

The maps in Equations 1 and 2, π_1 and π_2 are pure order 2 and 3 respectively.

Prop 0.7. *Let $\pi : V \to W$ be a finite branched holomorphism from $W : V_0 \to W_0$.*

If W_0 is locally connected, $w \in W$ and $\lambda \in pi^{-1}(w)$ then there are arbitrarily normal neighbourhoods U of λ and $A = \pi(V)$ of w such that $\pi : U \to A$ is a finite branched holomorphic cover of pure order.

Definition 0.8. Let $\pi : V \to W$ be a finite branched holomorphism from $W : V_0 \to W_0$ and U and A be as in Proposition 0.7.

Then the pure order stabilizes on sufficiently small neighbourhoods of λ and is called the *branching order* of π at λ.

We shall now review polynomial theory.

Definition 0.9. If an integral domain is integrally closed in its field of fractions then we say its a *normal domain*.

An example of a normal domain is given in the next theorem.

Theorem 0.10. *UFD's are normal domains.*

Definition 0.11. Let k be a field and $p \in k[x]$. Then discriminant of p is

$$d_p = \prod_{i \neq j} (x_i - x_j)^2,$$

where $x_1, \ldots, x_n$ are the roots of p in some splitting field.

Theorem 0.12. *If A is a normal domain and k is the field of fractions of A then for all $p \in k[x]$, $d_p \in A$ and $d_p = 0$ if and only if p has multiple roots.*

Now we shall return to developing our tools for proving the analytic Nullstellensatz. Our main tool is:

Prop 0.13. *Choose coordinates on* $\mathbb{C}^n$ *and let* $m < n$.

Let $\pi : \mathbb{C}^n \to \mathbb{C}^m$ *be projected onto the first* m *coordinates.*

Let W *be connected and open in* $\mathbb{C}^m$ *and* $U = \pi(W)$.

For $j = m+1, \ldots, n$ *let* $p_j \in \mathcal{H}(W)[z_j]$ *be monic of degree greater than or equal to 1.*

Write $p_j(z', z_j)$ *to evaluate the coefficients of* p_j *at* z' *and* p_j *at* z.

If

$$V = \{(z_1, \ldots, z_n) \in U : p_j(\pi(z), z_j) = 0, \forall j = m+1, \ldots, n\}$$

then $\pi : V \to W$ *is a finite branched holomorphic cover.*

What is key for us is the construction of V_0 and W_0. Let $d_j \in \mathcal{H}(W)$ be the image of p_j. Let $D = \cup_{j=m+1}^{n} V(d_j)$. Then we set $W_0 = W \setminus D$ and $V_0 = \pi^{-1}(W_0)$.

Lemma 0.14. *Given positive integers* n *and* r *there exists a finite set of linear functionals,* $\{f_1, \ldots, f_r\}$ *such that for any set of distinct points* $\{z_1, \ldots, z_r\} \subseteq \mathbb{C}^n$ *there is some* i *such that* $f_i(z_i) \neq f_j(z_i)$ *for all* j.

Prop 0.15 (Propsition 4.4.6)**.** *Let* $W \subseteq \mathbb{C}^m$ *be connected and open and* D *be a proper subvariety of* W.

Let $W_0 = W \setminus D$ *and* $\pi : \mathbb{C}^n \to \mathbb{C}^m$ *be the projection with coordinates chosen in Prop 0.13.*

If

- V_0 *is a subvariety of* $\pi^{-1}(W_0)$ *with* $\bar{V}_0 \subseteq \pi^{-1}(W)$,
- $\pi : V_0 \to W_0$ *is a holomorphic covering map of order* r *and*
- $\pi : \bar{V}_0 \to W$ *is a proper map*

then

1. $\bar{V}_0$ *is a subvariety of* $\pi^{-1}(W)$,
2. $\pi : \bar{V}_0 \to W$ *is a finite branched holomorphic cover,*
3. *for all* $w \in W$ *there are at most* r *elements in* $\pi^{-1}(w) \cap \bar{V}_0$ *and*
4. *each* $f \in \mathcal{H}(\bar{V}_0)$ *is a root of a monic polynomial of degree* r *with coefficients in* $\mathcal{H}(W)$.

From this theorem we get two Corollaries (ref) used to prove the Nullstellensatz.

The final piece to prove the Nullstellensatz is:

Theorem 0.16 (Theorem 4.5.4)**.** *Let* $\mathcal{P}$ *be a prime ideal of* ${}_n\mathcal{H}_0$.

Suppose $m < n$ *and coordinates have been chosen* $\mathbb{C}^n$ *so that* ${}^{{}_n\mathcal{H}_0}/_{\mathcal{P}}$ *is a finite of* ${}_n\mathcal{H}_0$ *then* $loc\mathcal{P} = V' \cup V''$ *where* V' *and* V'' *are subvarieties of* $loc\mathcal{P}$ *such that* $\pi : \mathbb{C}^n \to \mathbb{C}^m$ *gives* V' *as the germ of a finite branched holomorphic cover of pure order on a neighbourhood of 0 in* $\mathbb{C}^n$.