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Summary of Ch. 8: Overview of analytic methods for multivariate generating functions

Big Goal: how do we make smart choices of contours of integration so that we can asymptotically evaluate the Cauchy integral

$$a_r = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z} \qquad ?$$

Generally... push the chain of integration down to a *critical point*, and then interpret the integral locally as an instance of a type that is classically understood *(e.g. saddle point methods, generating functions)*

Steps at end of Section 1.3:

(1) Use the multidimensional Cauchy integral to express a_r as an integral over a d-dimensional torus T in $\mathcal{C}^d$.
(2) Observe that T may be replaced by any cycle homologous to $[T]$ in $H_d(\mathcal{M})$ where $\mathcal{M}$ is the domain of holomorphy of the integrand.
(3) Deform the cycle to lower the modulus of the integrand as much as possible; use Morse theoretic methods to characterize the minimax cycle in terms of *critical points.*
(4) Use algebraic methods to find the critical points; these are points of $\mathcal{V}$ that depends on the direction $\hat{\mathbf{r}}$ of the asymptotics, and are saddle points for the magnitude of the integrand.
(5) Use topological methods to locate one of more contributing critical points, z_j and replace the integral over T by an integral over *quasi-local* cycles $\mathcal{C}(z_j)$ near each z_j.
(6) Evaluate the integral over each $\mathcal{C}(z_j)$ by a combination of residue and saddle point techniques.

- Chapter 8: **An Overview!**
- Chapter 9: Smooth point asymptotics
- Chapter 10: Multiple point asymptotics
- Chapter 11: Cone point asymptotics

1. Exponential rate

Can we get a first nontrivial estimation of a_r?

- exponential level: $\log|a_r| \sim g(\mathbf{r})$ as $\mathbf{r} \to \infty$.
- Possible violation: if an oscillatory term exists and the modulus (size) of the oscillatory factor in a_r is too small.

Better: when uniform as $\hat{\mathbf{r}}$ varies over some neighbourhood of $\hat{\mathbf{r}}_*$.

- Thus: smooth the exponential rate by replacing the rate function$g$ by the lim sup neighbourhood rate function $\overline{\beta}$.

Definition 1.1.

$$\overline{\beta}(\hat{\mathbf{r}}_*) = \inf_{\mathcal{N}} \lim_{\mathbf{r}\to\infty} \sup \left(|\mathbf{r}|^{-1} \log |a_{\mathbf{r}}|\right)$$

where $\mathcal{N}$ varies over a system of open neighbourhoods of $\hat{\mathbf{r}}_*$ whose intersection is the singleton $\{\hat{\mathbf{r}}_*\}$.

The above definition had a typo in the lecture notes that included a sum. This has been fixed here which makes the inequality $\overline{\beta}(\mathbf{r}) \le \beta^*(\mathbf{r})$, seen later, simple to prove.

Example 1.2. $a_{rs} = \binom{r+s-1}{s} - \binom{r+s-1}{r}$ *corresponds to the bivariate generating function*

$$\sum_{i,j} a_{i,j} x^i y^j = \frac{(x-y)}{(1-x-y)}.$$

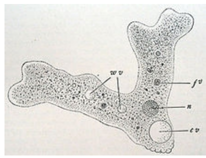
Naively: $a_{rr} = \binom{r+r-1}{r} - \binom{r+r-1}{r} = 0$ *and so we'd get* $\log|0| = -\infty$.
New definition: $\log 2$

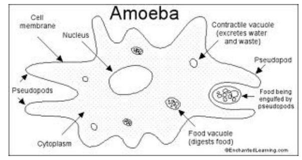
An important upper bound for $\overline{\beta}$: for any $\mathbf{x} \in B$, convergence of $\sum_{\mathbf{r}} a_{\mathbf{r}} z^{\mathbf{r}}$ implies the magnitude of the terms goes to zero.

This is because: $\mathbf{z} = \exp(\mathbf{x}+i\mathbf{y})$ and $a_{\mathbf{r}} = o(\exp(-\mathbf{r}\cdot\mathbf{x}))$ as $\mathbf{r} \to \infty$ so the set of $\mathbf{r}$ such that $a_{\mathbf{r}} \geq \epsilon \exp(-\mathbf{r}\cdot\mathbf{x})$ is finite for any $\epsilon > 0$.

We interested in the function $F(\mathbf{x})) = \frac{G(\mathbf{x})}{H(\mathbf{x})}$, where G and H are holomorphic.

An **amoeba** is a genus of Protozoa consisting of shapeless unicellular organisms.





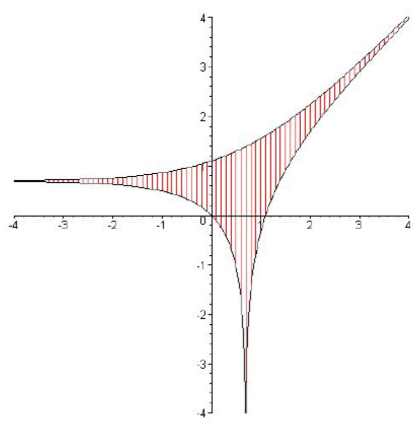
In complex analysis, an amoeba is a set associated with a polynomial in one or more complex variables.
Properties:

- Any amoeba is a closed set.
- Any connected component of the complement $\mathbb{R}\backslash\mathcal{A}_p$ is convex.
- The area of an amoeba of a not identically zero polynomial in two complex variables is finite.
- A two-dimensional amoeba has a number of 'tentacles' which are infinitely long and exponentially narrowing towards infinity.

We just need to know that the amoeba of a function f is defined by

$$amoeba(f) := \{Re \log z : f(z) = 0\}$$

Example 1.3. *amoeba*$(2 - x - y)$



(a) amoeba$(2 - x - y)$

and that the amoeba Legendre transform is defined by

$$\beta^*(\mathbf{r}) := \inf\{-\mathbf{r} \cdot \mathbf{x} : \mathbf{x} \in B\},$$

where B is a connected component of the amoeba H.

This function depends on the polynomial H and component B of $amoeba(H)^c$, but we suppress this notation. Taking the infimum over $\mathbf{x} \in B$ gives:

$$\overline{\beta}(\mathbf{r}) \leq \beta^*(\mathbf{r}).$$

This β^* is semialgebraic (a semialgebraic set is a subset S of R^n for some real closed field R, e.g. $\mathbb{R}$, defined by a finite sequence of polynomial equations of the form $P(x_1, \ldots, x_n) = 0$, and inequalities of the form $Q(x_1, \ldots, x_n) > 0$ or any finite union of such sets) which apparently means that it is **computable**. This leads us to want to know when $\overline{\beta} = \beta^*$ so that we can compute the exponential rate.
Related to: can a *dominating point* be found?

Consider B a component of the boundary of an amoeba, which means it is convex. Let $\mathbf{r} \in (\mathbb{R}^d)^*$. First we try to find out if the infimum in

$$\beta^*(\mathbf{r}) := \inf\{-\mathbf{r} \cdot \mathbf{x} : \mathbf{x} \in B\}$$

is achieved on $\overline{B}$.
Easiest case: infimum is $-\infty$.

Proposition 1.4. *If H is a Laurent polynomial (has both positive and negative powers) and the infimum of $-\mathbf{r}\cdot\mathbf{x}$ on a component B of amoeba$(H)^c$ is $-\infty$, then $a_r = 0$.*

Proof.
- Notation: $B^{(c)}$ is set of points whose ϵ-neighbourhood is contained in B¿
- If $-\mathbf{r}\cdot\mathbf{x}$ is unbounded from below: choose $\mathbf{x}_n \in B^{(c)}$ with $-\mathbf{r}\cdot\mathbf{x}_n \leq -n$.
- Otherwise: $B^{(c)}$ and $-\mathbf{r}\cdot\mathbf{x}$ are semialgebraic so can choose $\{\mathbf{x}_n\}$ so that there is a polynomial lower bound $|H(\mathbf{x}_n)| \geq n^{-\alpha}$ for some α.
- So for some polynomial $P(n)$ and the torus $\mathbf{T}(\mathbf{x_n}) := \exp(\mathbf{x} + i\mathbb{R}^d)$,
$$|\mathbf{z}^{-1}F(\mathbf{z})| \leq P(n)$$
on that torus.
- Using Cauchy's integral formula on $\mathbf{T}(\mathbf{x_n})$ and using that $|z^{-r}| \leq e^n$ on $\mathbf{T}(\mathbf{x_n})$ gives:
$$|a_r| = \left|\left(\frac{1}{2\pi i}\right)^d \int_{\mathbf{T}(\mathbf{x_n})} \mathbf{z}^{\mathbf{r-1}} F(\mathbf{z}) d\mathbf{z}\right|$$
$$\leq \frac{1}{(2\pi)^d} |\mathbf{T}(\mathbf{x_n})| e^{-n} P(n).$$
- The $(2\pi)^{-d}$ cancels the volume of the torus
- Notice: $e^{-n}P(n) \to 0$.
- Done.

$\square$

What if we don't have such a nice quotient of Laurent polynomials and instead have a more general meromorphic function G/H with H analytic and power series converging on a domain B? Then, when $-\mathbf{r}\cdot\mathbf{x}$ is unbounded from below on B, the a_r coefficients decay super-exponentially. Pemantle and Wilson suggest trying a saddle point method directly (see Section 3.2?)

Notation: $\Xi :=$ the set $\mathbf{r}$ when $\beta^*(\mathbf{r}) > -\infty$. A cone.

If $\beta^*(\mathbf{r}) := \inf\{-\mathbf{r}\cdot\mathbf{x} : \mathbf{x} \in B\}$ is finite, it will be achieved unless B has an asymptote in the direction normal to $\mathbf{r}$. Assuming this is not the case, the infimum is achieved uniquely unless B fails to be strictly convex and its boundary contains a line segment. Most common case: the *non-flat case*

Definition 1.5. *The direction $\hat{\mathbf{r}}_* \in \Xi$ is* non-flat *if the infimum $\beta^*(\mathbf{r}) := \inf\{-\mathbf{r}\cdot\mathbf{x} : \mathbf{x} \in B\}$ is attained at a unique point $\mathbf{x}_{\min}(\hat{\mathbf{r}}_*)$, of $\overline{B}$, which we call the* minimizing point *for $\hat{\mathbf{r}}_*$.*

Note that $\mathbf{x}_{\min}$ must lie on the boundary of B because extrema of linear functions are never in the interior of a set.
Deducing the domain of convergence for a nonflat direction $\hat{\mathbf{r}}_* \in \Xi$ with minimizing point $\mathbf{x}_{\min}$:
- The Cauchy integral chain of integration may be taken as the torus $\mathbf{T}(\mathbf{x})$ for any $\mathbf{x} \in B$.
- Sending $\mathbf{x} \to \mathbf{x}_{\min}$ gives another proof that $\overline{\beta}(\mathbf{r}) \leq \beta^*(\mathbf{r})$.
- Could observe: terms of a convergent power series must go to zero.
- More illuminating: using the multivariate Cauchy formula: if we deform the chain of integration beyond $\mathbf{T}(\mathbf{x}_{\min})$ to a chain on which $-\mathbf{r}\cdot\mathbf{x}$ is bounded above by some $c < \beta^*(\mathbf{r})$, we can deduce that $\overline{\beta} \leq c < \beta^*$.
- If not? The strong evidence that $\overline{\beta}(\mathbf{r}) = \beta^*(\mathbf{r})$.
- Be careful: it is not true that $\overline{\beta}(\mathbf{r}) = \beta^*(\mathbf{r})$ iff the chain $\mathbf{T}(\mathbf{x})$ for $\mathbf{x} \in B$ cannot be deformed into a chain supported by $\{\mathbf{x} : -\mathbf{r}\cdot\mathbf{x} < \beta^*(\mathbf{r})\}$.

They conject that there is some modified version of the above that is correct. However, looking into these deformations means you need to leave behind nice 'easy-to-visualize' chains (like tori) and hang out in a more topological (as opposed to geometrical) realm. The rest of the chapter does this....

2. Morse theory

The domain of holomorphy of a rational function F is the complement of the zero set of the denominator.

Example 2.1. $F(\mathbf{z}) = \frac{1}{1-x-y}$ *has the zero set in the denominator* $x + y = 1$*, so the domain of holomorphy is* $\mathbb{C}^2$ *without that.*

It is also an open subset of $\mathbb{C}^d$, namely the manifold

$$\mathcal{M} := \mathbb{C}^d \backslash \{\mathbf{z} : (z_1 \cdots z_d) : H(z) = 0\}$$

which is obtained by removing the coordinate hyperplanes and the singular variety $\mathcal{V}$.

Stokes' Theorem: Let ω be a $p-1$-form ($p \geq 1$) on a manifold $\mathcal{M}$ of dimension at least p and let $\mathcal{C}$ be a p-chain on $\mathcal{M}$. Then

$$\int_{\partial \mathcal{C}} \omega = \int_{\mathcal{C}} d\omega.$$

So... Stokes' theorem tells us that if ω is a d-form, i.e. exactly what we have and holomorphic on a domain $\mathcal{D}$ in $\mathbb{C}^d$ then $\int_C \omega$ depends only on the homology class of C in $H_d(\mathcal{D})$. There is a theorem in the Appendix that proves this. Here, our ω is the d-form $\omega := \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z}$ and our domain is the manifold $\mathcal{M}$ so we are able to get that $\int_C \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z}$ depends only on the homology class of C in $H_d(\mathcal{M})$.

So what have we done? We have gotten to step (2) in those steps outlined in Section 1.3. Today we are going to get to step (4). Let's move on.

We can fix $\hat{\mathbf{r}}_*$, the arbitrary direction, and define the height function

$$h(\mathbf{x}) := h_{\hat{\mathbf{r}}_*}(\mathbf{x}) := -\hat{\mathbf{r}}_* \cdot Re \log \mathbf{x}.$$

We want to use this height function as $|\mathbf{r}|h$ instead of the log magnitude of our original integrand ($\omega := \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})$) because it gets the part that goes to infinity with $\mathbf{r}$ and leaves only $\mathbf{z}^{-\mathbf{1}} F(\mathbf{z})$ which is bounded on compact subsets of $\mathcal{M}$ (our manifold).

So instead of step (3) (which says: Deform the cycle to lower the modulus of the integrand as much as possible; use Morse theoretic methods to characterize the minimax cycle in terms of *critical points.*), we can instead ask:

- What chain $\mathcal{C}$ of integration, homologous in $H_d(\mathcal{M})$ to $\mathbf{T}(\mathbf{x})$ for $\mathbf{x} \in B$ achieves the least value of $\max_{\mathbf{x} \in \mathcal{C}} h(\mathbf{x})$?
- *Can we make this maximum less than* β^**?*

The topology of our manifold $\mathcal{M}$ can be related (by duality?) to the topology of our variety $\mathcal{V}$. This is good, because $\mathcal{V}$ is a more classically understood object in Morse theory. Visualization hints:

- Think of a 2D example where the singular variety $\mathcal{V}$ is drawn where 'up' is height.
- This can't be done for $\mathbb{C}^2$, but the complex algebraic curve $\mathcal{V}$ is a surface of two real dimensions which sit in 3D space. $\mathcal{V}$ intersects the coordinate axes in a finite number of points, where the height function h will be infinite, and there are finitely many points at infinity where the height tends to $-\infty$..

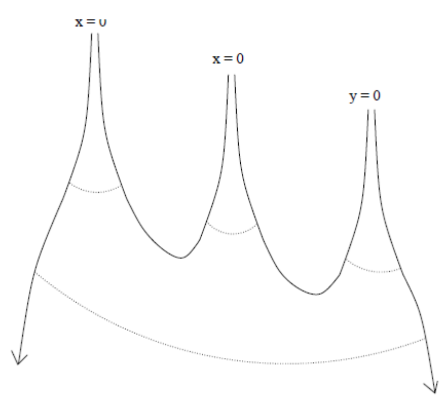


Figure 8.1 Simplified drawing of $\mathcal{V}$, with the vertical axis representing h.

Recall: Homotopic vs. Homologous

- In topology, two continuous functions from one topological space to another are called **homotopic** if one can be continuously deformed into the other.
- Homology is a certain general procedure to associate a sequence of modules with a topological space. Homology groups are generally easier to compute than homotopy groups. Original motivation for defining homology groups is the observation that shapes are distinguished by their holes. Sometimes homology can't see the holes, in which case we need homotopy.
- Here: we only need that chains are homologous, and Morse theory only guarantees that chains are homologous, so sometimes work with homology instead of homotopy. But usually we get homology through homotopy, so don't worry about it too much.

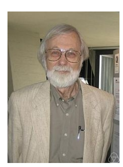
Now with that set up, let us consider some background to Morse theory in general...

Smooth Morse Theory

This is Marston Morse (1892-1977):

an American mathematician known for introducing the technique of differential topology no known as Morse theory. Morse theory enables one to analyse the topology of a manifold by studying differentiable functions on that manifold. Morse said that a typical differentiable function on a manifold will reflect the topology quite directly. A manifold of dimension n is a topological space that near each point resembles n-dimensional Euclidean space. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n. Lines and circles are manifolds. The figure eight is not. A 2-D manifold is called a surface. Examples are the sphere, plane and torus.

It was more fully developed by John Milnor, an American mathematician born in 1931 who is one of only three mathematicians to have won each of the Fields medal, the Abel prize and the Wolf prize.



The book says that basic Morse theory concerns a compact manifold $\mathcal{V}$ (A compact manifold is a manifold that is compact as a topological space (every open cover has a finite subcover)) endowed with a smooth height function $h : \mathcal{V} \to \mathbb{R}$.

Denote: $\mathcal{V}^c := \{x \in \mathcal{V} : h(x) \leq c\}$, the subset of points at height at most c.

Traditional purpose of Morse theory is to tell how this subset $\mathcal{V}^c$ changes as c increases from its minimum to maximum value.

Fundamental Morse Lemma: the topology does not change between critical values. Lemma B.1.2 Appendix B.

Second main result in Morse theory: a description of how the topology changes at critical values. Theorem B.1.3 Appendix B.

Important: the implications for the minimax height of a cycle representing a given homology class. Consider picture again.

Suppose $\mathcal{V}$ has k critical points with *distinct* critical values $c_1 > \ldots > c_k$. The Morse Lemma (above) is proved by showing that for any interval $[a, b]$ containing no critical values of h, the set $\mathcal{V}^b$ retracts homotopically into $\mathcal{V}^a$. Any cycle $\mathcal{C}$ supported by $\mathcal{V}^b$ is carried by this retraction into $\mathcal{V}^a$.

$h_*(\mathcal{C}) :=$ (A critical value of h) the infimum of c given a d-cycle $\mathcal{C}$ such that $\mathcal{C}$ is homologous in $H_d^*(\mathcal{V})$ to a cycle supported on $\mathcal{V}$; $\rightarrow h_*(\mathcal{C})$ is always a critical value c_j of h.

Consider $\mathcal{C}$ a cycle in $\mathcal{V}$. Then:

(1) $h_*(\mathcal{C}) = c_1$ **or**
(2) $\mathcal{C}$ is homologous to a cycle supported on $\mathcal{V}^{c_1-\epsilon}$.

If (2) then:

- the homology class $[\mathcal{C}]$ vanishes when projected to the relative homology group $H_d(\mathcal{V}, \mathcal{V}^{c_1-\epsilon})$ for some $\epsilon > 0$.
- If this homology class is nonzero this is a *'topological obstruction'* at height c_1.
- Then inductively: $h_*(\mathcal{C}) = c_j$ where j is the least index where a topological obstruction occurs.
- This obstruction is local to the critical point $\mathbf{p}$ which is at height c_j:
  - All trajectories of $\mathcal{V}^{c_j-\epsilon}$ reach $c_j - \epsilon$ except those in a neighbourhood of $\mathbf{p}$.

Sum up lemma:

Lemma 2.2. (quasi-local cycle) *Let $\mathcal{V}$ be a compact manifold of homology dimension d and let $h : \mathcal{V} \to \mathbb{R}$ be smooth. Suppose that h has finitely many critical points with distinct critical values $c_1 > \ldots > c_k$. Let $\mathcal{C}$ be any cycle in $H_d(\mathcal{V})$. Then*

(1) *The minimax height $h_*(\mathcal{C})$ of the class $[\mathcal{C}]$ is equal to c_j for some j.*
(2) *Let $\mathbf{p}$ be the unique critical point of h at height h_*. Then $\mathcal{C}$ is homologous to a cycle $\mathcal{C}_*$ supported on the union of $\mathcal{V}^{c_j-\epsilon}$ and an arbitrarily small neighbourhood of $\mathbf{p}$. The value of j and the homology class of $\mathcal{C}_*$ in the space $X^{p,loc}$ is uniquely determined by $\mathcal{C}$.*
(3) *j bay be characterized as the least index i for which the image of C in $H_d(\mathcal{V}, \mathcal{V}^{c_j})$ vanishes.*

Example 2.3. *Consider the following example of a homology class travelling down a surface with two saddles:*

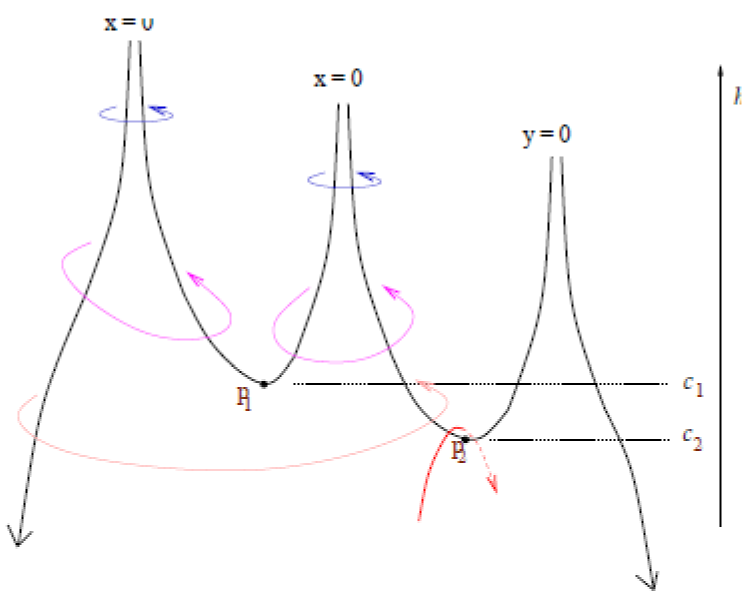


Figure 8.2 A homology class traveling down a surface that has two saddles, p_1 and p_2 at respective heights $c_1 > c_2$

Here $\mathcal{C}$ is the upper two blue cycles made up of two components and their are two critical values, c_1 and c_2. If $c > c_1$, we can retract $\mathcal{V}$ to $\mathcal{V}^c$ by carrying $\mathcal{C}$ to a cycle that is homotopic to $\mathcal{C}$, like the middle two circles drawn in magenta. Since the two components are in opposite directions near the saddle p_1 at height c_1 there is no topological obstruction when we continue below p_1, and those components can merge and form *the pink cycle (homology vs. homotopy). When p_2 is hit, this is a topological obstruction and the maximum height of $\mathcal{C}$ cannot be pushed below c_2.*

Stratified Morse Theory
1988: (Goresky and MacPherson) basic Morse theory extended to

(1) stratified spaces
(2) noncompact spaces (eg. complements of manifolds)

Recall:

Definition 2.4. (Whitney stratification) *Let Z be a closed subset of a smooth manifold $\mathcal{M}$. A* Whitney stratification *of Z is an I-decomposition such that*

(1) *Each S_α is a manifold in $\mathbb{R}^n$.*
(2) *If $\alpha < \beta$, if the sequences $\{x_i \in S_\beta$ and $\{y_i \in S_\alpha$ both converge to $y \in S_\alpha$ if the lines $l_i = \overline{x_i y_i}$ converge fo a line l and the tangent planes $T_{x_i}(S_\beta)$ converge to a plane T of some dimension, then both l and $T_y(S_\alpha)$ are contained in T.*

Also:

- Every algebraic variety is a Whitney stratified space.
- $\mathcal{V}$ is the disjoint union of manifolds of various dimensions.
- $h : \mathcal{V} \to \mathbb{R}$ is 'smooth' if it is smooth when restricted to each stratum.
- $\mathbf{p} \in \mathcal{V}$ is critical if it is a critical point of $h|_S$ where S is the stratum of $\mathcal{V}$ with $\mathbf{p}$.
- Let h be a *proper map* (inverse image of any compact set is compact).

Lemma 2.5. (Fundamental Lemma for stratified spaces *Let $\mathcal{V}$ be a stratified space and let $h : \mathcal{V} \to \mathbb{R}$ be a smooth, proper function with finitely many distinct critical values. If h has no critical values in $[a, b]$, then X^a is a strong deformation retract of X^b. In particular, the homotopy types of X^t are all naturally identified for $a \le t \le b$ and any cycle in $\mathcal{V}^b$ is homotopic to a cycle in $\mathcal{V}^a$.*

Idea (again): Any cycle may be pushed down until it reaches some topological obstruction at a critical point, then it becomes a quasi-local cycle having height at most $h_*(\mathcal{C}) - \epsilon$ except in some neighbourhood of that critical point ($\mathcal{C}$ is the cycle).

Non-proper Morse theory

We actually want to deform chains of integration in the complement of an algebraic variety $\mathcal{V}$ in $\mathbb{C}^d$. The height function $h : \mathcal{M} \to \mathbb{R}$ is never proper.

Expect from topological duality? the critical values of h on $\mathcal{V}$ are the only values of c at which the topology of $\mathcal{M}^{c+\epsilon}$ and $\mathcal{M}^{c-\epsilon}$ can differ. Indeed!

Lemma 2.6. (Fundamental Lemma for complements of stratified spaces) *Let $\mathcal{M}$ denote the complement in $(\mathcal{C}^d)^*$ of a stratified space $\mathcal{V}$. If the smooth proper function $h : \mathcal{V} \to \mathbb{R}$ has no critical values in $[a, b]$ in $\mathcal{V}$, then X^a is a strong deformation retract of X^b. Any cycle in $\mathcal{M}^b$ is homotopic to a cycle in $\mathcal{M}^a$ and three original conclusions hold.*

How do we use this?

- Ideally for step (3) of the original 6 steps in section 1.3.
- But since h is not usually proper on $\mathcal{V}$ since h can sometimes h approach a finite limit as $x \to \infty$ on $\mathcal{V}$ (particularly when $d \ge 3$.).
- However, this holds (mostly, strata at infinity might have new critical points) if there is a compactification of $\mathcal{V}$ such that h extends continuously as a function to the extended reals $[-\infty, \infty]$.
- Such a compactification is conjectured by Pemantle (2010).
- In the meantime: proceed but verify deformations for each new class of problems.

Non-Morse Morse Theory

Q: What does it actually mean for a funciton to be Morse?

A: The critical points are nondegenerate and the critical values are distinct.

When we assume that height function h has isolated critical points, the Morse Lemma remains true in all forms. But if this is not the case, the pair $(X^c, X^{c-\epsilon})$ will in general be homotopy equivalent to a direct sum of local pairs, $X^{\mathbf{p},loc}$ and defined as follows:

Lemma 2.7. (quasi-local cycles when critical values are not distinct) *Let $\mathcal{M} = (\mathbb{C}^d)^* \backslash \mathcal{V}$ be the complement of a stratified space and let $h : (\mathbb{C}^d)^* \to \mathbb{R}$ be smooth and proper. Let $\mathbf{p}_{i,1}, \ldots, \mathbf{p}_{i,n}$ denote the set of critical points with critical values c_i. Then,*

(1) *The minimax height $h_*(\mathcal{C})$ of the class $[\mathcal{C}]$ is equal to c_j for some J.*
(2) *The cycle $\mathcal{C}$ is homologous to a cycle $\mathcal{C}_*$ supported on the union of $\mathcal{M}^{c_j - \epsilon}$ and an arbitrary small neighbourhood of $\{\mathbf{p}_{j,1}, \ldots, \mathbf{p}_{j,n_j}\}$.*
(3) *j may be characterized as the least index i for which the image of $\mathcal{C}$ in $H_d(\mathcal{M}, \mathcal{M}^{c_i})$ vanishes.*
(4) *The homology group $H_d(\mathcal{M}^{c_j+\epsilon}, \mathcal{M}^{c_j-\epsilon})$ is naturally the direct sum, induced by inclusion, of the homology group $H_d(X^{\mathbf{p}_{j,i}, loc})$ for $1 \le i \le n_j$, ($X^{\mathbf{p}, loc}$ is the pair $(X^{c_j - \epsilon} \bigcup N(\mathbf{p}), X^{c_k - \epsilon})$ and $N(\mathbf{p})$ is an arbitrarily small neighbourhood of $\mathbf{p}$).*
(5) *The cycle $\mathcal{C}_*$ may be written as a sum $\sum_{contrib} \mathcal{C}_*(z)$ where contrib is the set of $z = \mathbf{p}_{i,j}$ for which the projection of $\mathcal{C}$ to $H_d(\mathcal{M}^{z, loc})$ is nonzero.*

What is the same? What is different?

- Same: Cauchy integral is unchanged if the chain of integration is varied over a homology class in $H_d(\mathcal{M})$.
- Different: Relative homology classes are a coarser partition of chains of integration and do not leave integrals invariant.
- However: chains in same relative homology class in $H_d(\mathcal{M}, \mathcal{M}^{c_j - \epsilon})$ have integrals that only differ by $O(\exp[(c - \epsilon)|\mathbf{r}|])$
- Thus, these are exponentially smaller than $\exp[(c_j + o(1))|\mathbf{r}|]$ ... *error term within a relative homology class is negligible.*

Conclude: The topological invariant $\mathcal{C}_* = \sum_{z \in contrib} \mathcal{C}_*(z)$ determines the asymptotics of a_r up to an exponentially smaller remainder.

3. Critical Points

Goal: analyze a Laurent series for $F = G/H$ which is convergent on a component of B of the complement of the amoeba of H.

Steps:

(1) Let $T = \mathbf{T}(\mathbf{X})$ (the torus $\exp(\mathbf{x} + i\mathbb{R}^d)$) for some $\mathbf{x} \in B$.
(2) When $\hat{\mathbf{r}}_*$ is fixed, the height function $h = h_{\hat{\mathbf{r}}_*}$ is constant on T with common value $b := -\hat{\mathbf{r}}_* \cdot \mathbf{x}$.
(3) List the critical values of h that are at most b in descending order $b > c_1 > c_2 \ldots$.
(4) (Bit hand wavy): the cycle $\mathcal{C}$ is homologous to the sum of one or more quasi-local cycles at critical points at height $h_*(\mathcal{C})$.
(5) We can compute these from H (the denominator)! (unless we are in the degenerate case..) Why?
    - There is a computable Whitney stratification for $\mathcal{V}$.
    - The strata of $\mathcal{V}$ are smooth manifolds each of dimension $k \le d - 1$.
    - Any stratum S of dimension k is a k-dimensional algebraic variety $\overline{S}$ possibly minus some varieties of smaller dimensions.
    - Any irreducible k-dimensional complex algebraic variety can be represented as the intersection of $d - k$ algebraic hypersurfaces which intersect 'transversely.' (*set crossways??*)
    - Thinking of $\overline{S}$ this way, all points at which the intersection is not transverse are in $\overline{S} \backslash S$, i.e. are in lower dimensional strata.
    - So... $\overline{S}$ can be represented as the intersection of $d - k$ algebraic hypersurfaces $\mathcal{V}_{f_1}, \ldots, \mathcal{V}_{f_{d-k}}$ which intersect transversely at every point of S.
    - **The polynomial f_j can be computed and have nonvanishing gradient at every point of S.**
(6) How do we actually do this?
    - Let M be the $(d - k + 1) \times d$ matrix whose rows are the the $d - k$ gradients with $\hat{\mathbf{r}}_*$ (looking for $\hat{\mathbf{r}}_*$ to be in the span of the $d - k$ gradient vectors of f with respect to $\log \mathbf{x}$).
    - At points of S: the submatrix of M made up of the first $d - k$ rows has rank $d - k$.
    - The span of the $d - k$ gradients contain $\hat{\mathbf{r}}_* \leftrightarrow$ the k determinants M_{d-k+i}, $1 \le i \le k$ where M_{d-k-i} has the first $d - k$ columns of M together with the $(d - k + i)^{th}$ column.

- **This gives the d-*critical point equations:***

$$f_i = 0, \qquad 1, \ldots, d-k;$$
$$\det(M_{d-k+i}) = 0, \qquad i = 1, \ldots, k.$$

The most common special case is when the critical points are smooth and we can be even more explicit.

- the defining equation for $\overline{S}$ is $f_1 := H = 0$.
- Being in the span of the gradient of f_1 with respect to $\log \mathbf{x}$ leads to $d-1$ equations for vanishing 2×2 subdeterminants:

$$\begin{aligned} H &= 0 \\ r_1 x_2 \frac{\partial H}{\partial x_2} &= r_2 x_1 \frac{\partial H}{\partial x_1} \\ &\vdots \\ r_1 x_d \frac{\partial H}{\partial x_d} &= r_2 x_1 \frac{\partial H}{\partial x_1} \end{aligned}$$

Example 3.1. *The generating function $F = G/H = \frac{1}{1-x-y}$ has binomial coefficients $\binom{r+s}{r,s}$. Here $H = 1 - x - y$. We compute its gradient, and get $\bigtriangledown H = (-1,-1)$. Great thing: this never vanishes and so $\mathcal{V}$ is smooth. Then our equations become:*

$$1 - x - y = 0$$
$$-sx = -ry$$

and solving we get $x = \frac{r}{r+s}$ and $y = \frac{s}{r+s}$. Thus our critical point is

$$\left(\frac{r}{r+s}, \frac{s}{r+s} \right).$$

Example 3.2. *Let $\mathcal{A}_r$ be the lattice paths from the origin to $\mathbf{f}$ in $\mathbb{Z}^2$ that only use steps that go N, E or NE. Its generating function is called the Delannoy generating function and is*

$$F(x,y) = \frac{1}{1 - x - y - xy}.$$

Here, $H = 1 - x - y - xy$ and we can easily compute its gradient as $(-1-y, -1-x)$.

We must also check that $\mathcal{V}$ is smooth. The strategy for this is to verify that $-1-y$ and $-1-x$ and H never simultaneously vanish. This can be done by checking the Groebner basis for all three. You can do this in Maple. Then the critical point equations are:

$$1 - x - y - xy = 0$$
$$sx(1+y) = ry(1+x)$$

Solving this.... (with some Maple help) gives that

$$y = \frac{-r \pm \sqrt{r^2+s^2}}{s} \qquad x = \frac{-s \pm \sqrt{r^2+s^2}}{r}$$

Of course, there are then four possible pairings, but only two actually solve the equations above: the two positive roots and the two negative roots. Thus the critical points are:

$$\left(\frac{\sqrt{r^2+s^2} - s}{r}, \frac{\sqrt{r^2+s^2} - r}{s} \right) \text{ and } \left(\frac{-\sqrt{r^2+s^2} - s}{r}, \frac{-\sqrt{r^2+s^2} - r}{s} \right).$$