ASSIGNMENT 6 SOLUTIONS

MATH 303, FALL 2011

If you find any errors please let me know

Manipulation

(M1) By rule C we have that

$$((c = d) \wedge (d = c')) \to c = c'$$

is valid. Then by rule F applied with $A(x)$ being $((c = x) \wedge (x - c'))$ and B being $c = c'$ we get that

$$\exists x((c = x) \wedge (x = c')) \to c = c'$$

is valid, which is what we were supposed to show.

(M2) First notice that by Cohen's definition of "derive" the question is asking if

$$(\forall x A(x) \wedge (A(c) \to B)) \to B$$

is valid.

By rule E we know that $\forall x A(x) \to A(c)$ is valid. Consider the following formula

$$(\forall x A(x) \to A(c)) \to ((\forall x A(x) \wedge (A(c) \to B)) \to B)$$

Letting C be $\forall x A(x)$ this is the formula

$$(C \to A(c)) \to ((C \wedge (A(c) \to B)) \to B)$$

This is a propositional function in the letters C, $A(c)$, and B, so we can apply rule A. To do this build a truth table (unfortunately a bit of a large one)

$A(c)$	B	C	$C \to A(c)$	$A(c) \to B$	$C \wedge (A(c) \to B)$	$(C \wedge (A(c) \to B)) \to B$	whole thing
F	F	F	T	T	F	T	T
F	F	T	F	T	T	F	T
F	T	F	T	T	F	T	T
F	T	T	F	T	T	T	T
T	F	F	T	F	F	T	T
T	F	T	T	F	F	T	T
T	T	F	T	T	F	T	T
T	T	T	T	T	T	T	T

We see that the propositional function in question is identically true, so by rule A it is a valid statement. Finally since $CA(c)$ is valid and $(C \to A(c)) \to ((C \wedge (A(c) \to B)) \to B)$ by rule B we conclude that

$$(C \wedge (A(c) \to B)) \to B$$

which is what we wanted to show.

(M3) $\{a,b,c,d\}$ is both maximal and largest in X. $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$ are all minimal, but there is no smallest element (if there were it would also have to be the unique minimal element, but there are 4 minimal elements).

(M4) $((\omega^+)^+)^+ = \{0,1,2,\dots,\omega,\omega^+,(\omega^+)^+\}$ Take any two elements m and n of $((\omega^+)^+)^+$, I will describe precisely when $m \leq n$ in $((\omega^+)^+)^+$.

If m and n are equal then $m \leq n$ in $((\omega^+)^+)^+$.

If m and n are both natural numbers then $m \leq n$ in $((\omega^+)^+)^+$ if $m \leq n$ in ω.

If m is a natural number and n is not then $m \leq n$. (If n is a natural number and m is not then $m \not\leq n$.)

If neither m and n are natural numbers but they are distinct, then set $\omega \leq \omega^+$, $\omega^+ \leq (\omega^+)^+$ (and hence $\omega \leq (\omega^+)^+$) to determine when $m \leq n$.

(M5) $\omega3 = \{0,1,2,3,\dots,\omega,\omega+1,\omega+2,\dots,\omega2,\omega2+1,\omega2+2,\dots\}$. Define $f : \omega3 \to \omega$ as follows.

$$f(x) = \begin{cases} 3x & \text{if } x \in \omega \\ 3k+1 & \text{if } x = \omega + k \\ 3k+2 & \text{if } x = \omega2 + k \end{cases}$$

Now we just need to show that f is one-to-one and onto.

One-to-one: Suppose $f(a) = f(b)$. If $f(a)$ is divisible by 3 we have $3a = 3b$ so $a = b$. If $f(a)$ is congruent to 1 modulo 3 then we have $3k+1 = 3\ell+1$ where $a = \omega + k$ and $b = \omega + \ell$, so $k = \ell$ and so $a = b$. Similarly if $f(a)$ is congruent to 2 modulo 3 then we have $3k+1 = 3\ell+1$ where $a = \omega2 + k$ and $b = \omega3 + \ell$ so again $k = \ell$ and $a = b$.

Onto: Take any $n \in \omega$. If n is divisible by 3 then $n = 3x$ and $f(x) = n$. If $n = 3k+1$ then $f(\omega + k) = n$. If $n = 3k+2$ then $f(\omega2 + k) = n$. This exhausts the possibilities so f is onto.

Pure Math

(P1) (a) a a least element of X means that for all $x \in X$, $a \leq x$. Suppose $b \in X$ with $b \leq a$, then since a is least we also have $a \leq b$ and so $a = b$. Thus a is minimal.

(b) Suppose a and b are both least elements of X. Then $a \leq b$ and $b \leq a$, so $a = b$. Thus if X has a least element then it has a unique least element.

(P2) (a) Let R be addition, that is $(a,b,c) \in \bar{R}$ if and only if $a + b = c$. Then the sentence of S says $\forall x \forall y \forall z(\forall w(x + y = w \leftrightarrow x + z = w) \to y = z)$ equivalently $\forall x \forall y \forall z(x + y = x + z \to y = z)$. This is true in $\mathbb{Z}$ (just subtract x from both sides). Thus with this interpretation $\mathbb{Z}$ is a model for S.

(b) *most possible choices for $\bar{R}$ will make S false, here is one possibility:* Let R be the statement that the first and last arguments agree, that is $(a,b,c) \in \bar{R}$ if and only if $a = c$. Then the sentence of S says $\forall x \forall y \forall z(\forall w(x = w \leftrightarrow x = w) \to (y = z))$. $x = w \leftrightarrow x = w$ is always true, so this is just the sentence $\forall x \forall y \forall z \forall w(y = z)$, equivalently $\forall y \forall z(y = z)$ which is false in $\mathbb{Z}$ since $\mathbb{Z}$ has at least two elements. Thus with this interpretation $\mathbb{Z}$ is not a model for S.

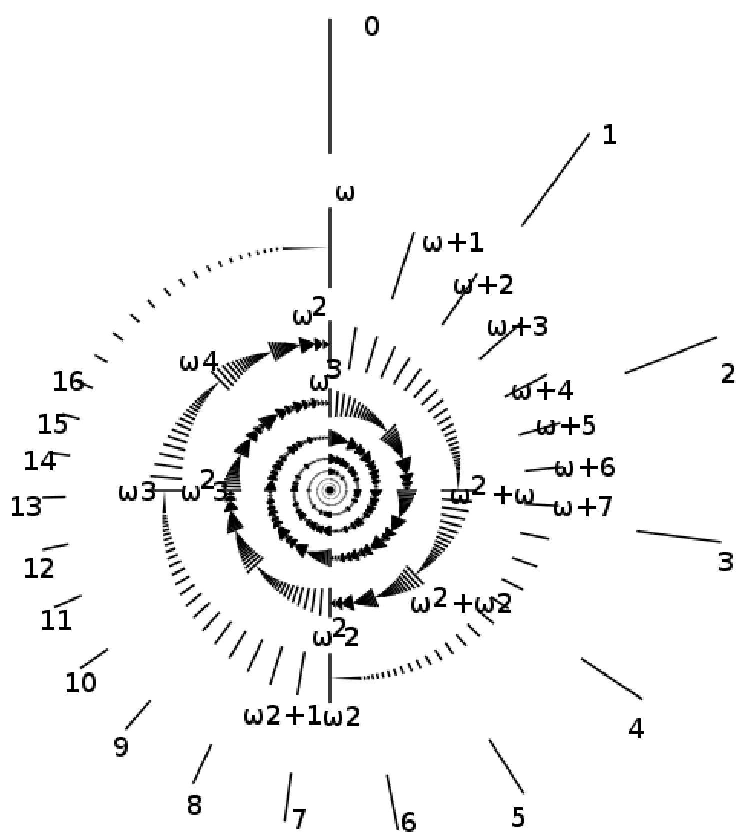
(P3) Countable things can be listed indexed by ω. Let the sets be A_0, A_1, $A_2, \ldots$. Let $A_i = \{a_{i,0}, a_{i,1}, a_{i,2}, \ldots\}$. Put these in an infinite matrix

$$\begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ \vdots & & \vdots & \ddots \end{bmatrix}$$

We can count the elements of the union of the A_i by zigzagging down the antidiagonals (as we did to show $\omega \times \omega$ was countable), that is first take $a_{0,0}$ then $a_{0,1}$ then $a_{1,0}$ then $a_{2,0}$ then $a_{1,1}$ then $a_{0,2}$ then $a_{0,3}$ then $a_{1,2}$ etc.

This still doesn't give a bijection of the union of the A_i with ω because some elements may be in more than one A_i and so they get counted too many times. However this is easy to fix, just skip any elements which we have already seen in the above list. This finally gives the map we need.

Ideas



(I1)

(I2) *answers will vary*