De Bruijn-Erdős theorem

Set Up. By a *graph* I mean a set of vertices and a set of edges that connect pairs of vertices. We say a graph can be *coloured* with k colours if each of its vertices can be given one of k colours so that no two vertices joined by an edge have the same colour.

The De Bruijn-Erdős theorem says that for every infinite graph G and every finite integer k, G can be coloured with k colours if and only if each finite subgraph of G can be coloured with k colours.

The proof of this result requires the axiom of choice.

What to do. Give the class an example of a graph, an example of colouring a graph (you'll probably want to give finite examples here), then explain the statement of the theorem. Explain why the theorem seems reasonable to you or seems unreasonable to you.

References.

- `http://en.wikipedia.org/wiki/De_Bruijn%E2%80%93Erd%C5%91s_theorem_%28graph_theory%29`
- `http://en.wikipedia.org/wiki/Graph_theory`
- `http://en.wikipedia.org/wiki/Graph_coloring`

The axiom of choice itself

Set Up. The way Halmos formulates the axiom of choice is not the most typical. More usual is:

Let $\mathcal{F}$ be a set of nonempty sets. Then there is a *choice function*, that is a function f from $\mathcal{F}$ to $\bigcup \mathcal{F}$ such that for each $X \in \mathcal{F}$, $f(X) \in X$.

What to do. Explain why we call this a choice function (and hence why this axiom is called the axiom of choice). Explain why this is or is not a reasonable axiom in your view. If you have time, explain how this is equivalent to the axiom of choice as Halmos gives it. Alternately you could also talk about one other related formulation given in Devlin p56 (see the references).

References.

- Halmos p59 and p60 (Halmos proves this is equivalent to his first formulation of the axiom of choice)
- Keith Devlin, *The Joy of Sets*, p56 and p57.
- `http://en.wikipedia.org/wiki/Axiom_of_choice`

The well ordering principle

Set Up. We say a set A is totally ordered if we have defined $\leq$ on the set (not necessarily in a way which relates to its usual meaning) such that for all $a, b, c \in A$:

- $a \leq a$
- If $a \leq b$ and $b \leq a$ then $a = b$
- If $a \leq b$ and $b \leq c$ then $a \leq c$
- Either $a \leq b$ or $b \leq a$

If $\leq$ also has the property that every non-empty subset of A has a smallest element, then we say $\leq$ is a well order.

A statement which is equivalent to the axiom of choice is the statement that every set can be well-ordered. Note that this means in particular that the set of real numbers can be well-ordered.

What to do. Explain what a well ordering is to the class. Show how the usual order on the real numbers is not a well order, and imagine what a well order of the reals would look like.

References.

- Halmos section 17
- Thomas Jech, *The Axiom of Choice*, p9
- Keith Devlin, *The Joy of Sets*, p58
- `http://en.wikipedia.org/wiki/Well-ordering_theorem`
- `http://en.wikipedia.org/wiki/Axiom_of_choice`

Zorn's Lemma

Set Up. We say a set A is partially ordered (or A is a *poset*) if we have defined $\leq$ on the set (not necessarily in a way which relates to its usual meaning) such that for all $a, b, c \in A$:

- $a \leq a$.
- If $a \leq b$ and $b \leq a$ then $a = b$
- If $a \leq b$ and $b \leq c$ then $a \leq c$

If it is also true that either $a \leq b$ or $b \leq a$ then we call $\leq$ a *total order*. A *chain* C in A is a totally ordered subset $C \subseteq A$. An upper bound for C is an $a \in A$ such that $a \geq c$ for all $c \in C$. A maximal element of A is an $a \in A$ such that $a \leq b$ implies $a = b$.

Then the following statement is equivalent to the axiom of choice: (Zorn's Lemma) If X is a partially ordered set such that every chain in X has an upper boundm then X contains a maximal element.

Zorn's Lemma seems a bit unmotivated from our perspective, but often it is the most useful version of the axiom of choice, particularly in abstract algebra.

What to do. Explain the difference between a partial order and a total order with an example. Give an example of a chain in a partial order. Explain the difference between upper bounds and maximal elements. Explain the statement of Zorn's lemma. If you have time, give an idea of the proof that Zorn's lemma is equivalent to the axiom of choice, or perhaps just give one direction, the full details will be too long.

References.

- Halmos sections 14 and 16
- Keith Devlin, *The Joy of Sets* p60,61
- Thomas Jech, *The Axiom of Choice*, p9-11
- `http://en.wikipedia.org/wiki/Axiom_of_choice`
- `http://en.wikipedia.org/wiki/Zorn%27s_lemma`

Vector space bases

Set Up. Recall that a *basis* of a vector space V is a set of linearly independent vectors that spans V. The bases you have probably seen before are finite sets, because in most first linear algebra courses one only works with finite dimensional vector spaces.

If we have the axiom of choice, then everything you knew about bases of finite dimensional vector spaces is also true for infinite dimensional vector spaces. In particular every vector space has a basis. If we assume the negation of the axiom of choice then we can show there is a vector space with no basis.

What to do. Remind the class what basis, linearly independent, and span mean. Give an example of an infinite dimensional vector space. Explain why you think it is plausible or not plausible that every vector space has a basis.

References.

- Thomas Jech, *The Axiom of Choice* p12, p145
- `http://en.wikipedia.org/wiki/Axiom_of_choice`
- `http://en.wikipedia.org/wiki/Basis_%28linear_algebra%29`
- `http://en.wikipedia.org/wiki/Examples_of_vector_spaces`

The Banach-Tarski paradox

Set Up. The Banach-Tarski paradox says that a solid sphere in 3-space can be cut into a finite number of pieces so that the pieces can be translated and rotated to give two spheres of the same size as the original sphere.

The trick is that the pieces must be chosen using the axiom of choice. This result is often thought to be the most unbelievable consequence of the axiom of choice.

What to do. Explain the paradox to the class.

References.

- Karl Stromberg, "The Banach-Tarski paradox" American Mathematical Monthly, **86**, 3, (1979) pp151-161. `http://www.jstor.org/stable/2321514`
- `http://en.wikipedia.org/wiki/Banach-Tarski_paradox`

No countable subset

Set Up. What if we don't have the axiom of choice? If we can construct some sort of objects in which the Zermelo-Fraenkel axioms other than the axiom of choice hold and some other fact P of interest to us is true, then we say there is *a model of ZF in which P is true*. Model theory is a whole area in its own right, and we won't go into how these models are constructed, but they show us some of the consequences of not having the axiom of choice.

There is a model of ZF which has an infinite set of real numbers without a countable subset. This can never happen if we assume the axiom of choice.

What to do. Describe to the class why this statement seems reasonable or unreasonable. What does this say about whether we should take the axiom of choice to be true?

References.

- Thomas Jech, *The Axiom of Choice*, p141
- `http://en.wikipedia.org/wiki/Axiom_of_choice`

Sequentially continuous vs continuous

Set Up. What if we don't have the axiom of choice? If we can construct some sort of objects in which the Zermelo-Fraenkel axioms other than the axiom of choice hold and some other fact P of interest to us is true, then we say there is *a model of ZF in which P is true.* Model theory is a whole area in its own right, and we won't go into how these models are constructed, but they show us some of the consequences of not having the axiom of choice.

Let $\mathbb{R}$ be the set of real numbers. We say a function f from $\mathbb{R}$ to $\mathbb{R}$ is *sequentially continuous* at $a \in \mathbb{R}$ if for every sequence $\{a_n\}$ converging to a we have $\lim_{n\to\infty} f(a_n) = f(a)$.

There is a model of ZF which has a function f from $\mathbb{R}$ to $\mathbb{R}$, and an $a \in \mathbb{R}$ such that f is not continuous at a, but f is sequentially continuous at a. This can never happen if we assume the axiom of choice.

In calculus you probably didn't even distinguish between continuous and sequentially continuous, you just used whichever was convenient at the time. This shows how deeply the axiom of choice is embedded in calculus.

What to do. Remind the class about continuous and tell them about sequentially continuous. Is is plausible that a function could be sequentially continuous but not continuous at a point.

References.

- Thomas Jech, *The Axiom of Choice*, p142
- `http://en.wikipedia.org/wiki/Axiom_of_choice`

Tarski's theorem

Set Up. Another statement equivalent to the axiom of choice is the following.

For every infinite set A, there is a one-to-one and onto map between the sets A and $A \times A$.

For a countable set A, this doesn't require the axiom of choice, and a consequence is the usual proof that the integers and the rational numbers are the same size (in the sense that there is a one-to-one and onto map between them).

What to do. Tell the class what you think of this statement. Can you see how to relate it to the axiom of choice?

References.

- `http://en.wikipedia.org/wiki/Axiom_of_choice`