Math 303, Fall 2011, Lecture 17

① A brief introduction to model theory

(Following Cohen ch I section 4)

To use logic to talk about set theory we only needed one special symbol (as opposed to the general logical symbols)

Namely

$\in$

$\in$ is a relation symbol

- it takes some number of inputs
  2 in this case
- and makes a statement about a relationship between its inputs $a \in b$

If we want to talk about other areas of mathematics on their own terms (not as built of set theory) then we need other relations

eg for integers we have $\leq$

this relation also takes 2 inputs

eg I could make a relation to encode addition

$R_+(x, y, z)$ which says $x+y=z$

so $R_+(1,2,3)$ true in integers

and $R_+(1,1,3)$ false in integers

eg we can make any function into a relation in the same way

say $f(x_1, x_2, \dots, x_n)$ is a function of n variables

then let $R_f(x_1, x_2, \dots, x_n, x_{n+1})$ say $f(x_1, \dots, x_n) = x_{n+1}$

eg We can also write relations like $\leq$ or $\in$ in this format

say $R_{\leq}(a,b)$ means $a \leq b$

$R_{\in}(a,b)$ means $a \in b$

So $R_{\leq}(1,2)$ is true in the integers

but $R_{\leq}(2,1)$ is false in the integers

We can expand our well formed formulas to include such relations by expanding rule (1) to say that for any relation symbol R that takes n inputs

$$R(t_1, t_2, \ldots, t_n)$$

is a well formed formula where each t_i is either a variable symbol or a constant symbol.

Now say we have a set S of sentences in the formal language involving constant symbols

$$c_i \quad \text{for} \quad i \in I \qquad \text{with } I \text{ some index set}$$

and relations

$$R_j \quad \text{for} \quad j \in J \qquad \text{with } J \text{ some index set.}$$

We may assume all the sentences in S are good.

eg
$$S = \{ \forall x \forall y \forall z \left(R_+(x,y,z) \longleftrightarrow R_+(y,x,z) \right),$$
$$\forall x \forall z \left(R_+(x,0,z) \longleftrightarrow z = x \right) \}$$

this is a set of sentences involving

the constant symbol 0

the relation symbol R_+

in more usual notation these formulas say the following in usual notation they say

for the first: $\forall x \forall y \forall z \left(R_+(x,y,z) \longleftrightarrow R_+(y,x,z)\right)$

says $\forall x \forall y \forall z \left(x+y=z \leftrightarrow y+x=z \right)$

equivalently $\forall x \forall y \left(x+y = y+x \right)$

commutative law of addition

for the second: $\forall x \forall z \left(R_+(x,0,z) \longleftrightarrow z=x \right)$

says $\forall x \forall z \left(x+0=z \leftrightarrow z=x \right)$

equivalently $\forall x \left(x+0 = x \right)$

additive identity

What we have so far: $c_i \quad i \in I$

$R_j \quad j \in I$

S

Next say we have a set M

and maps $f(c_i) = \overline{c_i} \in M$

$$g(R_j) = \overline{R_j} \subseteq \underbrace{M \times \ldots \times M}_{n \text{ times}}$$

where R_j takes n inputs.

These give us interpretations for the constant symbols and relation symbols within M.

eg let S, c_i, R_j be as in the previous example

let $M = \mathbb{Z}$ the set of integers

then we can let

$$f(0) = 0$$

symbol 0

The element 0 of $\mathbb{Z}$

$$g(R_+) = \{(a,b,c) : a,b,c \in \mathbb{Z}, \ a+b=c\}$$

then checking if $x+y=z$
is the same as checking
if $(x,y,z) \in g(R_+)$

With this interpretation the sentences of S are true

eg We could also make a silly interpretation

eg $f(0) = 5$

$g(R_+) = \emptyset$

lets check if the sentences of S are true in this interpretation.

means $(x,y,z) \in g(R_+) = \emptyset$ so $R_+(x,y,z)$ is always false in this interpretation

$$\forall x \forall y \forall z \left(\underset{F}{R_+(x,y,z)} \overset{\longleftrightarrow}{\leftrightarrow} \underset{F}{R_+(y,x,z)} \right)$$ True in this interpretation

$$\left.\forall x \forall z \left(\underset{F}{R_+(x,0,z)} \leftrightarrow z = x \right)\right\}$$ False in this interpretation

if $z = x$ $R_+(x,0,z)$ false, $z = x$ true so second sentence is false

For the break

try this interpretation

$M = \mathbb{Z}$

$f(0) = 1$

$g(R_+) = \{(a,b,c) : a,b,c \in \mathbb{Z},\ ab = c\}$

Are the sentences of S true in this interpretation?

lets check yo they say

- $\forall a \forall b (ab = ba)$ True in $\mathbb{Z}$
- $\forall a (a1 = a)$ True in $\mathbb{Z}$

So far when checking if sentences are true in a given interpretation we have relied on our intuitive understanding of truth.

In practice this is often the best way, but we don't need to rely on it

Definition Let A be a good formula with free variables among $x_1, \dots, x_n$. Let $\bar{x}_1, \dots, \bar{x}_n$ be elements of M.

The truth of A evaluated at $\bar{x}_1, \dots, \bar{x}_n$ is defined recursively as follows

(1) If A is of the form

$x_i = x_j$ or $x_i = c_j$ or $c_i = c_j$

Then A is true at $\bar{x}_1, \dots, \bar{x}_n$ if

$\bar{x}_i = \bar{x}_j$, $\bar{x}_i = \bar{c}_j$, $\bar{c}_i = \bar{c}_j$ respectively, in M

same element in M

interpretation of c_j

(2) If A is $R(t_1, .., t_m)$ where R is a relation symbol with m inputs and each t_i is a constant symbol or one of the x_j, then A is true at $\bar{x}_1, .., \bar{x}_n$

$$\text{if } (\bar{t}_1, \ldots, \bar{t}_m) \in \bar{R}$$

$\bar{R} = g(R)$

(3) If A is a propositional function of formulas, then use the propositional calculus to evaluate the truth of A at $\bar{x}_1, \ldots, \bar{x}_n$

(4) If A is of the form

$\forall y\, B(y, x_1, .., x_n)$ or $\exists y\, B(y, x_1, .., x_n)$

then A is true at $\bar{x}_1, .., \bar{x}_n$ if

for all $\bar{y}$ in M $B(y, x_1, .., x_n)$ is true at $\bar{y}, \bar{x}_1, \ldots, \bar{x}_n$, there is a $\bar{y}$ in M such that $B(y, x_1, .., x_n)$ is true at $\bar{y}, \bar{x}_1, \ldots, \bar{x}_n$

respectively.

eg $\forall x \forall z (R_+(x,0,z) \leftrightarrow z = x)$ in $\mathbb{Z}$ with the usual (first) interpretation

is true if for all $\bar{x}$ in $\mathbb{Z}$

$\forall z (R_+(x,0,z) \leftrightarrow z = x)$ is true evaluated at $\bar{x}$

Thats the case if for all $\bar{x}$ in $\mathbb{Z}$ and all $\bar{z}$ in $\mathbb{Z}$

$R_+(x,0,z) \leftrightarrow (z = x)$ is true evaluated at $(\bar{x}, \bar{z})$.

Now use propositional calculus.

2 cases

if $\bar{x} = \bar{z}$

$R_+(x,0,z)$ is true at $(\bar{x},\bar{z})$ because $(\bar{x},0,\bar{x}) \in g(R_+)$

if $\bar{x} \neq \bar{z}$

$R_+(x, 0, z)$ is false

at $(\bar{x}, \bar{z})$

because $(\bar{x}, 0, \bar{z}) \notin g(R_+)$

so only two rows are possible in the truth table

$R_+(x, 0, z)$	$x = z$	$R_+(x,0,z) \leftrightarrow x = z$
T	T	T
F	F	T

so in all cases $R_+(x,0,z) \leftrightarrow x=z$ is true evaluated at $(\bar{x}, \bar{z})$

so $\forall z(R_+(x,0,z) \leftrightarrow x = z)$ is true evaluated at $\bar{x}$

so $\forall x \forall z(R_+(x,0,z) \leftrightarrow x = z)$ is true in $\mathbb{Z}$ with the usual interpretations

The point of all this is the following definition

Definition If S is a set of sentences involving

- constant symbols c_i, $i \in I$
- relation symbols R_j, $j \in J$

And M is a set with an interpretation given by

- $f(c_i) = \overline{c}_i \in M$
- $g(R_j) = \overline{R}_j \subseteq \underbrace{M \times \dots \times M}_{n \text{ times}}$ where R_j takes n inputs

Then we say M (with the interpretation) is a **model** for S if all sentences of S are true in M

eg If $S = \{ \forall x \forall y \forall z (R_+(x,y,z) \leftrightarrow R_+(y,x,z)),$
$\forall x \forall z (R_+(x,0,z) \leftrightarrow z = x) \}$

Then $\mathbb{Z}$ with the usual interpretation is a model for S

eg $\mathbb{Z}$ with the third interpretation is also a model for S

Trying to understand how the set of sentences influences the possible models is called model theory.

In this class we are mostly interested in
$\in$ as our only relation
and S the set of axioms of set theory

We have an intuitive model in mind

but are there other models?
we'll come back to that at the end of the course (Skolem's Paradox)

② Next time

- Important results of model theory
- Back to Halmos to review partial orders and well orders

Please read Cohen ch 1, sections 4 and 5 (don't worry about the proofs)
And Halmos chapter 14