

# Math 303, Fall 2011, Lecture 18

## ① Important results of model theory

Here are some important results of model theory. We won't prove them (see Cohen if you are interested).

**Definition** A set of statements is **consistent** if the statement  $A \vee \sim A$  cannot be derived from  $S$  for any  $A$

① If  $A$  is a valid statement then  $A$  is true in any model

② If a set of statements has a model then it is consistent

③ (Gödel's Completeness Theorem)

Let  $S$  be any consistent set of statements  
There exists a model  $M$  for  $S$  for which

$$\#(M) \leq \begin{cases} \#(S) & \text{if } \#(S) \geq \#(\omega) \\ \#(\omega) & \text{if } \#(S) < \#(\omega) \end{cases}$$

this  $\#$  is

④ Let  $S$  be any set of statements

If  $A$  is not derivable from  $S$  then there is a model for  $S$  in which  $A$  is false

(5) (Compactness theorem) Let  $S$  be a set of statements  
If every finite subset of  $S$  has a model, then  
 $S$  has a model

Definition

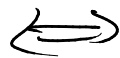
Let  $M_1 \subseteq M_2$  be models of  $S$

where the interpretations of the constants are the same  
and for each relation  $R$

$$(\bar{R} \text{ in } M_2) \cap M_1 = \bar{R} \text{ in } M_1$$

If for every formula  $A(x_1, \dots, x_n)$  and every  $\bar{x}_1, \dots, \bar{x}_n$  in  $M_1$

$A$  is true in  $M_1$   
at  $\bar{x}_1, \dots, \bar{x}_n$



$A$  is true in  $M_2$   
at  $\bar{x}_1, \dots, \bar{x}_n$

Then  $M_1$  is an elementary submodel of  $M_2$

## Idea

⑥ (Löwenheim - Skolem) let  $T$  be a set of constant symbols and relation symbols.

let  $M$  be a model of these symbols (ie

Then there is an elementary submodel  $N$  of  $M$  with

$$\#(N) \leq \begin{cases} \#(T) & \text{if } \#(T) \geq \#(w) \\ \#(w) & \text{if } \#(T) < \#(w) \end{cases}$$

This is all we're going to say about logic for now  
Next - back to Halmos

② Review of partial orders  
(Halmos chapter 14)

Recall that if  $X$  is a set and  $\leq$  is a relation on  $X$   
and the following properties are satisfied

for all  $a, b, c$  in  $X$

①

②

③

then  $\leq$  is called a **partial order** and  $X$  is called  
a **partially ordered set** (or **poset**)

We've seen 2 examples of this

①

(see lecture 10)

②

this was P2 on assignment 4

These are the two most important examples and the ones you should base your intuition on.

**Definition** let  $X$  with  $\leq$  be a partially ordered set

①  $a \in X$  is a **least** or **smallest** element of  $X$   
if

②  $a \in X$  is a **greatest** or **largest** element of  $X$   
if

③  $a \in X$  is a **minimal** element of  $X$   
if

equivalently

④  $a \in X$  is a **maximal** element of  $X$  if

eg Take  $X = \{2, 3, 4, \dots\}$

eg let  $E$  be a set with at least 2 elements  
let  $X = \mathcal{P}(E) - E$  (so  $X$  is the set of  
proper subsets of  $E$ )

Then

sub eg let  $E = \{a, b, c\}$

here is a picture

In contrast if  $X = \mathcal{P}(E)$  then the picture is



### ③ Next time

- Review of well orders
- Transfinite induction
- Ordinals

Please read Halmos sections 17 and 19