Math 303, Fall 2011, Lecture 18

① Important results of Model theory

Here are some important results of model theory. We won't prove them (see Cohen if you are interested).

Definition A set of statements is **consistent** if the statement $A \vee \sim A$ cannot be derived from S for any A

① If A is a valid statement then A is true in any model

② If a set of statements has a model then it is consistent

③ (Gödel's Completeness theorem)

Let S be any consistent set of statements
There exists a model M for S for which

$$\#(M) \leq \begin{cases} \#(S) & \text{if } \#(S) \geq \#(\omega) \\ \#(\omega) & \text{if } \#(S) < \#(\omega) \end{cases}$$

this # is

④ Let S be any set of statements
If A is not derivable from S then there is a model for S in which A is false

(5) (Compactness theorem) Let S be a set of statements. If every finite subset of S has a model, then S has a model

Definition Let $M_1 \subseteq M_2$ be models of S where the interpretations of the constants are the same and for each relation R

$$(\overline{R} \text{ in } M_2) \cap M_1 = \overline{R} \text{ in } M_1$$

If for every formula $A(x_1, \ldots, x_n)$ and every $\overline{x}_1, \ldots, \overline{x}_n$ in M_1

$$A \text{ is true in } M_1 \text{ at } \overline{x}_1, \ldots, \overline{x}_n \iff A \text{ is true in } M_2 \text{ at } \overline{x}_1, \ldots, \overline{x}_n$$

Then M_1 is an elementary submodel of M_2

Idea

⑥ (Löwenheim - Skolem) Let T be a set of constant symbols and relation symbols.

Let M be a model of these symbols (ie)

Then there is an elementary submodel N of M with

$$\#(N) \leq \begin{cases} \#(T) & \text{if } \#(T) \geq \#(\omega) \\ \#(\omega) & \text{if } \#(T) < \#(\omega) \end{cases}$$

This is all we're going to say about logic for now

Next - back to Halmos

② Review of partial orders
(Halmos chapter 14)

Recall that if X is a set and $\leq$ is a relation on X and the following properties are satisfied

for all a, b, c in X

①

②

③

then $\leq$ is called a partial order and X is called a partially ordered set (or poset)

We've seen 2 examples of this

①

(see lecture 10)

(2)

this was P2 on assignment 4

These are the two most important examples and the ones you should base your intuition on.

Definition let X with $\leq$ be a partially ordered set

(1) $a \in X$ is a least or smallest element of X if

(2) $a \in X$ is a greatest or largest element of X if

(3) $a \in X$ is a minimal element of X if

equivalently

(4) $a \in X$ is a maximal element of X if

eg Take $X = \{2, 3, 4, \dots\}$

eg let E be a set with at least 2 elements
let $X = \mathcal{P}(E) - E$ (so X is the set of proper subsets of E)

Then

sub eg let $E = \{a, b, c\}$

here is a picture

In contrast if $X = \mathcal{P}(E)$ then the picture is

3 Next time

- Review of well orders
- Transfinite induction
- Ordinals

Please read Halmos sections 17 and 19