Math 303, Fall 2011, Lecture 5

(1) Numbers

We want to build nonnegative integers out of sets.
As with ordered pairs our construction must have the properties we want but may also have spurious properties

Idea

Start with 0

Define

How about 1?

We need a set with 1 element

There are lots of those

$\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\{\emptyset, \{\emptyset\}\}\}$

etc.

but we want the simplest one we can make out of 0. So

$$\boxed{\text{Define} \quad 1 = \{\emptyset\}}$$

Notice $1 = \{0\}$

What about 2?

Define

What about 3?

Define

How do we write this in general?

given $0, 1, \ldots, n-1$ define

$n =$

One way to phrase it:

Definition For every set x, define the successor of x to be

$$x^+ = x \cup \{x\}$$

eg $2^+ =$

eg $\{\{\phi\}\}^+$

Note

We call $0, 1, 2, \ldots$ natural numbers

Note

Some weird properties of natural numbers

1

2

Some good properties of natural numbers

1

2

2 An infinite set

So far (see homework 1) we don't have any infinite sets. In order to guarantee one we need to build it into our axioms

$\boxed{\begin{array}{l}\text{Axiom of infinity} \\ \text{There exists a set containing}\overset{\text{as an element}}{\swarrow}\ 0 \text{ and containing}\overset{\text{as an element}}{\swarrow} \text{the} \\ \text{successor of each of its elements}\end{array}}$

That is

the smallest

But wait

To answer these questions

Say a set A is a successor set if $0 \in A$ and $x^+ \in A$ wherever $x \in A$

For the break

Is $\omega = \{0, 1, 2, \ldots\}$?

In fact we can take ω as the rigorous definition of $\{0, 1, 2, \ldots\}$

Historically people took a long time to accept infinite sets.

(cf. Aristotle's potential infinite)

(cf Aristotle's actual infinite)

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ω is a place we can hide Russell's paradox

Another

2. Berry's paradox gets its name from G. G. Berry, an Oxford University librarian who communicated it to Russell. It concerns "the smallest integer that cannot be expressed in less than thirteen words." Since this expression has 12 words, to which set does the integer it describes belong: the set of integers that can be expressed in English with less than 13 words, or the set of integers that can be expressed only with 13 words or more? Either answer leads to a contradiction.

3. The philosopher Max Black expressed the Berry paradox in a fashion similar to the following version: Various integers are mentioned in this book. Fix your attention on the smallest integer that is not referred to in any way in the book. Is there such an integer?

From Aha Gotcha again

(3) Next time

using ω - induction