

Math 303, Fall 2011, Lecture 8

① Our axioms so far

from lecture 1

Axiom of extension two sets are equal if and only if they have the same elements

Axiom of Specification or Subset Selection

For every set A and every condition $S(x)$ there is a set B consisting of exactly the elements of A for which $S(x)$ holds

$$\text{ie } B = \{x \in A \mid S(x) \text{ is true}\}$$

from lecture 2

Axiom of the empty set There is a set, written \emptyset , which contains no elements

Axiom of pairing (or unordered pairs)

For any two sets A and B , there is a set C
with $A \in C$ and $B \in C$ and nothing else

Axiom of Unions

Let \mathcal{C} be a set of sets. Then there is a set
which contains all elements which belong to at least
one set from \mathcal{C} , and nothing else

from lecture 3

Axiom of Power sets

For every set E , there is a set \mathcal{P} consisting of precisely
the subsets of E

that is $A \subseteq E$ if and only if $A \in \mathcal{P}$

from lecture 5

Axiom of infinity

There exist a set containing 0 and containing the successor of each of its elements

as an element as an element

Where does this get us?

Today, the standard axiomatization of set theory used throughout mathematics is

Zermelo - Fraenkel Set theory

What is the history?

In 1908 Ernst Zermelo wrote a paper giving 7 axioms for set theory
His axioms were

Zermelo set theory

- ① Axiom of extensionality
- ② Axiom of elementary sets
- ③ Axiom of separation
- ④ Axiom of the power set
- ⑤ Axiom of the union
- ⑥ Axiom of choice
- ⑦ Axiom of infinity

(our axiom of extension)
(our axiom of the empty set
and axiom of pairing together)
(yet another name for the
axiom of subset selection)
(same as ours)
(same as ours)

next lecture

(his gave $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}$
rather than our ω , but
a very similar idea)

Zermelo's axiom of separation, like ours currently, was a bit vague on what sort of properties were allowed - the machinery of first order logic was not yet available

Also these axioms are not powerful enough to talk about bigger infinities

In 1922 Abraham Fraenkel and Thoralf Skolem added two more axioms

The axiom of regularity or foundation
and

The axiom of replacement or substitution

and formalized the axiom of separation.

giving

Zermelo - Fraenkel set theory

Before we get into all that lets finish Zermelo set theory.

② Families

To state the axiom of choice we need to return to Halmos chapter 9 and define families

A family of sets is something like
There are

indexes

eg

To formalize this

That is

eg Continuing the above eg

We have a formal notion of function in set theory, namely a function is the set of ordered pairs $(x, f(x))$

So define a family $\{A_i\}$ of subsets of X indexed by I
to be a function $f: I \rightarrow \mathcal{P}(X)$

- -
 -
- index set
index
(namely A_i)
term

We can use families to define larger cartesian products.

Recall $X \times Y$ is

We can rewrite this using families

Claim Z and $X \times Y$ are essentially the same
precisely, there is a one-to-one and onto map
between them.

proof of claim

How do we know Z is even a set?

Now we have the tools to define general cartesian products

Definition Let $\{Y_i\}$ be a family of sets indexed by I

Then the **cartesian product** of the family

$$\prod_{i \in I} Y_i$$

is the set of all families

$$y: I \rightarrow \bigcup_{i \in I} Y_i$$

with $y(i) \in Y_i$

eg let $I = \{a, b, c\}$

Let $Y_a = \{1, 2, 3\}$

Let $Y_b = \{4, 5\}$

Let $Y_c = \{6\}$

Then $\prod_{i \in I} Y_i$ is

What are the possibilities for y ?

③ Next time

The axiom of choice

Please read Halmos section 15

If you can easily bring a laptop (with wireless)
please do