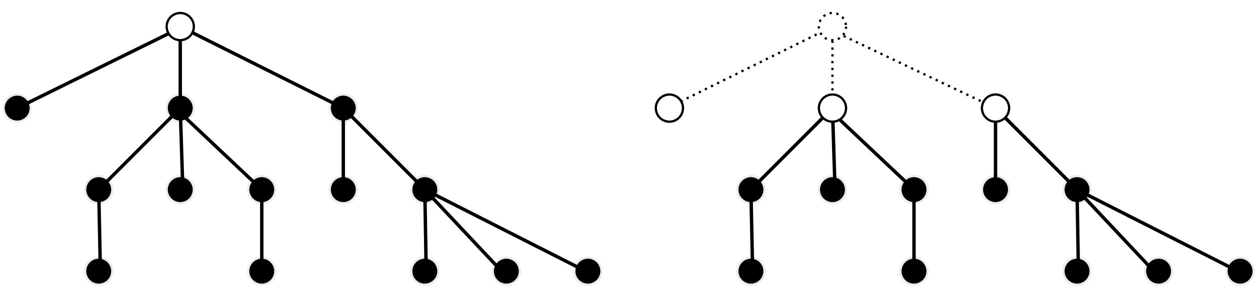
Trees and Lagrange inversion

Contents

1 Recursive specifications

The prototypical *recursive* structure is a tree.

Definition. A plane tree is the embedding of a graph without cycles into the plane. Such a tree is rooted if one of its vertices is specified (the root vertex). Since the tree is embedded in the plane, the children of each node have a unique ordering (say clockwise). The size of a rooted plane tree is the number of vertices it contains.

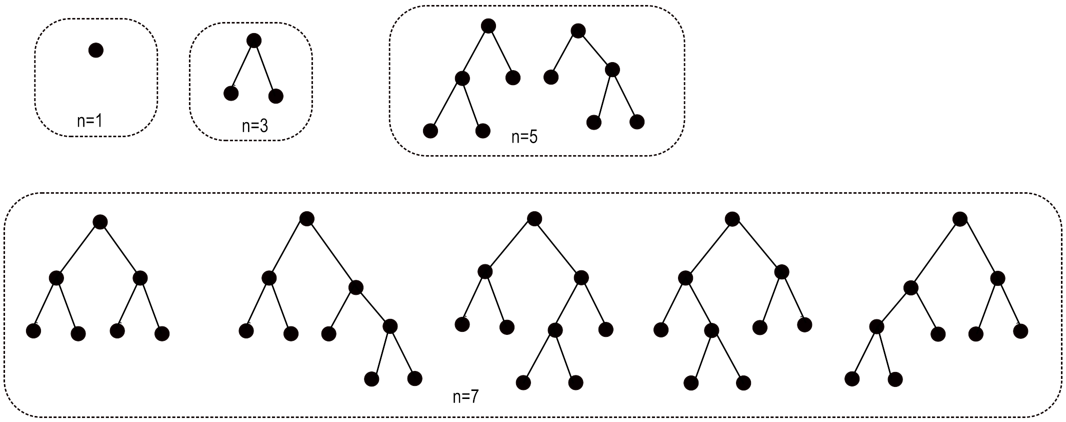


The enumeration of trees is best done recursively (there are other sneakier ways). Take any tree you like — and delete its root. One is left with a "forest" of trees — possibly empty. This forest consists of a (possibly empty) sequence of trees — each rooted at the vertex which was attached to the original root.

1.1 Binary Trees $\mathcal{B}$

(These are the binary trees from assignment 1, not the ones from lecture.)

We define the combinatorial class $\mathcal{B}$ as the set of all rooted trees in which each node has either 2 or 0 children. The size of a tree is the number of nodes. Recursively, a tree is defined as either a node, or a node and two subtrees.



$\mathcal{B}_n$ for small n

1.1.1 A recursive structure

We can formalize this recursive structure into a combinatorial decomposition. Consider the following example of a decomposition:

$$\mathcal{B}_{11} = \mathcal{Z} \times \mathcal{B}_1 \times \mathcal{B}_9 + \mathcal{Z} \times \mathcal{B}_3 \times \mathcal{B}_7 + \mathcal{Z} \times \mathcal{B}_5 \times \mathcal{B}_5 + \mathcal{Z} \times \mathcal{B}_7 \times \mathcal{B}_3 \cup \mathcal{Z} \times \mathcal{B}_9 \times \mathcal{B}_1.$$

This translates into the following counting formula.

$$b_{11} = b_1 b_9 + b_3 b_7 + b_5^2 + b_3 b_7 + b_9 b_1.$$

We see in general the counting formula is

$$b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k}.$$

1.1.2 Decomposition

In fact, we can describe a recursive combinatorial description. We are going to translate "A binary tree is either a node, or a node and two subtrees" into the following decomposition:

$$\mathcal{B} = \mathcal{Z} \quad + \quad \mathcal{Z} \times \mathcal{B} \times \mathcal{B}.$$

We apply the above sum and the product rules to recover the following generating function relation:

$$B(z) = Z(z) + Z(z)B(z)B(z) = z + zB(z)^2.$$

We can even solve for $B(z)$; $B(z)$ satisfies

$$zB(z)^2 - B(z) + z = 0.$$

We can use the quadratic formula to solve this [1]:

$$B(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}.$$

[1] Yes, there is a second solution, but it does not give rise to a formal power series solution. Evaluate the coefficients: they are negative. We can show that there is always a power series solution to a combinatorially derived algebraic equation.

1.1.3 Coefficients

We can use various series expansion tools to recover the initial terms of the generating function:

$$B(z) = z + z^3 + 2z^5 + 5z^7 + \dots.$$

We can also apply the binomial theorem to determine an explicit expression for the coefficient of z^n in $B(z)$. We did a comparable computation in Lecture 2. Follow it along to derive that, for n even the coefficent is 0, and when $n = 2m - 1$, we have

$$b_n = b_{2m-1} = \binom{2m-2}{m-1}\frac{1}{m}.$$

Exercise. Compute b_{2m-1} for small m, and verify that it gives the right number of binary trees.

1.2 Rooted Plane Trees

We can even consider general rooted plane trees. Every rooted plane tree is a root vertex attached to a sequence of rooted plane trees

$$\begin{aligned}\mathcal{T} &= \mathcal{Z} \times \textsc{Seq}(\mathcal{T})\\ T(z) &= \frac{z}{1-T(z)}\\ T(z)^2 - T(z) + z &= 0\\ T(z) &= \frac{1 \pm \sqrt{1-4z}}{2}\end{aligned}$$

If we take the positive branch the power series has negative coeffs, so

$$\begin{aligned}&= \frac{1-\sqrt{1-4z}}{2} = z + z^2 + 2z^3 + 5z^4 + \dots\\ &= \sum_{n\geq 0}\frac{1}{n}\binom{2n-2}{n-1}z^n\end{aligned}$$

We can step through this recursion slowly to generate the elements of $\mathcal{T}$. Rewrite the recursion as

$$\mathcal{T}^{[n+1]} = \mathcal{Z} \times \textsc{Seq}(\mathcal{T}^{[n]})$$

Start by taking $\mathcal{T}^{[0]} = \{\epsilon\}$, then

$$\begin{aligned}\mathcal{T}^{[1]} &= \{\circ\}\\ \mathcal{T}^{[2]} &= \{\circ, \circ[\circ], \circ[\circ\circ], \circ[\circ\circ\circ], \dots\}\end{aligned}$$

Thus $\mathcal{T}^{[n]}$ contains all trees of depth $< n$. One can quickly see that continuing this iteration gives all trees (any given tree will appear after a finite number of iterations).

Exercise. Did you notice that $[z^{2n+1}]B(z) = [z^n]T(z)$? Can you find the bijection?

2 Simple trees and Lagrange Inversion

2.1 Restricting the out-degree

We already talked about binary trees

$$\mathcal{B} = \mathcal{Z} + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$$

and rooted plane trees —

$$\mathcal{G} = \mathcal{Z} \times \textsc{Seq}(\mathcal{G}).$$

In each case the construction is made by remarking that if the root is deleted, what is left is an ordered sequence of (smaller) rooted trees. This gives generating function relations:

$$B(z) = z + zB(z)^2 \qquad G(z) = \frac{z}{1 - G(z)}.$$

We solved them using the quadratic formula, and both give Catalan numbers in the couting sequence:

$$B(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z} \qquad G(z) = \frac{1 - \sqrt{1 - 4z}}{2}.$$

For this example the nodes can have any out-degree, but in many cases (eg useful computer science ones) we need to restrict the possible outdegrees of the nodes. For example, let Ω be a subset of non-negative integers that contains zero. We can define the class of Ω-restricted trees $\mathcal{T}^{\Omega}$ to be the set of rooted plane trees whose node outdegrees lie only in Ω.

These classes will result in equations for the generating function which are of higher degree than quadratic, so we will need new tools beyond the quadratic formula to solve them. But first the classes themselves:

If we take $\Omega = \mathbb{Z}^+$ we get all rooted plane trees (boring), but we can also take $\Omega = \{0, 2\}$ to get binary trees, etc etc. Now define

$$\phi(u) = \sum_{\omega \in \Omega} u^{\omega}$$

Some examples

$$\begin{aligned} \phi(u) &= 1 + u^2 && \text{binary trees} \\ \phi(u) &= 1 + u + u^2 && \text{unary-binary trees} \\ \phi(u) &= 1/(1 - u) && \text{all trees} \end{aligned}$$

Then we have the following theorem about $\mathcal{T}^{\Omega}$:

Lemma. *The ogf, $T^{\Omega}(z)$, of Ω-restricted plane trees satisfies the following*

$$T^{\Omega}(z) = z\phi(T^{\Omega}(z))$$

where $\phi(u) = \sum_{\omega \in \Omega} u^{\omega}$.

A class of trees that satisfies such an equation is called a simple variety of trees.

2.2 Lagrange Inversion

So notice now that this functional relation implies

$$z = \frac{T}{\phi(T)}$$

ie — T takes some number z and turns it into $T(z)$. You can get back to where you started (ie functional inverse) by computing $T/\phi(T)$. Cute. But also, this functional form can be exploited in order to get an exact expression for T_n — this uses something called the Lagrange inversion formula.

Theorem (Lagrange inversion)**.** *The coefficients of an inverse function and all of its powers are determined by the coefficients of powers of the forward function. So if $z = T/\phi(T)$ then*

$$[z^n]T(z) = \frac{1}{n}[\omega^{n-1}]\phi(\omega)^n$$
$$[z^n]T(z)^k = \frac{k}{n}[\omega^{n-k}]\phi(\omega)^n$$

Note that this immediately gives

$$T_n^{\Omega} = \frac{1}{n}[\omega^{n-1}]\phi(\omega)^n$$

Considerably more general forms of this theorem exist, but this suffices for our purposes.

Example. Plane Trees

$$T(z) = \frac{z}{1 - T(z)} \qquad \phi(u) = \frac{1}{1-u}$$
$$\phi(u)^n = \frac{1}{(1-u)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} u^k$$
$$T_n = \frac{1}{n}[u^{n-1}]\phi(u)^n = \frac{1}{n}\binom{2n-2}{n-1}$$

Example. Binary Plane Trees

$$\mathcal{B} = \mathcal{Z} \times \left\{\mathcal{E} + \mathcal{B}^2\right\}$$
$$B(z) = z(1 + B(z)^2)$$
$$B(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z}$$