Math 343, Lecture 7

① More trees

We had binary rooted trees

What about ternary rooted trees

But now we run into a problem -

We can make lots of other classes of trees with this kind of problem

eg let $\Omega \subseteq \mathbb{Z}_{\geq 0}$ with $0 \in \Omega$

Suppose we are interested in plane rooted trees where the number of children of each vertex is in Ω

We want to exclude ε from such trees so

So $\mathcal{T} =$

so $T(x) =$

Write

subexps

Def A class of trees $\mathcal{T}$ is called simple if its generating function satisfies

for

The trees with number of children restricted to lie in Ω are always simple:

The ternary trees from the beginning of today

Simple classes of trees can't include the empty tree since

So rewrite

$$(\mathcal{T}-\mathcal{E}) =$$

let $\mathcal{U} = \mathcal{T} \setminus \mathcal{E}$

② Lagrange inversion

We still haven't solved the problem of how to get coefficients back out of these equations. We need the Lagrange inversion theorem, but first two formal power series definitions we skipped over before.

Def Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. Then define the formal derivative

$$\frac{d}{dx} A(x) =$$

Def Let $A(u) = \sum_{n=0}^{\infty} a_n u^n$ and $B(x) = \sum_{n=1}^{\infty} b_n x^n$ be formal power series, then

$$A(B(x)) =$$

Theorem (Lagrange inversion)

Let $T(x)$ & $\phi(u)$ be formal power series with $[u^0]\phi(u) \neq 0$ and $[x^0]T(x) = 0$

Suppose
$$x = \frac{T(x)}{\phi(T(x))}$$

In fact

Then
$$[x^n]T(x) = \frac{1}{n}[u^{n-1}]\phi(u)^n$$
$$[x^n]T(x)^k = \frac{k}{n}[u^{n-k}]\phi(u)^n$$

Before we can prove this we need to talk briefly about formal Laurent series.

Def A formal Laurent series is an expression of the form eg

Formal power series are

We can define the obvious operations for formal Laurent series as we did for formal power series

We need some Lemmas

Lemma 1 Let $F(x)$ be a formal Laurent series

$$[x^{-1}] \frac{d}{dx} F(x) = 0$$

proof

Lemma 2 Let $F(x)$ and $G(x)$ be formal Laurent series

$$[x^{-1}] F(x) G'(x) = -[x^{-1}] F'(x) G(x)$$

proof By Lemma 2 $0 =$

Lemma 3 (change of variables) Let $F(x)$ be a nonzero formal Laurent series. Let $G(x)$ be a formal power series with first nonzero term g_ℓ for some $\ell > 0$.

Then

$$[x^{-1}]F(G(x))G'(x) = \ell[u^{-1}]F(u)$$

proof First consider the special case when $F(u) = u^k$

Then $F(G(x))G'(x) =$ for $k \neq -1$

So by Lemma 1 for $k \neq -1$

Also if $k \neq -1$ then

If $k = -1$ then write $G(x) = x^\ell H(x)$

$H(x)$ is a power series with nonzero constant term and so

So $F(G(x))G'(x) =$

A16

Therefore the result holds for $F(u) = u^k$

In general write $F(u) = \sum_{n=I}^{\infty} f_n u^n$

$[x^{-1}] F(G(x)) G'(x) =$

Proof of Lagrange inversion

Let $F(u) = u^k$

Then $[x^n] T(x)^k =$

Now do a change of variables

let $P(u) = \dfrac{u}{\phi(u)}$ so $x = P(u)$, $u = T(x)$

let $H(u) = P(u)^{-n} F'(u)$

so $[x^n] F(T(x)) =$

For an example go to tutorial, also see the notes

3 Next line Random generation.