Math 800, Commutative Algebra, Lecture 15

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1 Proving theorem B

Theorem B: If R is an affine domain over a field F then $KdimR = trdeg_F R$.

Proof. By induction on $n = trdeg_F R$. $n = 0$ follows from theorem A. Take $n > 0$, by Noether normalization R is integral over $R_0 = F[b_1, b_2, \dots, b_n]$ and $R_0 \cong F[\lambda_1, \lambda_2, \dots, \lambda_n]$. We have that every maximal ideal of R_0 has height $\geq n$ so $KdimR_0 \geq n$ and integral extensions preserve Krull dimension so $KdimR \geq n$. Now suppose we have $P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n \supsetneq 0$ a chain in $Spec(R_0)$ which has length $> n$ let $\bar{R} = R_0/P_n$, so $trdeg\bar{R} < n$. So by induction $Kdim\bar{R} < n$ but $P_0/P_n \supsetneq P_1/P_n \supsetneq \cdots \supsetneq P_n/P_n = 0$ is a chain of length n in $Spec(\bar{R})$. □

2 Chain conditions and modules

For the rest of today R not necessary commutative.

Definition. *Let S be a poset, S satisfies the ascending chain condition (ACC) if there is no infinite strictly ascending chain in S $s_1 < s_2 < \cdots$, equivalently any weakly ascending chain in S $s_1 \leq s_2 \leq \cdots$ eventually stabilizes. i.e. $\exists n$ s.t $s_n = s_{n+1} = \cdots$.*
S satisfies the descending chain condition (DCC) if there is no infinite strictly descending chain in S $s_1 > s_2 > \cdots$, equivalently any weakly descending chain in S $s_1 \geq s_2 \geq \cdots$ eventually stabilizes. i.e. $\exists n$ s.t $s_n = s_{n+1} = \cdots$.
In the case M is a left R-module and $\mathcal{L}_R(M)$ is the lattice of submodules. Then say M is Noetherian(Artinian) if $\mathcal{L}_R(M)$ satisfies ACC(DCC).

Property. *Let S be a poset,*

(1) S satisfies ACC iff every nonempty subset of S has a maximal element

(2) S satisfies DCC iff every nonempty subset of S has a minimal element

Proof. (1) $\Leftarrow$: Suppose we have $s_1 \leq s_2 \leq \cdots$ an ascending chain in S, consider $\{s_1, s_2, \ldots\} \subseteq S$ this has a maximal element say s_i. But $s_i \leq s_{i+1} \leq \ldots$ so $s_i = s_{i+1} = s_{i+2} = \ldots$
$\Rightarrow$: Suppose there is a subset $S_0 \subseteq S$ which doesn't have a maximal element. Take $s_1 \in S_0$, given $s_1 < s_2 < \cdots < s_k \in S_0$ since s_k is not maximal in S_0, $\exists s_{k+1} \in S_0$, $s_k < s_{k+1}$ this builds an infinite strictly ascending chain contradicting ACC.
(2) same by flipping $\leq_S$ i.e by dual poset. □

Property. *Suppose $N \subseteq M$ submodule,*

(1) M is Noetherian iff N is Noetherian and M/N is Noetherian.

(2) M is Artinian iff N is Artinian and M/N is Artinian.

Proof. (1) $\Rightarrow$: Any infinite ascending chain in N is also infinite ascending chain in M so N is Notherian. An infinite ascending chain in M/N looks like $M_0/N \subseteq M_1/N \subseteq M_2/N \subseteq \cdots$. Then $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ is an infinite ascending chain in M so it eventually stabilizes, so $M_0/N \subseteq M_1/N \subseteq M_2/N \subseteq \cdots$ stabilizes.
$\Leftarrow$: let $M_0 \subseteq M_1 \subseteq \cdots$ be an ascending chain of submodules of M. Then $M_0 \cap N \subseteq M_1 \cap N \subseteq \cdots$ is an ascending chain of submodules of N and hence stabilizes. Also $(M_1 + N)/N \subseteq (M_2 + N)/N \subseteq \cdots$ is an ascending chain of submodules of M/N and hence stabilizes. Take i large enough that both these chains stablized. So $M_i \cap N = M_{i+1} \cap N$ and $(M_i + N)/N = (M_{i+1} + N)/N$ but modularity of modules says

$$\begin{aligned} M_i &= M_i + (N \cap M_i) \\ &= M_i + (N \cap M_{i+1}) = (M_i + N) \cap M_{i+1} \\ &= (M_{i+1} + N) \cap M_{i+1} \\ &= M_{i+1} \end{aligned}$$

So M is noetherian. (2) The same proof. □

Corollary. *Let M be a left R-module if M is Artinian and Noetherian then M has a composition series.*

Proof. Given any nonzero module M_i consider the set of proper submodules of M_i. This has a maximal element M_{i+1} so M_i/M_{i+1} is simple. Iterating we get $M = M_0 \supsetneq M_1 \supsetneq \cdots$ and this terminates by Artinianess. □

Property. *Let M be an left R module, M is Noetherian iff every submodule of M is finitely generated.*

Proof. $\Leftarrow$: Take $M_0 \subseteq M_1 \subseteq \cdots$ chain of submodules of M. $\bigcup M_i$ is finitely generated so there is some j such that all the generators (finitely many) are in M_j, so $M_j \subseteq M_{j+1} \subseteq \cdots \subseteq \bigcup M_i \subseteq M_j$ thus $M_j = M_{j+1} = \cdots$
$\Rightarrow$: Say N is a submodule which is not finitely generated. Consider the set of finitely generated submodules of N. This is nonempty since it contains 0. So it contains a maximal element call it N', $N' \subsetneq N$ since N is assumed not finitely generated. Take $a \in N - N'$ then $N' + Ra$ is submodule of N and is finitely generated and properly contains N' contradicting the maximality of N'. □

3 Noetherian and Artinian rings

Definition. *A ring R is left(right) Noetherian if R is Noetherian as a left(right) R module. A ring R is left(right) Artinian if R is Artinian as a left(right) R module.*
A ring R is Noetherian(Artinian) if it is both left and right Noetherian(Artinian). For R commutative left Noetherian(Artinian) and right Noetherian(Artinian) are the same.

Note. *(rephrasing one of the module results) A ring R is left Noetherian iff every left ideal is finitely generated.*

Property. *(1) Let R be a left Noetherian ring then every finitely generated left R-module is Noetherian.*

(2) Let R be a left Artinian ring then every finitely generated left R-module is Artinian.

Proof. (1): Any such module M has the form $M \cong R^{(n)}/K$ where K is a submodule of $R^{(n)}$. It suffice to show $R^{(n)}$ is Noetherian. Prove this by induction on n.
$n = 1$, given. Take $n > 1$, $R^{(n)}/R \cong R^{(n-1)}$ so $R^{(n-1)}$ is Noetherian by induction and R is given Noetherian. So $R^{(n)}$ is Noetherian.
(2) Same. □

A theorem we won't prove.
Theorem. (Corollary of Hopkins-Levitzki)
All left Artinian rings are left Noetherian.

Example. *(1) Let F be a field viewing F as a ring. Then F is both Noetherian and Artinian.*

(2) Let V be a vector space over a field viewing as a module. V is Noetherian iff V is finite dimensional iff V is Artinian.

(3) R a PID. Since every ideal is cyclic hence finitely generated, so R is Noetherian.

(4) $R = F[\lambda_1, \lambda_2, \ldots]$ *is not Noetherian since* $0 \subsetneq < \lambda_1 > \subsetneq < \lambda_1, \lambda_2 > \subsetneq \cdots$.

(5) Fix p prime, let $M = \{\frac{m}{n} : n \text{ is a power of } p\}$ *viewing M as a* $\mathbb{Z}$*-module (i.e an abelian group). Let* $N = M/\mathbb{Z}$ *suppose K is a submodule of N, if* $\frac{m}{n} + \mathbb{Z} \in K$ *with* $g.c.d(m, n) = 1$*. Write* $am + bn = 1$ *so* $a(\frac{m}{n} + \mathbb{Z}) = \frac{1}{n} - \frac{bn}{n} + \mathbb{Z} = \frac{1}{n} + \mathbb{Z}$*. Say* $\frac{1}{p^a} + \mathbb{Z} \in K$, $\frac{1}{p^b} + \mathbb{Z} \in K$ *with* $a < b$ *then* $p^{b-a}(\frac{1}{p^b} + \mathbb{Z}) = \frac{1}{p^a} + \mathbb{Z}$ *so every nonzero submodule of N is generated by the elements of the form* $\frac{1}{n} + \mathbb{Z}$.
We have a chain $(\frac{1}{p} + \mathbb{Z})N \subsetneq (\frac{1}{p^2} + \mathbb{Z})N \subsetneq \cdots$ *thus n N is not Noetherian.*
Given a set of submodules of N take the minimum denominator of generators, this generates the minimal element of the set. Thus N is Artinian.

4 Hilberts Basis theorem

Theorem. (Hilberts basis theorem)
If R is a left Noetherian ring then the polynomial ring $R[\lambda]$ is also left Noetherian.

Proof. Suppose $I \subseteq R[\lambda]$ is a left ideal take $f_1 \neq 0$, $f_1 \in I$ of least degree proceeding inductively on i, for $i \geq 1$, $< f_1, f_2, \ldots, f_i >= I$ then we are done. If not then pick $f_{i+1} \in I \setminus < f_1, \ldots, f_i >$ of least degree.
Let a_i be the leading coefficient of f_i. Since R is Noetherian so $< a_1, a_2, \ldots > \subseteq R$ generated by $a_1, \ldots, a_m$. Claim $f_1, \ldots, f_m$ generates I. Suppose not, $a_{m+1} = \sum_{i=1}^{m} c_i a_i$ so let $g = \sum_{i=1}^{m} c_i f_i \lambda^{(deg f_{m+1} - deg f_i)}$ by construction $deg f_{m+1} - deg f_i \geq 0$. Note $f_{m+1} - g$ has degree strictly lower than f_{m+1} but $f_{m+1} - g \notin < f_1, \ldots, f_m >$ contradicting the choice of f_{m+1}. Result follows. □

Corollary. *Every affine algebra is Noetherian.*

Proof. An affine algebra is of the form $F[\lambda_1, \ldots, \lambda_m]/A$, it suffices to show $F[\lambda_1, \ldots, \lambda_m]$ is Noetherian which is true by Hilberts basis theorem applied inductively. $\square$