COMMUTATIVE ALGEBRA, FALL 2013

ASSIGNMENT 4 SOLUTIONS

(1) These two questions end up being quite similar.

Rowen ch8 #2 Let $\Lambda = \{\lambda_s : s \in S\}$ be a set of commuting indeterminates. Let $R[\Lambda]$ be the polynomial ring in Λ and let

$$T = R[\Lambda]/\langle s\lambda_s - 1 : s \in S\rangle$$

Note that the map $\phi : R \to T$ which takes $r \in R$ to the constant polynomial r has each $\phi(s)$, $s \in S$, invertible with inverse λ_s, and central since S is central in R and R is central in $R[\Lambda]$.

Now suppose we have $f : R \to T'$ an algebra homomorphism with $f(s)$ invertible and central for all $s \in S$. Then we want a map $\hat{f} : T \to T'$ with $f = \hat{f}\phi$. To defne $\hat{f}$, first define $\hat{f} : R[\Lambda] \to T$ by $\hat{f}(r) = f(r)$ for $r \in R$ and $\hat{f}(\lambda_s) = f(s)^{-1}$ for $s \in S$. Extending this as an algebra homomorphism we get $\hat{f}(s\lambda_s - 1) = f(s)f(s)^{-1} - 1 = 0$ and so $\hat{f}$ also gives a well defined algebra homomorphism from F to F'. It is the unique such map with $f = \hat{t}\phi$ because that equation forces the behaviour of $\hat{f}$ on R and Λ.

Therefore by the universal property of localization $T = S^{-1}R$.

Rowen ch8 #4 Let $\mathcal{R}$ and f_s be as in the question. View R inside $\mathcal{R}$ via $\phi : r \mapsto (R1, \text{multiplication by } r)$. Let T be the subring of $\mathcal{R}$ generated by R and $\{(Rs, f_s) : s \in S\}$. Note that $(\phi(s))(Rs, f_s) = (Rs, \text{id})$ which is equivalent to $(R1, \text{id})$ since they agree on their intersection, and $(R1, \text{id}) = 1$ in T. Furthermore $\phi(s)$ is central in T since S is central in R.

Now suppose we have $f : R \to T'$ an algebra homomorphism with $f(s)$ invertible and central for all $s \in S$. Then we want a map $\hat{f} : T \to T'$ with $f = \hat{f}\phi$. Define $\hat{f} : T \to T'$ by $(R1, \text{mult by } r) \mapsto f(r)$ for $r \in R$ and $(Rs, f_s) \mapsto f(s)^{-1}$ for $s \in S$, annd extended as an algebra homomorphism. This is well defined as if some polynomial in the generators of R is zero then the analagous expression in $f(r)$, $f(s)^{-1}$ is zero. $\hat{f}$ is unique as the action on the generators is determined by $f = \hat{f}\phi$.

Therefore by the universal property of localization $T = S^{-1}R$.

(2) Take $P \in \text{Spec}C$. We can localize everything at $C \setminus P$ (this is still a multiplicative subset of R), and so can assume that C is local with maximal ideal P.

Suppose $1 \in PR$ so $1 = \sum_{i=1}^{t} p_i r_i$ for some $r_i \in R$, $p_i \in P$. Then let $R' = C[r_1, \ldots, r_t]$. Then $1 \in PR'$, so since PR' is an ideal then we have $R' = PR'$. But R' is a finitely generated C modulate, so by Nakayama's lemma $PR' \neq R'$ which is a contradiction.

Therefore $1 \notin PR$. So $C \cap PR$ is a proper ideal of C, and $P \subseteq C \cap PR$. Therefore as C is local $P = C \cap PR$.

Furthermore PR is maximal hence prime as R/PR is integral over C/P (simply mod out the polynomials) so C/P a field implies that for any $a \in R/PR$ we have $C/P[a]$ is a field, so $a^{-1} \in C/P[a] \subseteq R/PR$, and so R/PR is also a field.

Returning to the original C, we still have $C \cap PR = P$ as if it were larger then it would remain larger (hence equal to C) upon localization, since if $x \in (C \smallsetminus P) \cap PR$ then $x \in C$, $x \in PR$, so $x \in C \cap PR$ so $x \in P$ which would be a contradiction.

And finally since localization takes prime ideals to prime ideals and vice versa, PR remains prime in the original setup.

(3) Take P with height at least 2. Suppose there are only fininitely many height 1 prime ideals contained in P. Call them $P_1, \ldots, P_t$.

Suppose $P_1 \cup \cdots \cup P_t = P$. Then by prime exclusion $P = P_i$ for some i contradicting the height of P_i.

Since we haven't done prime exclusion let's prove it in the form we need here. Throw away P_j if necessary until $P_i \not\subseteq \bigcup_{j \neq i} P_j$. Assume $t \geq 2$. Take $a_i \in P_i \smallsetminus P_t$ for $i < t$ and take $a_t \in P_t \smallsetminus \bigcup_{i<t} P_i$.

Then $a_1 \cdots a_{t-1} \notin P_t$ since P_t is prime. If $a_t + a_1 \cdots a_{t=1} \in P_t$ then we get $a_1 \cdots a_{t-1} \in P_t$ which is a contradiction. On the other hand if $a_t + a_1 \cdots a_{t-1} \in P_1 \cup \cdots \cup P_{t-1}$ then $a_t \in P_1 \cup \cdots P_{t-1}$ which is also a contradiction. Thus we must have $t = 1$ and so $P = P_i$ for some i.

Now returning to the main argument, $P_1 \cup \cdots \cup P_t \neq P$ so there exists an $a \in P$ with $a \notin P_1 \cup \cdots \cup P_t$. So P is minimal over a which contradicts the principal ideal theorem.

(4) If R is a field then every R-module is a vector space and every exact sequence of vector spaces splits, so every R-module is projective.

If R is a domain and every R-module is projective. Take $a \neq 0$, $a \in R$ and consider aR. We have the exact sequence $0 \to R \to R \to R/aR \to 0$ where the map from R to R is multiplication by a; call this map f. This exact sequence is split by assumption, so by an old homework, f in particular is split, that is there exists a $g : R \to R$ such that $gf = 1_R$. So $g(f(1)) = 1$ so $g(a) = 1$. But g is a module homomorphism so $ag(1) = g(a) = 1$ and so a is a unit in R.

Therefore R is a field.

(5) Given

$$\begin{array}{ccccccccc}
\longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & \\
\longrightarrow & C'_{n+1} & \xrightarrow{d'_{n+1}} & C'_n & \xrightarrow{d'_n} & C'_{n-1} & \longrightarrow
\end{array}$$

and having defined $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(C'_\bullet)$ by $z_n + B(C_\bullet) \mapsto f_n z_n + B(C'_\bullet)$ we need to check the following:

First $f_n z_n$ is a cycle: We know z_n is a cycle so $d_n z_n = 0$ so $d'_n f_n z_n = f_{n-1} d_n z_n = 0$ so $f_n z_n$ is a cycle.

Next check that the definition is independent of choices. Say $z_n + B_n(C_\bullet) = z'_n + B_n(C_\bullet)$. So $z_n - z'_n \in B_n(C_\bullet)$. So $z_n - z'_n = d_{n+1} c$ for some $c \in C_{n+1}$, so $f_n z_n - f_n z'_n = f_n d_{n+1} c = d'_{n+1} f_{n+1} c \in B_n(C'_\bullet)$. So $f_n z_n + B_n(C'_\bullet) = f_n z'_n + B_n(C'_\bullet)$.

Next note that $H_n(1_{C_\bullet})$ is the identity by definition.

Finally $H_n(gf) : z_n + B_n \mapsto g_n(f_n(z_n + B_n)) = H_n(g)(H_n(f)(z_n + B_n))$ so H_n is a functor.

(6) This one is a diagram chase.

First I need to label the maps:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & K'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} & & \\
0 & \longrightarrow & P' & \xrightarrow{\eta} & P & \xrightarrow{\zeta} & P'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & A' & \xrightarrow{\eta} & A & \xrightarrow{\zeta} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Take $k' \in K'$. Suppose $\alpha(k') = 0$ so $k' \in \ker\alpha$. Then $\delta\alpha(k') = 0$ so $\eta\gamma(k') = 0$. But η is injective so $\gamma(k') = 0$. γ is injective so $k' = 0$ and hence $\ker\alpha = 0$ as it should.

Also for $k' \in K'$ (without other hypotheses), consider $\alpha(k')$. $\delta\alpha(k') = \eta\gamma(k')$ so $\epsilon\beta\alpha(k') = \zeta\delta\alpha(k') = 0\gamma(k') = 0$. But ϵ is injective so $\beta\alpha(k') = 0$. Therefore $\alpha(k') \in \ker\beta$.

Take $k \in K$. Suppose $\beta(k) = 0$ so $k \in \ker\beta$. Then $\epsilon\beta(k) = 0$ so $\zeta\delta(k) = 0$. So $\delta(k) \in \ker\zeta = \mathrm{im}\eta$. So there exists a $p' \in P'$ with $\eta(p') = \delta(k)$. Also $\mu\theta(p') = \kappa\eta(p') = \kappa\delta(k) = 0$. But μ is injective so $\theta(p') = 0$. Therefore $p' \in \ker\theta = \mathrm{im}\gamma$ so there exists $k' \in K'$ with $\gamma(k') = p'$. So $\delta\alpha = \eta\gamma(k') = \eta(p') = \delta(k)$ and δ is injective so $\alpha(k') = k$. Thus $k \in \mathrm{im}\alpha$.

Taking the two previous paragraphs together we have $\ker\alpha = \mathrm{im}\beta$.

Next take $k'' \in K''$. Then $\epsilon(k'') \in P''$. But ζ is onto so there exists $p \in P$ with $\zeta(p) = \epsilon(k'')$. Then $\pi\kappa(p) = \nu\zeta(p) = \nu\epsilon(k'') = 0$, so $\kappa(p) \in \ker\pi = \mathrm{im}\mu$. So there exists $a' \in A'$ with $\mu(a') = \kappa(p)$. But θ is onto so there exists $p' \in P'$ with $\theta(p') = a'$. So $\kappa\nu(p') = \mu\theta(p') = \kappa(p)$. So $p - \eta(p') \in \ker\kappa = \mathrm{im}\delta$. Say $k \in K$ with $\delta(k) = p - \eta(p')$. Then $\epsilon\beta(k) = \zeta\delta(k) = \zeta(p) - \zeta\eta(p') = \zeta(p) = \epsilon(k'')$ but ϵ is injective so $\beta(k) = k''$ and thus $\mathrm{im}\beta = K''$.

This completes the proof.

(7) G is free abelian means G is free as a $\mathbb{Z}$-module. Suppose G is free abelian. Then also G is projective as a $\mathbb{Z}$-module. So $Ext^1(G, F) = 0$ for all $\mathbb{Z}$-modules F. So in particular $Ext^1(G, F) = 0$ for F free abelian.

Suppose $Ext^1(G, F) = 0$ for all F free abelian. Then consider the map $p : \mathbb{Z}^{|G|} \to G$ defined by taking the generator indexed by g to g itself for $g \in G$. Then $0 \to \ker p \to \mathbb{Z}^{|G|} \to G \to 0$ is exact. Now $\ker p \subseteq \mathbb{Z}^{|G|}$ but subgroups of free abelian groups are free abelian, so $\ker p$ is free abelian. Thus by hypothesis this exact sequence splits. So G is also a subgroup of $\mathbb{Z}^{|G|}$ and hence is also free abelian.

(8) *answers will vary*