# COMPLETION AND RELATED CONSTRUCTIONS

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### Filtrations,the Artin Reese Lemma and Krull's Theorem

Abstract: Pete Clark says "I very much appreciate that finding the right bit of structure can make the solutions to your problem self evident." In this paper we will look at some bits of structure, that can at least help make some solutions to certain problems evident. The over arching idea, as I see it, is the construction of new rings and modules from known rings and modules that behave nicely can help us learn about the original ring in suitable circumstances. Specifically, we will look at constructions that arise from chains of ideals and or submodules. We start by considering the associated graded rings that arise give descending sequences of submodules and various constructions and prove the Artin Reese Lemma and Krull's Theorem. We then turn to completion, and define it in two different ways. Namely via Cauchy sequences and the inverse limit. After showing the definitions are equivalent we prove some things about completion. By and large we will follow Eisenbud's Commutative Algebra through chapters 5 and 7 and Atiyah McDondald's Introduction to Commutative Algebra with more of the details filled in.

**Definition.** Let $\{I_j\}_j$ be a descending sequence of ideals. We call such a sequence a filtration if $I_iI_j \subseteq I_{i+j}$. The obvious example is the $I - adic$ filtration $I_n = I^n$.

Some examples: Take $I = (x)$ in $\mathbb{C}[x]$ for concreteness. Then $I^n = (x^n)$. So $\cap_n I^n = (0)$. On the other hand, when is the intersection non-empty? Take $R = \mathbb{Z}^2$ Then take $I =< (1,0) >$. So $(1,0)^n = (1,0)$ for all $n$ and so $(1,0) \in I^n$ for all $n$. This means $I^n = I$. We can also make things strictly decreasing. Take, $(1,0)$ and $(0,2)$ in $R$ as above. Put $I = (1,0)\mathbb{Z}^2$ and $J = (0,2)\mathbb{Z}^2$. Now define $T = I + J$. Note that if $i \in I$ and $j \in J$ then $ij = 0$. Furthermore, $I^n = I$ but $J^n$ is strictly decreasing. So $\cap_{n\geq 1} T^n \neq (0)$ and the $T^n$ are strictly decreasing.

**Definition.** Let $M$ be an $R$ module and suppose that $\{I_n\}$ is a descending filtration.A descending sequence $\{M_n\}$ of submodules is called an $I$ filtration if $IM_n \subseteq M_{n+1}$ for all $n$. We say an $I$ filtration is $I$ stable if for all but finitely many $n$ we have that $IM_n = M_{n+1}$.

**Example.** Let $M$ be any $R$ module, and $I$ any ideal. Set $M_n = I^nM$. Then $M_n$ is $I$ stable.

**Definition.** Let $I$ be an ideal of a ring $R$. We define $gr_I R := \oplus_{n=0}^{\infty} I^n/I^{n+1}$. (Where $I^0 := R$) We make $gr_I R$ into a ring as follows. If $a \in I^n$ and $b \in I^m$ then we set $(a + I^{n+1}) \cdot (b + I^{m+1}) = ab + I^{n+m+1}$. More generally, let $\mathcal{F} := M = M_0 \supseteq M_1 \supseteq ....$ be a $I$ filtration, we define $gr_{\mathcal{F}} M := \oplus_{n=0}^{\infty} M_n/M_{n+1}$. We make $gr_{\mathcal{F}} M$ into a graded $gr_I R$ module in the obvious way. Namely, if $i_n + I_{n+1} \in I_n/I_{n+1}$ and $m_k \in M_k/M_{k+1}$ then $(i_n + I_{n+1}) \cdot (m_k + M_{k+1}) = i_n m_k + M_{n+k+1}$. This is well defined because $\mathcal{F}$ is an $I$ filtration.

**Proposition.** *Let $I$ be an ideal of a ring $R$ and $\mathcal{F}$ and $I$ stable filtration of a $R$ module $M$. If the sub modules of the filtration are finitely generated then so is the associated graded module over $gr_I R$.*

Proof: There is an $n$ such that for $i \geq n$ we have $IM_i = M_{i+1}$. In this case, note that if $m_{i+1} \in M_{i+1}$ then there is some $t \in I$ and $m_i \in M_i$ with $tm_i = m_{i+1}$. Then working in the graded structure gives $(t + I^2)(m_i + M_{i+1}) = tm_i + M_{i+2} = m_{i+1} + M_{i+2}$. That is, $(I/I^2)(M_i/M_{i+1}) = M_{i+1}/M_{i+2}$. So let $G$ be the generators of the first $n$ sub modules. So we can write, $(I/I^2)(M_n/M_{n+1}) = M_{n+1}/M_{n+2}$. Continuing inductively gives $(I^i/I^{i+1})(M_n/M_{n+1}) = M_{i+n}/M_{n+i+1}$. Since elements of $gr_{\mathcal{F}} M$ are finite sums, the generators of $M_n/M_{n+1}$ and the generators of the first $n$ sub modules are enough to give everything.

*Remark.* The intuition is that stable filtration's are kind of like finitely generated modules. The above translates the intuition into concrete facts about the graded constructions.

**Definition.** Let $M$ be an $R$ module with a filtration $\mathcal{F}$. Given $f \in M$ we set $s : M \to \mathbb{N} \cup \{\infty\}$ be the map that sends $f$ to the largest integer $m$ such that $f \in M_m$ and $f$ to infinity if $f \in \cap_{n=1}^{\infty} M_n$. Now set $in(f) = f + M_{s(f)+1} \subseteq gr_{\mathcal{F}} M$ if $s(f)$ is finite, and $in(f) = 0$ otherwise. Given an ideal $I$ of $R$ and a $I$ filtration of $M$ with $N \leq M$ we define $in(N)$ to the module generated by the initial forms of elements of $N$. I.E $in(N) =< in(f) : f \in N >$ in $gr_{\mathcal{F}} M$.

**Example.** The $in$ operator can behave in not so nice ways. Consider $J =< xy + y^3, x^2 > \subseteq k[x,y]$ and let $I = (x,y)$. Put the $I$-adic filtration on $k[x,y]$. $in(J) =< in(f) : f \in J >$. Notice that $in(x^2) = x^2 + (x,y)^3 k[x,y]$ and $in(xy + y^3) = xy + y^3 + (x,y)^2 k[x,y] = xy + (x,y)^2 k[x,y]$. So the question is whether, $in(J)$ is generated by the initial forms of the generators of $J$. But this is not the case as we can look at what the degrees of elements spanned by $xy + (x,y)^2 k[x,y]$ and $x^2 + (x,y)^3 k[x,y]$. Now, $y^5 \in J$ so $y^5 + I^6 k[x,y] \in in(J)$ but we cannot generate $y^5 + I^6 k[x,y]$ with our given generators.

**Definition.** Let $R$ be a ring and $I$ an ideal. We set $B_IR := \oplus_{n=0}^{\infty} I^n = R \oplus I \oplus I^2 \oplus .... \cong R[tI]$. If $\mathcal{F}$ is a $I$ filtration then the blow up will be $B_{\mathcal{F}}M = \oplus_{i=0}^{\infty} M_i$.

**Lemma.** $B_IR/IB_IR :\cong gr_IR$

Proof: Let $\pi_i : I^i \to I^i/I^{i+1}$ be the canonical projection. Consider $\pi : \sum \pi_i \to gr_IR$. $\pi$ is clearly a surjection. Now, suppose that $\pi(\sum_{k=0}^{n} i_k) = \sum_{k=0}^{n} \pi_k(i_k) = 0$. Then we must have that $\pi_k(i_k) = 0$ for each $k$ and so $i_k \in I^{k+1}$. Now, $i_k \in I^{k+1}$ means that $i_k \in IB_IR$ and so the sum of the $i_k$ are in $IB_IR$. So $\ker \pi \subseteq IB_IR$ . On the other hand, any element in $IB_IR$ is in the kernel so we have the result.

*Remark.* So we can regard $gr_IR$ as $R[tI]/IR[tI]$.(Where $R[tI] = \{\sum_{k=0}^{n} i_k x^k : i_k \in I_k\}$) Now, given $f \in R$ what is $in(f)$ intuitively? Well, if $in(f) \neq 0$ let $in(f) = f + I_{n+1}$. On the otherhand, in $B_IR$ we have that $f$ can be thought of as the element, $f + fx + .... + fx^n$. So in $gr_IR$ this is the element $\sum_{j=0}^{n} fx^j + IR[tI]$. Notice that $\sum_{j=0}^{n} fx^j \notin IR[tI]$. So in this sense, $in(f) = x^n f + IR[tI]$ and the initial term picks off the highest order the element $f$ in the blow up algebra, and then projects this down into the graded ring. So in this sense, the $in$ operator behaves somewhat like picking off the highest order term in a monomial ordering. ( If the reader is not familiar with monomial orders and wishes to learn then Eisenbud's chapter 15 treats them. For an easier read, there is Ideals,Varieties, and Algorithms by David A. Cox, John B. Little and Don O'Shea.)

*Note.* The blow up algebras have geometric interpretations and roles. For example, see Eisenbud 5.2 for a brief discussion.

The proposition below is interesting, it takes a question about the stability of filtrations of finitely generated modules, and then relates this to a module theoretic question about the blow up ring and blow up algebra. Since we have a large body of theory to draw upon when dealing with modules this can be useful.

**Proposition.** *Let $R$ be a ring and $I$ an ideal. Let $M$ be a f.g. $R$ module with $I$ filtration $\mathcal{F}$ by finitely generated $R$ modules. $\mathcal{F}$ is stable if and only if $B_{\mathcal{F}}M$is a finitely generated $B_IR$ module.*

Proof: Suppose that $B_{\mathcal{F}}M$ is finitely generated. We can take a finite set of homogenous elements of the $M_i$ for $i \leq n$ (for some $n$) with the homogenous elements also generating each $M_i$. Now, let $m \in M_{n+i}$ for some $i > 0$. Then there are some generators $m_{j,k}$ with $m_{j,k} \in M_j$ for $j \leq n$. Then, if $\sum r_{j,k} m_{j,k} = m$ we know, because of the graded structure, that $r_{j,k} m_{j,k} \in M_{n+i}$. That is, $r_{j,k} \in I^{n+i-j}$. But then, using the grading we can look at this as a sum of elements in $M_n$ multiplied by elements in $I^i$. So we have that, as a $B_IR$ module, $M_n$ generates $\oplus_{i=0}^{\infty} M_{n+i}$ or that $M_{n+i} = I^i M_n$ so that the filtration is $I$ stable. Conversely, suppose that the filtration is $I$ stable. Then, for all $i \geq n$ we have $IM_i = M_{i+1}$. So, take the generators of the $M_0, M_1, ..., M_n$ as homogenous elements. Then we know that $I^i M_n = M_{n+i}$ and so over the blow up algebra, we get everything.

We now come to the Artin Reese Lemma which concerns Noetherian rings.

**Theorem.** *Artin Reese*

*Let $R$ be a Noetherian ring, and $\mathcal{J} := M_0 \supseteq ...$ a $I$ stable filtration. Let $N$ be a submodule of $M_0$. Then, the induced filtration is also $I$ stable. That is, there is an $n$ such that for all $i \geq n$ we have $N \cap M_{i+1} = I(M_i \cap N)$.*

Proof: The filtration is $I$ stable. This means that $B_{\mathcal{J}}M$ is finitely generated as a $B_IR$ module. Now, $B_IR$ is in fact Noetherian as a ring since we can regard $B_IR \cong R[It] \subseteq R[t]$ which is Noetherian by Hilbert's Basis theorem. Now, $\oplus_{i=0}^{\infty} N \cap M_i$ is a sub module of $B_{\mathcal{J}}M$. Since, $B_{\mathcal{J}}M$ is finitely generated over a Noetherian ring, we have that $B_{\mathcal{J}}M$ is Noetherian, which means that $\oplus_{i=0}^{\infty} N \cap M_i$ is finitely generated over $B_IR$ and so stable as desired.

It is interesting how easy this was. Although I suppose it is not so surprising. We are dealing with chain conditions; the Noetherian hypothesis is one of the premier chain conditions, there must be a reason it is so popular.

**Theorem.** *Krull Intersection Theorem*

*Let $R$ be a Noetherian ring, and $I$ a proper ideal. If $M$ is a f.g. $R$ module then there is an element $r \in I$ such that $(1 - r) \in ann(\cap_{i=1}^{\infty} I^i M)$. If we further insist that $R$ is local, or a domain then we have $\cap_{j=1}^{\infty} I^j = \{0\}$.*

Proof: We prove the first statement about modules first. Consider the stable filtration $M_n = I^n M$. Then Artin Reese applies to $N = \cap_{i=1}^{\infty} I^i M$ so there is some $m$ such that for $i \geq m$ we have

$$(\cap_{i=1}^{\infty} I^i M) \cap I^{m+1} M = I((\cap_{i=1}^{\infty} I^i M) \cap I^m M)$$

But we have $(\cap_{i=1}^{\infty} I^i M) \cap I^{m+1} M = (\cap_{i=1}^{\infty} I^i M)$ and $(\cap_{i=1}^{\infty} I^i M) \cap I^m M) = \cap_{i=1}^{\infty} I^i M$ so by the above we have $IN = N$. This is a general situation. If $N$ is finitely generated (It is here since it is a submodule of a Noetherian module $M$) and $I$ a proper ideal, then $IN = N \Rightarrow \exists i \in I$ such that $(1 - i)N = 0$. For example, see corollary 2.5 of Atiyah McDonald. Since we are in this situation, we have the annihilator part of the proof. Now suppose that $R$ is a domain. Apply the theorem to $M = R$. Then $I^n R = I^n$ so there is some $i \in I$ with $(1 - i) \cap_{n=1}^{\infty} I^n = 0$. But, $I$ is proper so $1 - i \neq 0$. So $\cap_{n=1}^{\infty} I^n$ must only contain zero as $R$ is a domain. If $R$ is local, then apply the same argument, but note that $1 - i$ must be a unit.

**Corollary.** *Let $R$ be a Noetherian Local Ring and $I$ a proper ideal of $R$. If $gr_IR$ is a domain then $R$ is also a domain.*

Proof: In a certain sense, the proof writes itself if we buy that $in : R \to gr_I R$ is an interesting map. Then if $fg = 0$ in $R$ we have that $in(f)in(g) = 0$ in $gr_I R$. But then we have that $in(f) = 0$ or $in(g) = 0$ in $gr_I R$. But if say $in(f) = 0$ by definition we have $f \in \cap_{i=1}^{\infty} I^n$. But by the intersection theorem we know that $\cap_{i=1}^{\infty} I^n = 0$ so that $f = 0$ and so $R$ is a domain.

*Remark.* One immediately wonders if the converse holds. It does not. Take $R = k[x, y]/(x^2 - y^3)$ and $I = (x, y)$. It is well known that $(x^2 - y^3)$ is irreducible, so $R$ is a domain. Now let $I_n = (x, y)^n(k[x, y]/(x^2 - y^3))$. Note that $\bar{x} \notin (x, y)^2 k[x, y]/(x^2 - y^3)$. This means that in the associated graded ring we have $in(\bar{x}) = \bar{x} + I^2 R$. Now, take $\bar{x}^2$. In the graded ring, we are multiplying an element from $I_1/I_2$ by itself, so we get an element in $I_2/I_3$. So we have $in(x)^2 = \bar{x}^2 + I^3 R$. We want $\bar{x}^2 \in I^3 R$. But $y^3(1 + (x^2 - y^3)) = y^3 + (x^2 - y^3) = x^2 + (x^2 - y^3)$.(As elements of $R$) Since $y^3 \in I^3$ and $1 + (x^2 - y^3) \in R$ we have $x^2 + I^3 R = I^3 R$ which is the zero of this ring. Since $in(x)^2$ has no more components in the associated graded ring, $in(\bar{x})^2 = 0$ in $gr_I R$.

So certain nice properties of rings such as being a domain are not preserved by taking the graded structure. On the otherhand, (with suitable assumptions) nice properties of the graded structure can be reflected back to the original ring. On the other hand, some properties are preserved. As usual, being noetherian usually behaves fairly well.

**Proposition.** *Let $R$ be a Noetherian ring and $I$ an ideal. Then $gr_I R$ is Noetherian.*

Proof: We know that $R$ being Noetherian means that $B_I R$ is Noetherian by Hilbert's Basis theorem. So we know that any quotient is Noetherian. But then by an earlier lemma we have $B_I R/IB_I R \cong gr_I R$ is noetherian as desired.

*Remark.* This finishes our initial discussion on filtrations and the associated graded ring. One of the more interesting elements of the above work is (in my opinion) an example the quote in the abstract. We found that certain questions about filtrations under some suitable assumptions could be related to standard module theoretic questions, which allowed us to leverage results such as the Hilbert Basis Theorem on questions that were (at least not to me) obviously related.

## Completion

Completion is another construction that arises through filtrations. There are two definitions, both equivalent. One is purely algebraic, and the other at least feels more geometric. Throughout, it may be helpful to take as an example of all things we are doing as follows. Let $R = k[x_1, ..., x_n]$ and $\mathfrak{m} = (x_1, ..., x_n)$.

### Algebraic Definition

The algebraic definition is easier to state. Let $\{\mathfrak{m}_i\}$ be a descending sequence of filtrations of a ring $R$. We define the completion of $R$ with respect to the filtration $\{\mathfrak{m}_i\}$ to be

$$\hat{R}_{\{\mathfrak{m}_i\}} := \lim_{\leftarrow} R/\mathfrak{m}_i$$

Where the compatibility maps are the projections $\varphi_i^j(x + \mathfrak{m}_j) = x + \mathfrak{m}_i$. (for $j \geq i$ of course.) Note that $\hat{R}_{\{\mathfrak{m}_i\}} = \{(\bar{x}_i \in \prod_{i=1}^{\infty} R/\mathfrak{m}_i : \varphi_i^j(\bar{x}_j) = \bar{x}_i \text{ for all } j \geq i\}$.

**Example.** Let $p \in \mathbb{Z}$ be any prime number. Then then completion of $\mathbb{Z}$ with respect to the ideal $(p)$ is called the $p$ adic integers and (perhaps unfortunately given how many constructions bear this notion) written $\mathbb{Z}_p$. This ring can be obtained in a variety of ways. For example, one can take an analytic approach by putting a metric on $\mathbb{Q}$ and developing the theory from there. (For those interested Neil Koblitz has a accessible introduction)

This is all very well and good, and is a good definition to work with as it makes certain computations convenient. On the other hand, it is not immediately obvious (to me) why the process is called completion.

### Topological Definition

Let $\{\mathfrak{m}_i\}$ be a descending filtration. We can topologize $R$ as follows. Take $\mathcal{B} := \{x + \mathfrak{m}_i : x \in R, i \geq 0\}$ as a basis for a topology. This works because every element of $R$ is in some basis element, and if $y \in (u + \mathfrak{m}_i) \cap (v + \mathfrak{m}_j)$ then with $i \leq j$ then $y = u + m_i = v + m_j$. As $j \geq i$ we have $m_j \in \mathfrak{m}_i$ so that $u + m_i - m_j = v$ or that $u + \mathfrak{m}_i = v + \mathfrak{m}_i$. That is,

$$v + \mathfrak{m}_j \subseteq v + \mathfrak{m}_i = u + \mathfrak{m}_i$$

So $y \in v + \mathfrak{m}_j \subseteq (u + \mathfrak{m}_i) \cap (v + \mathfrak{m}_j)$. So $\mathcal{B}$ can be used as a basis for a topology in the usual sense. That is, a set $U$ is open if for all $u \in U$ there is some $b \in \mathcal{B}$ such that $u \in b \subseteq U$. Equivalently, the open sets of the topology are all unions of elements of $\mathcal{B}$. So we have a topology, which we call the $\mathfrak{m}_i$ topology. In fact, we have more. I claim that addition and multiplication are continuous in the $\mathfrak{m}_i$ topology. For exampled let $A : R \times R \to R$ be addition. Now let $U$ be an open set and suppose that $a + b = c \in U$. Choose some basis element, $x + \mathfrak{m}_i$ such that $x + \mathfrak{m}_i \subseteq U$ and $c \in x + \mathfrak{m}_i$. Then we have $x + \mathfrak{m}_i = c + \mathfrak{m}_i$. I claim that, $(a + \mathfrak{m}_i) \times (b + \mathfrak{m}_i) \subseteq A^{-1}(U)$. This is because if $u, v \in \mathfrak{m}_i$ then $(a + v) + (b + u) = a + b + (v + u) = c + (v + u) \in c + \mathfrak{m}_i \subseteq U$. This shows that every element of $A^{-1}(U)$ is contained in a open set contained in $A^{-1}(U)$ itself, which means $A^{-1}(U)$ is open and so $A$ is continuous. Multiplication is handled in a similar manner. So we have a topological ring.

**Definition.** Let $\{x_n\}$ be a sequence of elements of $R$. We say that the sequence is Cauchy if for all open sets $U$ there is some $N$ such that for $n, m \geq N$ we have $x_n - x_m \in U$. We say that the sequence converges to 0 if given any neighborhood of 0 there is some $N$ such that for all $n \geq N$ we have $x_n \in U$.

We say that two Cauchy sequences $(x_n)$ and $(y_n)$ are equivalent if $x_n - y_n$ converges to 0. This is easily seen to be an equivalence relation. We now let $\hat{R}$ be the set of all equivalence classes of Cauchy sequences.Now, given Cauchy sequences $x_n, y_n$ we have that $x_n + y_n$ is Cauchy, and so is the product $x_n y_n$. For example, if we wanted to prove this for multiplication it suffices to check for basis elements containing 0 which are of the form $\mathfrak{m}_i$ for some $i$. Then in this case, for large $n, m$ we have $x_n - x_m, y_n - y_m \in \mathfrak{m}_i$. Since $\mathfrak{m}_i$ is an ideal we have

$$x_n y_n - x_m y_m = x_n y_n + x_n y_m - x_n y_m - x_m y_m = x_n(y_n - y_m) + y_m(x_n - x_m) \in \mathfrak{m}_i$$

Addition is handled in a similar manner. Now let $\overline{x_n}$ , $\overline{y_n}$ be two equivalence classes. We define $\overline{x_n} + \overline{y_n} = \overline{x_n + y_n}$ and $\overline{x_n} \cdot \overline{y_n} = \overline{x_n y_n}$. This is well defined up to equivalence as if $x_n \sim u_n, y_n \sim v_n$ then we have $x_n + y_n - u_n - v_n = (x_n - u_n) + (y_n - v_n)$. Now given a neighborhood of 0 say $U$. $U$ contains a basis element containing 0 which is of the form $\mathfrak{m}_i$ for suitable $i$. Then for large enough $n, m$ we have $(x_n - u_n), (y_n - v_n) \in \mathfrak{m}_\mathfrak{i}$. But since the $\mathfrak{m}_i$ are ideals so is the sum. The proof that multiplication is well defined up to equivalence is similar. Use the identity $x_n y_n - v_n u_n = x_n(y_n - v_n) + v_n(x_n - u_n)$. It easily follows that $\hat{R}$ is a ring with unity. Notice that this set up generalizes the usual completion seen in analysis, and so it seems believe able that the ring of equivalence classes with the above operations deserves the name completion.

## Putting the definitions together

It remains to show that the two definitions are the same. Momentarily put $\mathcal{E}$ as the ring of equivalence classes defined above. Define $\varphi : \lim R/\mathfrak{m}_i \to \mathcal{E}$ by

$$\varphi((x_i + \mathfrak{m}_i)) = (x_i)$$

To see that $(x_i)$ is Cauchy, it suffices to check the Cauchy condition on basis elements. Given an basis element containing 0 say $\mathfrak{m}_i$ we have that if $n \geq m \geq i$ then $x_n - x_m \in \mathfrak{m}_m \subseteq \mathfrak{m}_i$ by the condition on the inverse limit. Furthermore, if $(x_i + \mathfrak{m}_i) = (y_i + \mathfrak{m}_i)$ then the sequence $x_i - y_i$ clearly converges to 0 as given a basis element $\mathfrak{m}_k$ then for $i \geq k$ we have $x_i - y_i \in \mathfrak{m}_i \subseteq \mathfrak{m}_k$. So $\varphi$ is well defined on equivalence classes, and is immediately a homomorphism.(For example, $(1 + \mathfrak{m}_i) \mapsto (1)_i$ which is the multiplicative identity) It remains to show that it is a bijection. Now $\varphi((x_i + \mathfrak{m}_i)) = 0$ means that $(x_i)$ is equivalent to the zero equivalence relation. So given $\mathfrak{m}_i$ there is some $N$ such that for all $j \geq N$ we have $x_j \in \mathfrak{m}_i$. But then, $0 \equiv x_j \equiv x_i \mod \mathfrak{m}_i$ so that $(x_i + \mathfrak{m}_i) = (\mathfrak{m}_i)$. So we know that $\varphi$ is an injection. On the otherhand, given a equivalence class represented by $(x_i)$ for each $i$ there is some integer $t_i \geq i$ such that for $m, n \geq t_i$ we have $x_n - x_m \in \mathfrak{m}_i$. Now consider $(x_{t_i})$. I claim that $(x_{t_i})$ is a Cauchy sequence. Consider a basis element $\mathfrak{m}_i$ . Then for $j \geq t_i$ we have $t_j \geq j \geq t_i$ so that if $k \geq j \geq t_i$ we have $x_{t_k} - x_{t_j} \in \mathfrak{m}_i$ . So $(x_{t_i})$ is a Cauchy sequence. I further claim that $(x_{t_i})$ is equivalent to $(x_i)$. To see why, given $\mathfrak{m}_i$ we have that for $j \geq t_i$ that $t_j \geq t_i$ so that $x_{t_j} - x_j \in \mathfrak{m}_i$ so that $(x_{t_j})$ is equivalent to $(x_j)$. Finally, if $j \geq i$ then $t_j \geq t_i$ so that $x_{t_j} - x_{t_i} \in \mathfrak{m}_i$ by construction. So $(x_{t_i} + \mathfrak{m}_i)$ is a element of $\hat{R}$. Hence, $\varphi(x_{t_i} + \mathfrak{m}_i) = (x_{t_i})$ and so $\varphi$ is a surjection as desired.

## Properties of completion

Now that we have the completion we want to know some things about it. Since we are dealing with filtrations it would be nice to obtain a related filtration in the completion, this is not difficult. We can form ideals in the completion, namely $\hat{\mathfrak{m}}_i := \{(x_i) \in \hat{R}_{\{\mathfrak{m}_i\}} : x_j = 0 \text{ for } j \leq i\}$ . Because we are working in a subring of the product, this is an ideal, and is decreasing with $i$. So we obtain a filtration on the new space. How are the filtrations related?

**Proposition.** $\hat{R}_{\{\mathfrak{m}_i\}}/\hat{\mathfrak{m}}_i \cong R/\mathfrak{m}_i$.

Proof: Define $p : \hat{R}_{\{\mathfrak{m}_i\}}/\hat{\mathfrak{m}}_i \to R/\mathfrak{m}_i$ as follows. Let $(x_n) + \hat{\mathfrak{m}}_i$ be given. We define $p((x_n) + \hat{\mathfrak{m}}_i) = x_i + \mathfrak{m}_i$. This is well defined, because if $(x_j) \equiv (y_j) \mod \hat{\mathfrak{m}}_i$ then in particular we have $x_i \equiv y_i \mod \mathfrak{m}_i$. The map is surjective because given $r + \mathfrak{m}_i$ consider the element, $(r + \mathfrak{m}_n)$. We can then map to $\hat{R}_{\{\mathfrak{m}_i\}}/\hat{\mathfrak{m}}_i$ and then project down to get the desired element. Finally, if $p((x_n) + \hat{\mathfrak{m}}_i) = 0$ then $x_i \equiv 0 \mod \mathfrak{m}_i$. But this forces all the $x_j = 0 \mod \mathfrak{m}_j$ for $j \leq i$ so that $(x_n) = 0$ in $\hat{R}_{\{\mathfrak{m}_i\}}/\hat{\mathfrak{m}}_i$. So we have an isomorphism.

*Note.* If $\mathfrak{m}_i = \mathfrak{m}^i$ is our filtration the completion will be written $\hat{R}_\mathfrak{m}$ and $\hat{\mathfrak{m}}_1 := \hat{\mathfrak{m}}$.

**Proposition.** *Suppose that $\mathfrak{m}$ is a maximal ideal, and we take a filtration $\mathfrak{m}_i = \mathfrak{m}^i$. Then $\hat{R}_\mathfrak{m}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$.*

Proof: By the previous proposition, $\hat{\mathfrak{m}}$ is maximal, so it remains to show that it is local. Now take $(x_n) \notin \hat{\mathfrak{m}}$. Then $x_1 \neq 0 \mod \mathfrak{m}$. This means that for $i \geq 1$ we we have $x_i \neq 0 \mod \mathfrak{m}^i$. Now, notice that $R/\mathfrak{m}^i$ is local with maximal ideal $\mathfrak{m}(R/\mathfrak{m}^i)$. Since $x_i \neq 0 \mod \mathfrak{m}$ we have that $x_i \notin \mathfrak{m}(R/\mathfrak{m}^i)$ so that $x_i$ is a unit in $R/\mathfrak{m}^i$. The result will now follow if we can show that $(x_i^{-1})_i$ is a element of the inverse limit. But this is easy as if $j \geq i$ then $x_i \equiv x_j \mod \mathfrak{m}_i$ so that by multiplying by the inverses gives $x_i^{-1} \equiv x_j^{-1} \mod \mathfrak{m}_i$. So we have the result.

We even get more from from the above. Notice, that $R/\mathfrak{m}^i \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^i$ as $R/\mathfrak{m}^i$ is local with maximal ideal $\mathfrak{m}/\mathfrak{m}^i$. So, if we localize at the maximal ideal, $\mathfrak{m}_\mathfrak{m}$ and then complete, we still get back to the same ring as all the factors in the direct limit are the same. We now turn to the classical example.

**Proposition.** *Let $S$ be a ring and $R = S[x_1, ...., x_n]$ and $\mathfrak{m} = (x_1, ..., x_n)$. Then $\hat{R}_\mathfrak{m} \cong S[[x_1, ..., x_n]]$*

Proof: Define $\varphi : k[[x_1, ..., x_n]] \to \hat{R}_\mathfrak{m}$ by $\varphi(f) = (f + \mathfrak{m}^n)_n$. This gives a well defined map into the inverse limit. To see it is injective, note that if $f$ is a non zero constant, then $\varphi(f) \neq 0$. So if $\varphi(f) = 0$ then $f = 0$ or is a not a constant. If $f$ is not constant, let $i$ be the least degree of monomials that appear in $f$. Then $f + \mathfrak{m}^{i+1} \neq \mathfrak{m}^{i+1}$ so $\varphi(f) \neq 0$ and so $\varphi$ is injective. On the other hand,let $f = (f_i + \mathfrak{m}^i)_i$ be an element of the inverse limit. Then we can write $f = (g_i + \mathfrak{m}^i)_i$ where $g_i$ only has monomials of degree $\leq i - 1$. Then, consider the element

$$h = g_1 + (g_2 - g_1) + (g_3 - g_2) + ...$$

$h$ is a well defined element of $S[[x_1, ..., x_n]]$ since $g_{i+1} - g_i$ only contains terms of degree $i$ since $g_{i+1} - g_i = 0 \mod \mathfrak{m}^i$ and $g_i, g_{i+1}$ terms of degree at most $i-1$, and $i$ respectively. In this case we have $g_1 + (g_2 - g_1) + .... \mod \mathfrak{m}^i = g_i + \mathfrak{m}_i$ because for $j > i$ we have $g_j - g_{j-1}$ contains monomials of degree $j - 1 \geq i$ so that these terms vanish. In this case, the rest of the terms telescope to $g_i$. So we have $\varphi(h) = (g_i + \mathfrak{m}^i) = f$ so that $\varphi$ is surjective.

**Corollary.** *If $k$ is a field then $k[[x_1, ..., x_n]]$ is a local ring with maximal ideal $\hat{\mathfrak{m}} = k[[x_1, ..., x_n]](x_1, ...., x_n)$*

Proof: $\mathfrak{m}$ is a maximal in this case, apply the above proposition.

*Remark.* This is a easy proof of the familiar fact that if a power series over say $\mathbb{C}$ has a non-zero constant term, then it can be inverted formally. Of course one can show this by solving systems of equations.

**Definition.** Let $\hat{\phi} : R \to \hat{R}_{\{\mathfrak{m}_i\}}$ be the map that sends $x \mapsto (x + \mathfrak{m}_i)_i$. We call this map the natural map, and we say $R$ is complete with respect to the $\mathfrak{m}_i$ if the map is an isomorphism.

*Remark.* It is immediate that $\ker \hat{\phi} = \cap_{i=1}^\infty \mathfrak{m}_i$

The above is obvious from the definition of $\hat{R}$. We see that the intersection of a filtration of ideals corresponds to the information lost when passing to the completion. So for example, if $R$ is noetherian and a domain, or local no information is lost by Krull's Intersection Theorem.

**Proposition.** *$R$ is Hausdorff as a topological space if and only if $\ker \hat{\phi} = \{0\}$*

Proof: Suppose that $R$ is Hausdorff. If $R = 0$ then were done. Towards a contradiction, suppose that $x \neq 0$ and $x \in \cap_{i \geq 1} \mathfrak{m}_i$. Then as $R$ is Hausdorff and $x \neq 0$ there are disjoint basis elements $y + \mathfrak{m}_i$ and $z + \mathfrak{m}_j$ with $x \in y + \mathfrak{m}_i$ and $0 \in z + \mathfrak{m}_j$. But then we have $y + \mathfrak{m}_i = x + \mathfrak{m}_i$ and $z + \mathfrak{m}_j = \mathfrak{m}_j$. But, $x \in \mathfrak{m}_i$ so $x + \mathfrak{m}_i = \mathfrak{m}_i$. But $\mathfrak{m}_i \cap \mathfrak{m}_j \neq \emptyset$ which is a contradiction. Conversely, suppose that $\ker \hat{\phi} = \{0\}$. Let $x \neq y$ . So $x - y \notin \cap_{i \geq 1} \mathfrak{m}_i$. So there is some $i$ with $x - y \notin \mathfrak{m}_i$. This means that $x + \mathfrak{m}_i$ and $y + \mathfrak{m}_i$ are disjoint which gives us that $R$ is Hausdorff.

*Remark.* Now, up until this point we have more or less used the algebraic definition to do things, but the topological descriptions is also useful. One way we can write a element of $\hat{R}$ is as follows. Set $S_n = \sum_{i=1}^n s_i$ with $s_i \in \mathfrak{m}_i$. Then the sequences $(S_n)$ is a Cauchy sequence because if $m \geq n$ we have $S_m - S_n = \sum_{i=n+1}^m s_i \in \mathfrak{m}_{n+1}$.

**Proposition.** *Let $R$ be complete being respect to $\mathfrak{m}$. Then the elements of $U = \{1 + a : a \in \mathfrak{m}\}$ are units.*

Proof: Identify $R$ with its completion. Given, $1 + a, a \in \mathfrak{m}$ set $S_n = \sum_{i=0}^n (-1)^i a^i$. Then, this is a Cauchy sequence by the above remarks. Now, consider $(1 + a) \sum_{i=0}^\infty (-1)^i a^i$. Using the Cauchy sequence definition, we have that $(1 + a)S_n = \sum_{i=0}^n (-1)^i a^i + \sum_{i=0}^n (-1)^i a^{i+1} = 1 + (-1)^n a^{n+1}$. Now consider the sequence $(1 + (-1)^n a^{n+1}) = (1) + ((-1)^n a^{n+1})$. Since, $(-1)^n a^{n+1})$ is equivalent to 0 as $a \in \mathfrak{m}$. We have that $(1 + (-1)^n a^{n+1}) = 1$. So $1 + a$ is a unit.

**Proposition.** *Let $R$ be a local ring with maximal ideal $\mathfrak{m}$. Then $R[[x_1, ..., x_n]]$ is local with maximal ideal $\mathfrak{m} + (x_1, ..., x_n)$.*

Proof: We can work in $\hat{R}_\mathfrak{m}$ which is complete with respect to $\mathfrak{m}$. Let $f$ be a element outside $\mathfrak{m} + (x_1, ..., x_n)$. Then if $f_0$ is the constant term of $f$ we have that $f_0 \notin \mathfrak{m}$ and so $f_0$ is a unit of $R$. This means that $f_0^{-1} f = 1 + g$ where $g \in (x_1, ..., x_n)$ so that $1 + g$ is a unit by the above proposition. But this means that $f$ is a unit and so $\hat{R}_\mathfrak{m}$ is local with maximal ideal $\mathfrak{m} + (x_1, ..., x_n)$

Finally, one may ask whether the completion behaves nicely in certain ways if the original ring. To this end we develop abit of machinery.This is a second application of convergence type thinking, that also is an example of the running theme. That passing to the associated construction can tell us things about the original ring in suitable circumstances.

**Lemma.** *Let $R$ be a ring and $\mathfrak{m}_i$ a filtration of ideals. Then given $f, g \in R$ we have $in(fg) = in(f)in(g)$ or $in(f)in(g) = 0$ in $grR$.*

Proof: Suppose that $\deg in(f) = p$ and $\deg in(g) = q$. Let $p \geq q$. Now, suppose that $in(f)in(g) \neq 0$. Then we know that $fg + \mathfrak{m}_{p+q+1} \neq 0$ because this is product $in(f)in(f)$. In this case we have that $fg \notin \mathfrak{m}_{p+q+1}$ but, $fg \in \mathfrak{m}_{p+q}$ certainly. Hence, $in(fg)$ is the image of $fg$ in $\mathfrak{m}_{p+q}/\mathfrak{m}_{p+q+1}$ which is exactly what we want.

**Proposition.** *Let $R$ be complete with respect to a filtration $\{\mathfrak{m}_i\}$. Suppose that $I$ is an ideal. If $a_1, ..., a_s \in I$ and $in(a_1), ..., in(a_s)$ generate $in(I)$ then $a_1, ...., a_s$ generate $I$.*

Proof: Since $R$ is complete with respect to the filtration $\hat{\phi}$ has trivial kernel. This means that $\cap_{i\geq 1}\mathfrak{m}_i = \{0\}$. So we may choose $d$ so that $a_i \notin \mathfrak{m}_d$ for $1 \leq i \leq s$. Now, identify $R$ with its completion. Let $f \in I$. Then we can regard $in(f)$ in $gr_{\{\mathfrak{m}_i\}}R$. Suppose that $f$ has degree $e$. (Meaning $in(f)$ has degree $e$ in the graded ring) So $in(f) = f + \mathfrak{m}_{e+1}$ and we can write $in(f) = \sum_{j=1}^{s} G_j in(a_j)$ where $G_j$ are homogenous of degree $\deg(in(f)) - \deg(in(a_j))$. We can assume that the $G_j in(a_j)$ are non zero and so choose some $g_j \in R$ such that $in(g_j) = G_j$. In this case we know that $in(g_j)in(a_j) = in(g_j a_j)$ by the above lemma so that $f - \sum_{j=1}^{s} in(g_j)in(a_j) \in \mathfrak{m}_{e+1}$. Now look at $f - \sum_{j=1}^{s} in(g_j)in(a_j)$ in $m_{e+1}$ and apply the same logic. At each step, we move up a degree so that eventually we obtain an element $g$ such that $f - g \in f - g \in \mathfrak{m}_{d+1}$. Furthermore, in this situation, we have that the elements $g_j$ computed above must have degree at most $e - d > 0$ and so must live in $\mathfrak{m}_{e-d}$ . Now apply the same procedure to $f - \sum_{j=1}^{s} g_j a_j$ which has degree at least $d+1$ to obtain $f - \sum_{j=1}^{s} g_j a_j - \sum_{j=1}^{s} g_j^1 a_j \in \mathfrak{m}_{d+2}$ such that $g_j^1 \in \mathfrak{m}_{e-d+1}$ for each $j$. Now iterate this process to obtain a sequence $g_j^i$ for $1 \leq j \leq s$ with $g_j^i \in \mathfrak{m}_{e-d+i}$ and $f - \sum_{k=0}^{i}\sum_{j=1}^{s} g_j^k a_j \in \mathfrak{m}_{d+i+1}$ . Now for fixed $j$ we have that the partial sums of $\{g_j^i\}_i$ is a Cauchy sequence. Furthermore we know that the sum of the sequences formed by the partial sums of the $g_j^i a_j$ are converging to $f$. This means that if $A_j$ is the limit of the Cauchy sequence $g_j^i$ then $f = \sum_{j=1}^{s} A_j a_j$ and so the $a_i$ generate $I$ as desired.

We now have the technology to prove that in the Noetherian case $\hat{R}$ inherits certain nice properties from $R$.

**Theorem.** *Let $R$ be Noetherian and $\mathfrak{m}$ an ideal then we have*

- $\hat{R}_{\mathfrak{m}}$ is noetherian.
- $\hat{R}_{\mathfrak{m}}/\mathfrak{m}^j\hat{R}_{\mathfrak{m}} \cong R/\mathfrak{m}^j$ and so $\hat{R}_{\mathfrak{m}}$ is complete with respect to $\mathfrak{m}\hat{R}_{\mathfrak{m}}$.
- $gr_{\mathfrak{m}}R \cong gr_{\mathfrak{m}\hat{R}_{\mathfrak{m}}}\hat{R}_{\mathfrak{m}}$.

Proof: If we look back at our earlier work we see that $gr_{\mathfrak{m}}R \cong gr_{\hat{\mathfrak{m}}_i}\hat{R}$ because $\hat{R}_{\{\mathfrak{m}_i\}}/\hat{\mathfrak{m}}_i \cong R/\mathfrak{m}_i$. Furthermore, since $R$ is noetherian we know that $gr_{\mathfrak{m}}R$ is noetherian by an earlier remark. Now let $I$ be an ideal of $\hat{R}_{\mathfrak{m}}$. We know that $\hat{R}_{\mathfrak{m}}$ is complete with respect to the $\hat{\mathfrak{m}}_i$. So we know that $gr_{\hat{\mathfrak{m}}_i}\hat{R}$ is noetherian, and in $gr_{\hat{\mathfrak{m}}}\hat{R}$ we have that $in(I)$ is finitely generated. But then by the above proposition, we know that $I$ is finitely generated in $\hat{R}_{\mathfrak{m}}$ which means that $\hat{R}_{\mathfrak{m}}$ is noetherian as desired. First, notice that $\hat{\mathfrak{m}}_n := \{(x_i + \mathfrak{m}^i) : x_i \in \mathfrak{m}^i \text{ for } i \leq n\}$. So consider the initial ideal $in(\hat{\mathfrak{m}}_n)$ inside $gr_{\hat{\mathfrak{m}}}\hat{R}_{\mathfrak{m}}$ which is all elements with degree $k \geq n$. On the otherhand, $in(\mathfrak{m}^n\hat{R}_{\mathfrak{m}})$ is again all elements in $gr_{\hat{\mathfrak{m}}}\hat{R}_{\mathfrak{m}}$ of degree $k \geq n$. Since we know that $\hat{R}_{\mathfrak{m}}$ is complete with respect to the $\hat{\mathfrak{m}}_i$ can apply our earlier results. Since $\hat{R}_{\mathfrak{m}}$ is noetherian, we know that $gr_{\hat{m}}\hat{R}_{\mathfrak{m}}$ is noetherian, and so $in(\hat{\mathfrak{m}}_n) = in(\mathfrak{m}^n\hat{R}_{\mathfrak{m}})$ is finitely generated by some elements. But then the proposition above kicks in and we have that the same elements generate $\hat{\mathfrak{m}}_n$ and $\mathfrak{m}^n\hat{R}_{\mathfrak{m}}$. So they are the same. That is, $\mathfrak{m}^n\hat{R}_{\mathfrak{m}} = \hat{\mathfrak{m}}_n$ and so by our earlier work we know that $R/\mathfrak{m}^n \cong \hat{R}_{\mathfrak{m}}/\hat{\mathfrak{m}}_n = \hat{R}_{\mathfrak{m}}/\mathfrak{m}^n\hat{R}_{\mathfrak{m}}$. This gives the second bullet point. Finally, since $\mathfrak{m}^n\hat{R}_{\mathfrak{m}} = \hat{\mathfrak{m}}_n$ we have that $gr_{\mathfrak{m}^n\hat{R}_{\mathfrak{m}}}\hat{R}_{\mathfrak{m}} = gr_{\hat{\mathfrak{m}}}\hat{R}_{\mathfrak{m}} \cong gr_{\mathfrak{m}}R$.

*Remark.* So we see that like taking the graded ring, completion behaves well with respect to the Noetherian hypothesis.

*Looking Back.* We have seen that when given a filtration of ideals, or perhaps more generally of submodules there are a number of ways one can construct related objects that can provide valuable information. The graded constructions provide information about filtration of ideals and submodules. We also looked at completion and rings complete with respect to some filtration of ideals. In this case we saw that completion inherits the Noetherian hypothesis from the original ring, and that questions about finite generation of ideals in complete rings can sometimes be solved by passing to the associated graded ring.

*Looking Forward.* I would be remiss to speak about completion and not mention certain facts that we did not have time to discuss.One important property is that $\hat{R}_{\mathfrak{m}}$ is flat as a $R$ module. In general one can define the completion of modules as an inverse limit. Namely $\hat{M} := \varprojlim M/\mathfrak{m}^j M$ for some ideal $\mathfrak{m}$. If we have a Noetherian ring and a finitely generated $R$ module $M$ then completion of modules can be related to the tensor product and in fact $\hat{M}_{\mathfrak{m}} \cong \hat{R}_{\mathfrak{m}} \otimes_R M$. Complete rings also satisfy Hensel's lemma which is very useful. Hensel's Lemma is as follows. Let $R$ be a noetherian ring and suppose that $\mathfrak{m}$ is a proper ideal with $R$ complete with respect to $\mathfrak{m}$. Let $f \in R[x]$. If $f(a) \equiv 0 \mod f'(a)^2\mathfrak{m}$ then there is some $b$ with $f(b) = 0$ and $b \equiv a \mod f'(a)\mathfrak{m}$. That is, if $a$ is a approximate root of $f$ with respect to $\mathfrak{m}$

there is some actual root lying close to $a$ mod $\mathfrak{m}$. Furthermore, if $f'(a)$ is a non zero divisor then $b$ is unique. Hensel's Lemma can be thought of an algebraic version of Newton's method. For example, Hensel's Lemma has a number of applications in the theory of $p$ adic numbers. Finally, in the case of finitely generated modules over Noetherian local rings the theory of completion behaves very well. Specifically, if $M, N$ are finitely generated modules over a Noetherian Local ring then $\hat{M} \cong \hat{N} \Rightarrow M \cong N$. So in this case, by looking at the completion, which has nice properties as we have seen above, one can answer whether two modules are isomorphic.

*References.*


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