MATH 817 ASSIGNMENT 2 SOLUTIONS

(1) (a) $G'' \lhd G$ so let G act on G'' by conjugation. The kernel of the action is $C_G(G'')$, and so $G/C_G(G'')$ is isomorphic to a subgroup of $\mathrm{Aut}(G'')$.
But G'' is cyclic by assumption, so the only automorphisms consist of raising the elements of G'' to powers relatively prime to $|G''|$ in the finite case and raising to powers ± 1 in the infinite case. In particular all such automorphisms commute, so $\mathrm{Aut}(G'')$ is abelian. Thus $G/C_G(G'')$ is abelian.
Thus (Isaacs Theorem 2.10) $G' \subseteq C_G(G'')$. So each element of G'' commutes with each element of G'. Thus $G'' \subseteq Z(G')$.
(b) Write $C'/G'' = \langle xG'' \rangle$. Write $G'' = \langle y \rangle$. Then G' is generated by x and y and $y \in G'' \subseteq Z(G')$ so $xy = yx$. Thus G' is abelian and so $G'' = 1$.

(2) Note that $[M, N] = 1$ since $M \lhd G$ and $N \lhd G$, so any commutator is in both M and N but $M \cap N = 1$.
Suppose $\phi : M \cong N$. Let $D = \{m\phi(m) : m \in M\}$.
D is a subgroup of G since $1 = 1\phi(1) \in D$ and if $m, n \in M$, then $m\phi(m)n\phi(n) = mn\phi(mn)$ since $[M, N] = 1$, which also gives that $(m\phi(m))^{-1} = \phi(m^{-1})m^{-1} = m^{-1}\phi(m^{-1}) \in D$.
Suppose $x \in D \cap M$ then $x = m\phi(m)$ for some $m \in M$. So $xm^{-1} \in M$ and $xm^{-1} = \phi(m) \in N$ but $M \cap N = 1$ which implies $\phi(m) = 1$ and so $m = 1$ and so $x = 1$. Thus $D \cap M = 1$. Similarly $D \cap N = 1$.
We always have $MD \subseteq G$. Take any $g \in G$. $G = MN$ so write $g = mn$, $m \in M$, $n \in N$. Then $g = (m\phi^{-1}(n^{-1}))(\phi^{-1}(n)n) \in MD$. So $G = MD$. Similarly $G = DN$.
Thus D is a diagonal subgroup of G.
Now suppose D is a diagonal subgroup of G.
Take $m \in M$. Since $N \cap D = 1$ and $M \subseteq G = DN$ we can uniquely write $m = dn$ with $d \in D$, $n \in N$. Thus we can define $\phi : M \to N$ by $\phi(m) = n^{-1}$.
Take $m, m' \in M$. Write $m = dn, m' = d'n'$, with $d, d' \in D$, $n, n' \in N$. Then $\phi(m)\phi(m') = n^{-1}(n')^{-1}$. Also

$$\begin{aligned} mm' &= dnm' \\ &= dm'n \text{ since } [M, N] = 1 \\ &= dd'n'n \end{aligned}$$

so $\phi(mm') = (n'n)^{-1} = n^{-1}n'^{-1}$. Thus ϕ is a homomorphism.
Since $M \cap D = 1$ and $N \subseteq G = DM$ we can uniquely write any $n \in N$ as $n = dm$ for some $m \in M$, $d \in D$ in which case $\phi(m) = n^{-1}$. Thus ϕ is onto. Further if $\phi(m') = n^{-1}$ for some $m' \in M$ then $m' = d'n$ for some $d' \in D$, and so $n = d'^{-1}m'$ so by uniqueness $m' = m$. Thus ϕ is one-to-one. Hence ϕ gives an isomorphism between M and N.

(3) Let P be a finite nonabelian p-group. By the class formula $Z(P)$ has size divisible by p since all other terms of the formula do. Thus $Z(P) \neq 1$. Suppose $H \subseteq P$ is a subgroup with the property that $HZ(P) = P$ and $H \cap Z(P) = 1$. Then $P = Z(P) \rtimes H$

where the implicit action is the conjugation action of H on $Z(P)$. But $Z(P)$ is the center of P so this action is trivial. Thus $P = Z(P) \dot{\times} H$. So $H \lhd P$. Then (Isaacs Theorem 5.21) $H \cap Z(P) \neq 1$ which is a contradiction. Thus P cannot split over its center.

(4) (a) Let G and G' exist with these properties. Let $\overline{H}$, $\overline{H}'$, $\overline{N}$, $\overline{N}'$ be the copies of H and N in G and G' as given in the theorem. Let $\phi : \overline{H} \to \overline{H}'$ and $\psi : \overline{N} \to \overline{N}'$ be the isomorphisms induced by the isomorphisms with H and N in each case. Consider the map $\theta : G = \overline{HN} \to G' = \overline{H}'\overline{N}'$ given by $\theta(\bar{h}\bar{n}) = \phi(\bar{h})\psi(\bar{n})$. This map is well defined by conditions (a) and (b) for G. Write $\bar{h}' = \phi(\bar{h})$ and $\bar{n}' = \phi(\bar{n})$ and likewise.

Check θ is a homomorphism

$$\begin{aligned}\theta(\bar{h}\bar{n}\bar{h_1}\bar{n_1}) &= \theta(\bar{h}\bar{h_1}\bar{n}^{\bar{h_1}}\bar{n_1}) \\ &= \theta(\bar{h}\bar{h_1}\overline{(n^{h_1})}\bar{n_1}) \\ &= \bar{h}'\bar{h_1}'\overline{(n^{h_1})}'\bar{n_1}' \\ &= \bar{h}'\bar{h_1}'(\bar{n}')^{\bar{h_1}'}\bar{n_1}' \\ &= \bar{h}'\bar{n}'\bar{h_1}'\bar{n_1}' \\ &= \theta(\bar{h}\bar{n})\theta(\bar{h_1}\bar{n_1})\end{aligned}$$

θ is onto by (a) for G' and one-to-one by (b) for G'.

(b) Suppose we have N and H and $\phi : H \to \mathrm{Aut}(N)$. Then H acts via automorphisms on N and so we can form G as in Isaacs Theorem 7.17, which is unique by the previous part. We can also form $N \rtimes H$ as discussed in class as the set $N \times H$ with the multiplication

$$(x_1, h_1)(x_2, h_2) = (x_1\phi(h_1)(x_2), h_1h_2)$$

To show $N \rtimes H \cong G$ it suffices by the preivous part to show that $N \rtimes H$ satisfies (a)-(d) of Isaacs' Theorem 7.17. Let $\overline{N} = \{(x, 1) | x \in N\}$, let $\overline{H} = \{(1, h^{-1}) | h \in H\}$ (with an inverse because of Isaacs' backwards conventions, but isomorphic none-the-less).

Then $\overline{HN} = N \rtimes H$ and $\overline{N} \cap \overline{H} = 1$. To check $\overline{N} \lhd N \rtimes H$ take $(x, 1) \in \overline{N}$ and $(n, h) \in N \rtimes H$. Then $(n, h)^{-1} = (\phi(h^{-1})(n^{-1}), h^{-1})$ so

$$\begin{aligned}(x, 1)^{(n,h)} &= (\phi(h^{-1})(n^{-1}), h^{-1})(x, 1)(n, h) \\ &= (\phi(h^{-1})(n^{-1}x), h^{-1})(n, h) \\ &= (\phi(h^{-1})(n^{-1}xn), 1) \\ &\in \overline{N}\end{aligned}$$

To check (d) take $n \in N$ and $h \in H$. Then $\overline{n^h} = \overline{\phi(h)(n)} = (\phi(h)(n), 1)$ and $\overline{n}^{\overline{h}} = (n, 1)^{(1,h^{-1})} = (1, h)(n, 1)(1, h^{-1}) = (\phi(h)(n), 1)$.

(5) P is cyclic of order p so P is generated by a product of disjoint p-cycles, write $P = \langle a \rangle$. The orbits of P are these p-cycles along with an orbit of size 1 for each element that P fixes. We also have that $x^{-1}ax = a^n$ for some $1 < n < p$.

Suppose i and j are both fixed by x and are in the same orbit of a. So i and j are in a p-cycle of a. So we can find $1 \le k < p$ such that a^k maps i to j. Then $x^{-1}a^kx = a^{nk}$ also maps i to j. But, since p is prime, a nontrivial power of a p-cycle is also p-cycle and differs on each point of the underlying set. This implies $nk \equiv k \mod p$. Thus $(n-1)k \equiv 0 \mod p$ which is impossible since $1 \le k < p$ and $1 < n < p$. Thus x can fix at most one point of each orbit of P.

(6) Certainly powers of a common element commute. For the other direction suppose F is a free group. Then elements of F are reduced words on some underlying set. Suppose $a, b \in F$ commute, are not powers of a common element, and a and b are of minimim total length so that this occurs.

Then $[a, b] = a^{-1}b^{-1}ab = 1$ so the word $a^{-1}b^{-1}ab$ must not be reduced. So either a ends with some letter x and b begins with x^{-1} or similarly for a^{-1} and b^{-1}, or similarly for b^{-1} and a. Pick one such letter x and write

$$a = x^k w x^\ell \qquad b = x^m w' x^n$$

where w and w' neither begin nor end with any power of x. Then

$$1 = [a, b] = x^{-\ell} w^{-1} x^{-k-n} (w')^{-1} x^{k-m} w x^{\ell+m} w' x^n$$

For the first w^{-1} to reduce further, at any stage of the reduction, we must have that $x^{-k-n} = 1$ so $k = -n$. For the last w' to reduce further, at any stage of the reduction we must have that $x^{\ell+m} = 1$ so $\ell = -m$. Thus

$$1 = [a, b] = x^{-\ell} w^{-1} (w')^{-1} x^{k-m} w w' x^n$$

There are now two cases. Either $k = m$ or $ww' = 1$. Suppose $k - m$, so $k = m = -n = -\ell$ and thus

$$1 = [a, b] = x^{-n} [w, w'] x^n$$

Hence $[w, w'] = 1$ and so by minimality of the pair (a, b) we must have that $w = y^s$, $w' = y^t$ for some $y \in F$. This gives

$$a = x^{-n} y^s x^n = (y^{x^n})^s \qquad b = x^{-n} y^t x^n = (y^{x^n})^t$$

so a and b are powers of a common element contradiction.

Now suppose $ww' = 1$. So

$$a = x^k w x^\ell \qquad b = x^{-\ell} w^{-1} x^{-k} = a^{-1}$$

which is again a contradiciton. Thus commuting elements must be powers of a common element.