MATH 821, Spring 2013, Lecture 17

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Outline: Coproduct in $\Lambda(\mathbf{x})$, $\Lambda(\mathbf{x})$ is a Hopf algebra, other bases for $\Lambda(\mathbf{x})$.

1 Δ on Symmetric Functions

The ring of symmetric functions $\Lambda(\mathbf{x})$ has a coproduct. First, we need to understand what $\Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{x})$ is.

Lemma 1. *We have a ring homomorphism $R(\boldsymbol{x}) \otimes R(\boldsymbol{x}) \to R(\boldsymbol{x}, \boldsymbol{y})$ defined by $f(\boldsymbol{x}) \otimes g(\boldsymbol{x}) \mapsto f(\boldsymbol{x})g(\boldsymbol{y})$, where $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$.*

This gives a map $\Lambda(\boldsymbol{x}) \otimes \Lambda(\boldsymbol{x}) \to R(\boldsymbol{x}, \boldsymbol{y})^{S_\infty \times S_\infty}$, where $R(\boldsymbol{x}, \boldsymbol{y})^{S_\infty \times S_\infty}$ is the set of elements of $R(\boldsymbol{x}, \boldsymbol{y})$ which are invariant under symmetric action in the x variables and in the y variables. Then,

(a) This map is an isomorphism.

(b) We have an inclusion as follows:

$$\begin{tikzcd}
\Lambda(\boldsymbol{x}, \boldsymbol{y}) \arrow[r, hook] \arrow[r, bend right, "\iota"'] & R(\boldsymbol{x}, \boldsymbol{y})^{S_\infty \times S_\infty} \cong \Lambda(\boldsymbol{x}) \otimes \Lambda(\boldsymbol{x})
\end{tikzcd}$$

Call this injection ι.

Proof. Part (a) is immediate as it is just two different ways of looking at the same space, while part (b) follows from (a). $\square$

This gives us the comultiplication on symmetric functions:

Definition. Define the coproduct on $\Lambda(\mathbf{x})$ as $\Delta : \Lambda(\mathbf{x}) \to \Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{x})$, mapping $f(x_1, x_2, \ldots)$ to $\iota \circ f(x_1, x_2, \ldots, y_1, y_2, \ldots)$.

Example. Compute $\Delta(m_{(3,1)})$.

Recall that $m_{(3,1)}(\mathbf{x}) = x_1^3x_2 + x_1x_2^3 + x_1^3x_3 + x_1x_3^3 + \cdots$. So by the definition of the coproduct, $\Delta(m_{(3,1)})$ contains the following items:

- $x_1^3x_2 + x_1x_2^3 + \cdots$: both parts x-variables
- $x_1^3y_1 + x_2^3y_1 + \cdots$: cubed part is an x-variable
- $x_1y_1^3 + x_2y_1^3 + \cdots$: cubed part is a y-variable
- $y_1^3y_2 + y_1y_2^3 + \cdots$: both parts y-variables

Therefore, we have

$$\Delta(m_{(3,1)}) = m_{(3,1)}(\mathbf{x}) + m_{(3)}(\mathbf{x})m_{(1)}(\mathbf{y}) + m_{(1)}(\mathbf{x})m_{(3)}(\mathbf{y}) + m_{(3,1)}(\mathbf{y}).$$

The last step is to apply the map ι to send this to $\Lambda(\mathbf{x}) \otimes \Lambda(\mathbf{x})$. This results in the coproduct

$$\Delta(m_{(3,1)}) = m_{(3,1)} \otimes \mathbb{1} + m_{(3)} \otimes m_{(1)} + m_{(1)} \otimes m_{(3)} + \mathbb{1} \otimes m_{(3,1)}.$$

Here we see that the coproduct just takes some parts of the partitions and assigns them x-variables, assigns y-variables to the remaining parts, does this for all possible groups of parts of the partition, and then writes the sum as tensors. This is the statement of the next proposition:

Proposition 2. *Let λ be a partition. Then $\Delta(m_\lambda) = \sum_{\substack{(\mu,\nu) \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$ where $\mu \sqcup \nu$ is the partition whose parts are the multiset union of the parts of μ and ν.*

Proof. Take a monomial in $\Delta(m_\lambda)$, viewed in $\Lambda(\mathbf{x},\mathbf{y})$. Let μ be the partition of the powers of elements of $\mathbf{x}$ and let ν be the partitions of the powers of elements of $\mathbf{y}$.

By the symmetric group action, all other permutations leading to μ and ν also appear in $\Delta(m_\lambda)$. Hence $m_\mu \otimes m_\nu$ appears in $\Delta(m_\lambda)$. Furthermore, its coefficient is 1 by considering any possible monomial.

All μ, ν with $\mu \sqcup \nu = \lambda$ when viewed as $m_\mu(\mathbf{x})m_\nu(\mathbf{y})$ in $\Lambda(\mathbf{x},\mathbf{y})$ are in $\Delta(m_\lambda)$, so $\Delta(m_\lambda) = \sum_{\substack{(\mu,\nu) \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$. $\square$

To check that this coproduct indeed works, we will prove that Δ is coassociative.

Lemma 3. *Δ on $\Lambda(\boldsymbol{x})$ is coassociative.*

Proof. Let's take $(\Delta \otimes id)\Delta(f)$ for some $f \in \Lambda(\mathbf{x})$. Then we have

$$\begin{aligned}(\Delta \otimes id)\Delta(f) &= (\Delta \otimes id) f(x_1, x_2, \ldots, z_1, z_2, \ldots) \\ &= f(x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots) \\ &\quad \text{(applying } \Delta \text{ to } x\text{'s and } id \text{ to } z\text{'s)}\end{aligned}$$

If we instead compute $(id \otimes \Delta)\Delta(f)$, we get

$$\begin{aligned}(id \otimes \Delta)\Delta(f) &= (id \otimes \Delta) f(x_1, x_2, \ldots, y_1, y_2, \ldots) \\ &= f(x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots) \\ &\quad \text{(applying } id \text{ to } x\text{'s and } \Delta \text{ to } y\text{'s)}\end{aligned}$$

Therefore, Δ is coassociative. $\square$

Note that the variables are indistinguishable because of the action of S_n so it does not matter whether we split on the left or on the right. Another way to prove coassociativity is to observe that multiset union is associative, and then to prove it using the previous proposition by looking at basis elements.

Now, we will show that $\Lambda(\mathbf{x})$ is in fact a Hopf algebra:

Proposition 4. *The ring of symmetric functions $\Lambda(\boldsymbol{x})$ is a graded, connected finite type Hopf algebra.*

Proof. $\Lambda(\mathbf{x})$ is graded by degree: $\Lambda_i(\mathbf{x}) = \{f \in \Lambda(\mathbf{x}) : f \text{ is homogeneous of degree } i\}$. Multiplication in $\Lambda(\mathbf{x})$ is just multiplication of power series; this respects the grading since $\Lambda_i \cdot \Lambda_j \subseteq \Lambda_{i+j}$.

Also, $\Lambda(\mathbf{x})$ is connected because the degree 0 elements are just the constants.

Comultiplication is an algebra homomorphism because it is a composition of algebra homomorphisms.

Since $\Lambda(\mathbf{x})$ is graded and connected, its counit map is ε defined as $\varepsilon|_{\Lambda_0} = id|_{\Lambda_0 = \Bbbk}$ on the degree 0 elements, and $\varepsilon|_{\oplus_{n=1}^{\infty}\Lambda_n} = 0$ on the rest. We already checked before that this will work in the graded, connected case.

Therefore $\Lambda(\mathbf{x})$ is a bialgebra. Since $\Lambda(\mathbf{x})$ is also graded and connected, then it is a Hopf algebra. $\square$

2 Other Bases

Part of the fun of symmetric functions is the interplay of different bases.

Definition. Given $n \in \mathbb{Z}_{>0}$, the *power sum symmetric function indexed by* n is

$$p_n = x_1^n + x_2^n + x_3^n + \cdots = m_{(n)}.$$

By convention, $p_0 = 1$. Futhermore, for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$.

Example. Compute $p_{(3,1)}$.

$$\begin{aligned} p_{(3,1)} &= p_3 p_1 \\ &= (x_1^3 + x_2^3 + x_3^3 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\ &= (x_1^3 x_2 + x_1^3 x_3 + x_1 x_2^3 + \cdots) \quad \text{(different variables from each part)} \\ &\quad + (x_1^4 + x_2^4 + x_3^4 + \cdots) \qquad \text{(same variable from each part)} \\ &= m_{(3,1)} + m_{(4)} \end{aligned}$$

Example. Compute $p_{(1,1,1)}$.

$$\begin{aligned} p_{(1,1,1)} &= p_1^3 \\ &= (x_1 + x_2 + x_3 + \cdots)^3 \\ &= (x_1^3 + x_2^3 + \cdots) + 3(x_1^2 x_2 + x_1 x_2^2 + \cdots) + 6(x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots) \\ &= m_{(3)} + 3m_{(2,1)} + 6m_{(1,1,1)} \end{aligned}$$

Definition. Given $n \in \mathbb{Z}_{>0}$, the *elementary symmetric function indexed by n* is

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = m_{\underbrace{(1,1,\ldots,1)}_{n}}.$$

By convention, $e_0 = 1$. We also extend multiplicatively to partitions: for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$.

Example. Compute $e_{(3,1)}$.

$$\begin{aligned}
e_{(3,1)} &= e_3 e_1 \\
&= m_{(1,1,1)} m_{(1)} \\
&= (x_1x_2x_3 + x_1x_2x_4 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\
&= (x_1^2x_2x_3 + x_1x_2^2x_3 + \cdots) + 4(x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots) \\
&= m_{(2,1,1)} + 4m_{(1,1,1,1)}.
\end{aligned}$$

Observe that we get the monomial symmetric function for the partition $(2,1,1)$, which is the conjugate of $(3,1)$.

Definition. Given $n \in \mathbb{Z}_{>0}$, the *complete homogeneous symmetric function indexed by n* is

$$h_n = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\lambda \text{ a partition of } n} m_\lambda$$

By convention, $h_0 = 1$. Also extend multiplicatively: for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$.

Example. Compute $h_{(3,1)}$.

$$\begin{aligned}
h_{(3,1)} &= h_3 h_1 \\
&= (x_1^3 + x_2^3 + \cdots \\
&\quad + x_1^2x_2 + x_1x_2^2 + \cdots \\
&\quad + x_1x_2x_3 + x_1x_2x_4 + \cdots) \cdot (x_1 + x_2 + x_3 + \cdots) \\
&= (x_1^4 + x_2^4 + \cdots) + 2(x_1^3x_2 + x_1x_2^3 + \cdots) \\
&\quad + 2(x_1^2x_2^2 + x_1^2x_3^2 + \cdots) + 3(x_1^2x_2x_3 + x_1^2x_2x_4 + \cdots) \\
&\quad + 4(x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots) \\
&= m_{(4)} + 2m_{(3,1)} + 2m_{(2,2)} + 3m_{(2,1,1)} + 4m_{(1,1,1,1)}
\end{aligned}$$

Now we want to check that these are bases. We will now prove this fact for the power sum and elementary symmetric functions. The general strategy will be to show that the transformation matrices from the monomial basis functions are triangular and invertible. To make this straightforward we need the following definition:

Definition. Suppose λ and μ are partitions of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$. Then write $\lambda \trianglerighteq \mu$, read as '$\lambda$ *dominates* μ', or 'λ *is larger than* μ', if $\lambda_1 + \lambda_2 + \cdots + \lambda_j \geq \mu_1 + \mu_2 + \cdots + \mu_j$ for all $j = 1, 2, \ldots, n$, where $\lambda_i = 0$ for $i > l$ and $\mu_i = 0$ for $i > t$.

Example. Given $\lambda = (5, 2, 2, 1)$ and $\mu = (3, 3, 2, 2)$, it is easy to check that $\lambda \trianglerighteq \mu$:

$$\begin{gathered} 5 \geq 3 \\ 5 + 2 \geq 3 + 3 \\ 5 + 2 + 2 \geq 3 + 3 + 2 \\ 5 + 2 + 2 + 1 \geq 3 + 3 + 2 + 2 \end{gathered}$$

Proposition 5. *The set $\{e_\lambda\}$, where λ runs over all partitions, is a basis for $\Lambda(\boldsymbol{x})$. If char$(\Bbbk) = 0$, then $\{p_\lambda\}$, λ running over all partitions, is also a basis for $\Lambda(\boldsymbol{x})$.*

Proof. In each case, we restrict to showing that these two sets, where we let λ run over all partitions of n, form a basis for Λ_n.

Part 1: First, for p_λ, assume char$(\Bbbk) = 0$. We claim that:

$$p_\lambda = \sum_\mu b_{\lambda,\mu} m_\mu$$

where $b_{\lambda,\mu}$ is the number of ways to partition the parts of λ into blocks, where the sums of the blocks give μ.

To see this, let $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$. For any monomial in this expression, we need to know which p_{λ_i} contributes the same variable, i.e. we need to know a partition of the parts of λ into blocks, each corresponding to a variable in the monomial. Such a variable appears in the monomial to the power of the sum of the parts of λ in the block, so this implies the result.

This claim gives us a transformation matrix from the m_λ's to the p_λ's. The next step is showing that if $\mu \not\trianglerighteq \lambda$, then $b_{\lambda,\mu} = 0$. This will imply that this matrix is triangular.

So assume that $\mu \not\trianglerighteq \lambda$. Then there is a minimal i such that $\lambda_1 + \lambda_2 + \cdots + \lambda_i > \mu_1 + \mu_2 + \cdots + \mu_i$, while $\lambda_1 + \cdots + \lambda_j \leq \mu_1 + \cdots + \mu_j$ for $j < i$.

Consider now λ_i, and which block it goes into. It cannot go into the block which becomes μ_j for $j \leq i$ by the preceding inequalities, while it cannot go into a block which becomes μ_j for $j > i$ because $\mu_j < \mu_i$ which is already too small. Therefore there is no such way of building μ from λ, and so $b_{\lambda,\mu} = 0$.

Now, we can write $p_\lambda = \sum_{\mu \trianglerighteq \lambda} b_{\lambda,\mu} m_\mu$, giving the desired triangular transformation matrix. We just have to check that no entry on the diagonal is 0. But $b_{\lambda,\lambda}$ is nonzero because we can just take the parts of λ as blocks (of size 1). Note that $b_{\lambda,\lambda}$ is not necessarily 1 because of multiplicities arising from repeated parts – for instance in $p_{(1,1,1,1)}$ we get a $6m_{(1,1,1,1)}$ term. But because the characteristic of the field $\mathbb{k}$ is zero, $b_{\lambda,\lambda} \neq 0$ for any λ.

Thus this transformation matrix is invertible as well. Since $\{m_\lambda\}$ is a basis, then so is $\{p_\lambda\}$.

Part 2: Now we prove that the e_λ's are also a basis by proving a claim similar to above:

$$e_\lambda = \sum_\mu a_{\lambda,\mu} m_\mu$$

where $a_{\lambda,\mu}$ is the number of 0-1 matrices with row sums equal to λ and column sums equal to μ.

To see this, expand $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$ and consider a monomial $x_{j_1} x_{j_2} \cdots x_{j_{\lambda_i}}$ in the e_{λ_i} factor. If I choose this monomial when expanding out, then I am contributing $x_{j_1}, \ldots, x_{j_{\lambda_i}}$ to μ, corresponding to placing 1's on the ith row of the matrix at columns $j_1, j_2, \ldots, j_{\lambda_i}$:

$$\begin{array}{cc} & \begin{matrix} j_1 & j_2 & \cdots & j_{\lambda_i} & \cdots \end{matrix} \\ \text{row } i: & \begin{bmatrix} \vdots & \vdots & & \vdots & \\ 1 & 1 & \cdots & 1 & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix} \\ & \begin{matrix} \mu_1 & \mu_2 & \cdots & \text{(column sums)} \end{matrix} \end{array}$$

For example, for $\lambda = (3,1,1)$ and $\mu = (2,2,1)$, we have two such 0-1 matrices, so $a_{(3,1,1),(2,2,1)} = 2$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

Now observe that if we push all the 1's in the matrix to the left, then we get the Ferrers diagram of λ. Suppose that $\tilde{\lambda} \not\trianglerighteq \mu$. Then we claim $a_{\lambda,\mu} = 0$. Let $\mu = (\mu_1, \ldots, \mu_l)$ and $\tilde{\lambda} = (\eta_1, \ldots, \eta_t)$. Then there is a minimal i such that $\mu_1 + \mu_2 + \cdots + \mu_i > \eta_1 + \eta_2 + \cdots + \eta_i$.

If $a_{\lambda,\mu} \neq 0$ then there should be some 0-1 matrix with row sums λ and column sums μ. Look at column i of this matrix. To get the column sum of μ_i, we should have $\mu_1 + \mu_2 + \cdots + \mu_i$ in the first i columns of the matrix.

Now if we push all 1's in the matrix to the left, then we can only get more 1's in the first i columns. However since pushing left gives the Ferrers diagram of λ, the first i columns in the resulting matrix will just be $\eta_1, \eta_2, \ldots, \eta_i$. This contradicts the assumption that $\mu_1 + \mu_2 + \cdots + \mu_i > \eta_1 + \eta_2 + \cdots + \eta_i$, and therefore $a_{\lambda,\mu} = 0$ if $\tilde{\lambda} \not\trianglerighteq \mu$.

From this we get $e_\lambda = \sum_{\tilde{\lambda} \trianglerighteq \mu} a_{\lambda,\mu} m_\mu$, and so the transformation matrix is triangular. Moreover, checking the diagonal, we have $a_{\lambda,\tilde{\lambda}} = 1$, since there is exactly one way to place 1's in a matrix so that row sums are λ and column sums equal $\tilde{\lambda}$. This is to place the Ferrers diagram as 1's in the matrix, with the top left corner aligned. Observe that we do not need $\Bbbk$ to have characteristic zero here because the diagonal is all 1's. As in the other case, we conclude that the transformation matrix is invertible, and therefore $\{e_\lambda\}$ is a basis for $\Lambda(\mathbf{x})$. $\square$

References: Reiner, Sections 2.1 and 2.2