Math 821, Spring 2013, Lecture 3

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1 Constructions coming from cycle index polynomial

Recall:

Definition. *Let A be a permutation group on $\{1, 2, \ldots, n\}$, $Z(A; s_1, s_2, \ldots, s_n) = \frac{1}{|A|} \sum_{\sigma \in A} \prod_{k=1}^{n} s_k^{j_k(\sigma)}$. Where $j_k(\sigma) =$ the number of cycles of length k in the disjoint cycle representation of σ.*

Theorem 1. *Let $\mathcal{C}$ be a combinatorial class, A a permutation group on $\{1, 2, \ldots, n\}$, let $\mathcal{B} = \mathcal{C}^{\{1,2,\ldots,n\}}/A$ then $B(x) = Z(A; C(x), C(x^2), \ldots, C(x^n))$.*

From this we get some formulas for some constructions.

Proposition 2. $Z(A; s_1, s_2, \ldots, s_n) = \sum_{\substack{j_1, j_2, \ldots, j_n \\ 1j_1 + 2j_2 + \cdots + nj_n = n}} \frac{s_1^{j_1} s_2^{j_2} \cdots s_n^{j_n}}{j_1! \cdot 1^{j_1} \cdot j_2! \cdot 2^{j_2} \cdots \cdot j_n! \cdot n^{j_n}}.$

Proof. Given a cycle structure with j_k cycles of size k, where $k = 1, 2, \ldots, n$. Let's count how many permutations have this cycle structure. That is we are trying to assign $1, 2, \ldots, n$ to the cycle. Among all $n!$ permutations of $1, 2, \ldots, n$, we overcount for all the ways of rearranging the cycles of a given size, i.e. we need to divide by $j_1! j_2! \cdots j_n!$. And we also overcount by each cyclic rotation within a given cycle, so we need to divide by $1^{j_1} 2^{j_2} \cdots n^{j_n}$. Then the $n!$ cancels with the $\frac{1}{|S_n|} = \frac{1}{n!}$ giving the proposition. $\square$

Definition. *Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \emptyset$, define $MSet(\mathcal{B})$ the combinatorial class of multisets of elements of $\mathcal{B}$ to be $MSet(\mathcal{B}) = \sum_{n=0}^{\infty} \mathcal{B}^{\{1,2,\ldots,n\}}/S_n$.*

Proposition 3. *Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \emptyset$, let $\mathcal{A} = MSet(\mathcal{B})$. Then $A(x) = exp(\sum\limits_{k=1}^{\infty} \frac{B(x^k)}{k})$. (Sometimes called Pólya exponential)*

Proof. We expand the right side.

$$
\begin{aligned}
exp(\sum_{k=1}^{\infty} \frac{B(x^k)}{k}) &= \sum_{l=0}^{\infty} (\sum_{k=1}^{\infty} \frac{B(x^k)}{k})^l \cdot \frac{1}{l!} \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{(i_1,\ldots,i_l) \\ ordered\ list}} (\frac{B(x^{i_1})}{i_1} \frac{B(x^{i_2})}{i_2} \cdots \frac{B(x^{i_l})}{i_l}) \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{j_1,j_2,\ldots \\ \sum j_i = l}} \frac{\prod\limits_{i=1}^{\infty} B(x^i)^{j_i}}{\prod\limits_{i=1}^{\infty} i^{j_i}} \cdot \frac{l!}{\prod\limits_{i=1}^{\infty} j_i!} \\
&= \sum_{n=0}^{\infty} \sum_{1j_1+2j_2+\cdots+nj_n=n} \frac{B(x)^{j_1} B(x^2)^{j_2} \cdots B(x^n)^{j_n}}{1^{j_1} 2^{j_2} \cdots n^{j_n} j_1! j_2! \cdots j_n!} \\
&= \sum_{n=0}^{\infty} Z(S_n; B(x), B(x^2), \ldots, B(x^n))
\end{aligned}
$$

Where j_k=the number of ks in the ordered list $(i_1, \ldots, i_l)$, $k = 1, 2, \ldots$. Note in the third equality from the bottom the double sum is over all ordered list by length first. And in the second equality from the bottom the double sum is over all ordered list by sum first. □

Example. *Rooted trees (without empty tree) (not plane)*

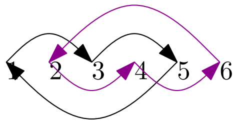
$\mathcal{T} = \mathcal{Z} \times MSet(\mathcal{T})$ *and* $T(x) = xexp(\sum\limits_{k=1}^{\infty} \frac{T(x^k)}{k})$.

Proposition 4. *Let C_n be the cyclic group on $\{1, 2, \ldots, n\}$, then*

$Z(C_n; s_1, s_2, \ldots, s_n) = \frac{1}{n} \sum\limits_{k|n} \phi(k) s_k^{\frac{n}{k}}$. *Where ϕ is the Euler ϕ function namely $\phi(d) :=$ the number of integers in $\{1, 2, \ldots, d\}$ coprime to d.*

Proof. The elements of C_n which takes 1 to m ($m = 2, \ldots, n, n+1$, here when $m = n+1$, it means 1 goes to 1 since we don't want $g.c.d(n, m-1) = 0$) has $g.c.d(n, m-1)$ cycles each of length $\frac{n}{g.c.d(n,m-1)}$. For any $k|n$, we have $\phi(k)$ such m having $g.c.d(n, m-1) = \frac{n}{k}$. Thus $Z(C_n; s_1, s_2, \ldots, s_n) = \frac{1}{n} \sum\limits_{k|n} \phi(k) s_k^{\frac{n}{k}}$. □

Example. *Take $n = 6$, $m = 3$,*



we have $g.c.d(6, 2) = 2$ cycles of length $\frac{6}{2} = 3$. And there are $\phi(3) = 2$ having $g.c.d(6, m - 1) = 2$, i.e $m = 3$ or $m = 5$.

Definition. *Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \emptyset$, define $DCyc(\mathcal{B})$ be the combinatorial class of directed cycles of elements of $\mathcal{B}$ by $DCyc(\mathcal{B}) = \sum_{n=1}^{\infty} \mathcal{B}^{\{1,2,\dots,n\}}/C_n$.*

Note there's no empty cycle (a convention). Also note in Flajolet and Sedgewick's book, this is denoted as $Cyc(\mathcal{B})$. But it's better to write DCyc since they're directed cycles.

Proposition 5. *Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \emptyset$, let $\mathcal{A} = DCyc(\mathcal{B})$. Then $A(x) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} log(\frac{1}{1-B(x^k)})$.*

Proof.

$$\begin{aligned}
A(x) &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k|n} \phi(k) B(x^k)^{\frac{n}{k}} \\
&= \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \sum_{\substack{l=1 \\ (kl=n)}}^{\infty} \frac{1}{l} B(x^k)^l \\
&= \sum_{k=1}^{\infty} \frac{\phi(k)}{k} (-log(1 - B(x^k))) \\
&= \sum_{k=1}^{\infty} \frac{\phi(k)}{k} log(\frac{1}{1 - B(x^k)}).
\end{aligned}$$

□

Example. *Binary necklaces without flipping being allowed. $\mathcal{B} = DCyc(\mathcal{Z}_0 + \mathcal{Z}_1)$.*

Note Seq fits into this framework. Let E_n be the group $\{(1)(2)\cdots(n)\}$, then $Z(E_n; s_1, s_2, \cdots, s_n) = s_1^n$. Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \emptyset$, then

$$\begin{aligned}\mathcal{A} &= Seq(\mathcal{B})\\ &= \sum_{n=0}^{\infty} \mathcal{B}^n\\ &= \sum_{n=0}^{\infty} \mathcal{B}^{\{1,2,\cdots,n\}}/E_n\end{aligned}$$

Thus,

$$\begin{aligned}A(x) &= \sum_{n=0}^{\infty} Z(E_n; B(x), \ldots, B(x^n))\\ &= \sum_{n=0}^{\infty} B(x)^n\\ &= \frac{1}{1-B(x)}.\end{aligned}$$

Notation. *We can also restrict all of these which we write in the subscripts.*

$$DCyc_\Omega(\mathcal{B}) = \sum_{\omega\in\Omega} \mathcal{B}^{\{1,2,\ldots,\omega\}}/C_\omega,$$

for $\Omega \subseteq \mathbb{Z}_{>0}$. And similarly for $Mset$ $(\Omega \subseteq \mathbb{Z}_{\geq 0})$ etc.

Example. *Binary rooted trees with no extra information (not plane, not keeping track of left vs. right).*

$$\begin{aligned}\mathcal{B} &= \mathcal{Z} \times Mset_{\leq 2}(\mathcal{B})\\ B(x) &= x(Z(S_0) + Z(S_1; B(x)) + Z(S_2; B(x), B(x^2)))\\ &= x(1 + B(x) + \frac{1}{2}(B(x) + B(x^2))).\end{aligned}$$

Example. *Rooted trees where each vertex has an even number of children.*

$$\mathcal{T} = \mathcal{Z} \times MSet_{\{2n:\ n\geq 0\}}(\mathcal{T}).$$

References.

Flajolet and Sedgewick, Analytic combinatorics, Cambridge (2009), Cha I.2
Harary and Palmer, Graphical Enumeration, Cha 2

2 Partitions

Let $\mathcal{I}$ be the combinatorial class of $\{1, 2, \ldots\}$ with the size of a nonnegative integer being itself. Then $\mathcal{I} = \mathcal{Z} \times Seq(\mathcal{Z})$.

Definition. *A partition λ of a nonnegative integer n is a list $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0)$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_t = n$. The λ_i are the parts of λ and n is the size of λ.*

Partitions form a combinatorial class $\mathcal{P}$, a specification $\mathcal{P} = MSet(\mathcal{I})$, $P(x) = exp(\sum\limits_{k=1}^{\infty} \frac{I(x^k)}{k}) = exp(\sum\limits_{k=1}^{\infty} \frac{x^k}{k(1-x^k)})$.

Here's another way to think of partitions. A partition is some number of parts of size 1, some number of parts of size 2 and so on.

$\mathcal{P} = Seq(\mathcal{Z}) \times Seq(\mathcal{Z}^2) \times \cdots$

But is this infinite product legitimate ? The answer is Yes!

Lemma 6. *Let $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$, $\ldots$ be combinatorial classes with $\varepsilon \in \mathcal{A}^{(i)}$ and $|a_i| \geq i$ for $a_i \in \mathcal{A}^{(i)}$ and $a_i \neq \varepsilon$. Then we can define $\prod\limits_{i=1}^{\infty} \mathcal{A}^{(i)}$ to be the combinatorial class whose elements are infinite sequences $(a_1, a_2, \cdots)$ such that $a_i \in \mathcal{A}^{(i)}$ and eventually all $a_j = \varepsilon$. From this fact we have $|(a_1, a_2, \cdots)| = |a_1| + |a_2| + \cdots$ which is a finite sum. Furthermore, $A(x) = \prod\limits_{i=1}^{\infty} A^{(i)}(x)$*

Proof. $\mathcal{A}_n = (\mathcal{A}^{(1)} \times \mathcal{A}^{(2)} \times \cdots \times \mathcal{A}^{(n)})_n$ by $|a_i| \geq i$ for $a_i \in \mathcal{A}^{(i)}$ and $a_i \neq \varepsilon$. So

$$\begin{aligned} a_n &= (\mathcal{A}^{(1)} \times \mathcal{A}^{(2)} \times \cdots \times \mathcal{A}^{(n)})_n \\ &= [x^n] \prod_{i=1}^{n} A^{(i)}(x) \\ &= [x^n] \prod_{i=1}^{\infty} A^{(i)}(x) \end{aligned}$$

by the same reason above. So returning to partitions $P(x) = \prod\limits_{n=1}^{\infty} \frac{1}{1-x^n}$. □

How about restrictions?

Example. *Partitions with all parts$\leq k$.*

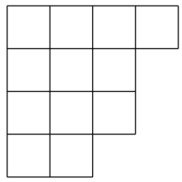
$\mathcal{P} = Seq(\mathcal{Z}) \times Seq(\mathcal{Z}^2) \times \cdots \times Seq(\mathcal{Z}^k)$, $P(x) = \prod\limits_{i=1}^{k} \frac{1}{1-x^i}$.

Example. *Partitions with at most k parts.*
$\mathcal{P} = MSet_{\leq k}(\mathcal{I})$.

By the cycle index polynomials we have an expression for this but here's something easier.

Definition. *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0)$ be a partition, the Ferrer's diagram of λ is the diagram with first row containing λ_1 boxes, next row containing λ_2 boxes etc. all left justified.*

Example. $\lambda = (4, 3, 3, 2)$

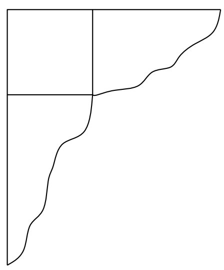


Definition. *Let $\mathcal{P}$ be the combinatorial class of partitions, define Φ to be the size-preserving automorphism of $\mathcal{P}$ given by reflecting the Ferrer's diagram about $y = -x$, where the top left corner is at $(0, 0)$.*

Note if λ has k parts then $\Phi(\lambda)$ has largest part$= k$.
There are many decompositions of partitions, some we'll see in the partitions section of the course. For now just one more.

Definition. *Let λ be a partition. The Ferrer's diagram of λ has a unique maximal square (beginning in the top left corner) called the Durfee square.*

The Durfee square gives us another decompositions.



$$\begin{aligned}
\mathcal{P} &= \sum_{k=1}^{\infty} (\mathcal{Z}^{k^2} \times \mathcal{P}_{at\ most\ k\ parts} \times \mathcal{P}_{parts \leq k}) \\
P(x) &= \sum_{k=1}^{\infty} x^{k^2} (\prod_{i=1}^{k} \frac{1}{1 - x^i})^2
\end{aligned}$$

Note we get some nontrivial identities of some power series by taking different ways to view the same combinatorial class $\mathcal{P}$ which is interesting.

References. *Flajolet and Sedgewick, Analytic combinatorics, Cambridge (2009), Cha I.3*