MATH 821, SPRING 2012, ASSIGNMENT 1

DUE THURSDAY JANUARY 31, 2013 IN CLASS

(1) Let $\mathcal{B}$ be a combinatorial class with no elements of size 0. Let $\mathcal{A} = \text{UC}\textsc{yc}(\mathcal{B})$ be the combinatorial class of undirected cycles of elements of $\mathcal{B}$, that is take a cycle as a graph and assign an element of $\mathcal{B}$ to each vertex.

Prove that

$$A(x) = \frac{1}{2}C(x) + \frac{1}{4}\sum_{n=1}^{\infty}\begin{cases} 2B(x)B(x^2)^{(n-1)/2} & \text{if } n \text{ is odd} \\ B(x)^2B(x^2)^{(n-2)/2} + B(x^2)^{n/2} & \text{if } n \text{ is even} \end{cases}$$

where $\mathcal{C} = \text{DC}\textsc{yc}(\mathcal{B})$.

(2) Let $\mathcal{B}$ be the class of binary rooted trees (with distinct left and right children) where every node has either 0 or 2 children.
   (a) Give a combinatorial specification for $\mathcal{B}$
   (b) Determine $[x^n]B(x)$
   (c) Shifting sizes appropriately there is a connection with the class of all binary rooted trees from lecture. Make this connection precise and find an explicit (non-recursive) bijection which demonstrates it.

(3) A composition of a positive integer n is a list of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots \lambda_k = n$. These are like partitions except that order matters. As for partitions the size of a composition will be the sum n.
   (a) Find a specification and find the generating function of the class of all compositions.
   (b) Find a specification and find the generating function of the class of compositions with at most k parts and where each part is odd.

(4) Let $\mathcal{C}$ be a combinatorial class formed (either iteratively or recursively) out of copies (potentially different) of $\mathcal{Z}$. All the combinatorial classes we have seen are like this – for trees the copies of $\mathcal{Z}$ are the vertices, for words they are the letters, for walks they are the steps. Call the copies of $\mathcal{Z}$ atoms.

Let $\Theta(\mathcal{C})$ be the combinatorial class whose elements are all pairs (C, z) with $C \in \mathcal{C}$ and z and atom of C, and with size function $|(C, z)| = |C|$. This is called the *pointing operator*.
   (a) Show that Θ is admissible
   (b) Let $\mathcal{A} = \Theta(\mathcal{C})$. Give an expression for $A(x)$ as a function of $C(x)$ using only functions and operators that you know from first year calculus.

(5) Pick three different combinatorial classes which we have studied. Calculate their counting sequences up to $n = 100$ (use a computer!) Plot the sequences on the same axes. How fast do they each seem to be growing?