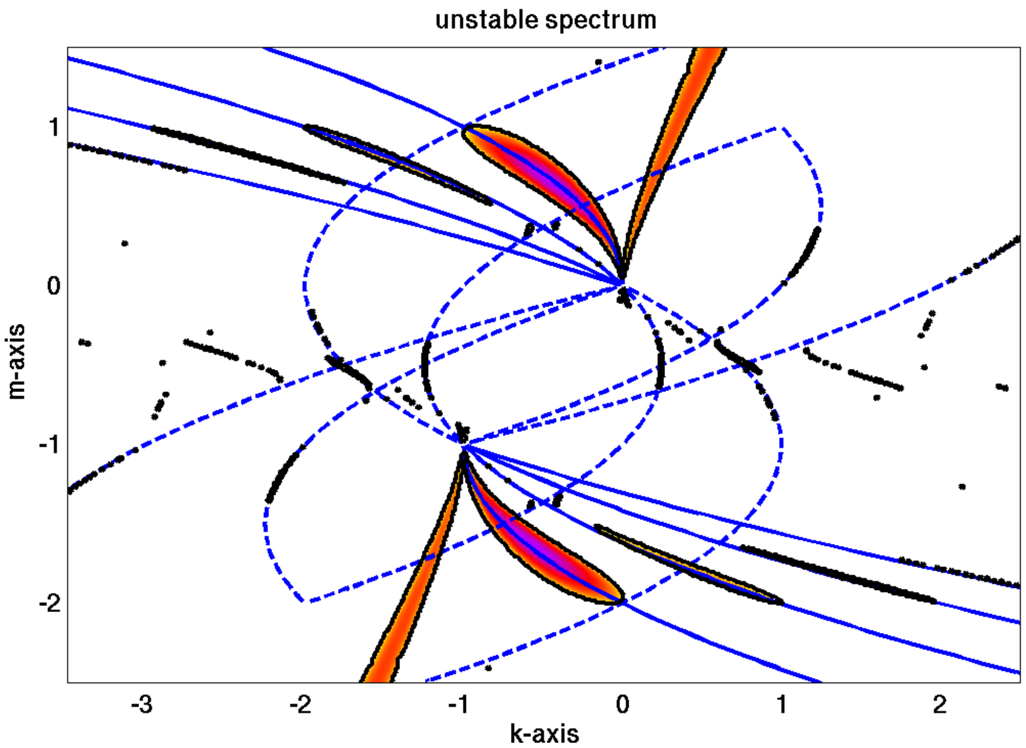
# Floquet Instability & Triad Resonance in a Stratified Flow

- ▷ linear instabilities of a single-mode gravity wave
- ▷ analysis from Floquet & resonant wave perspectives





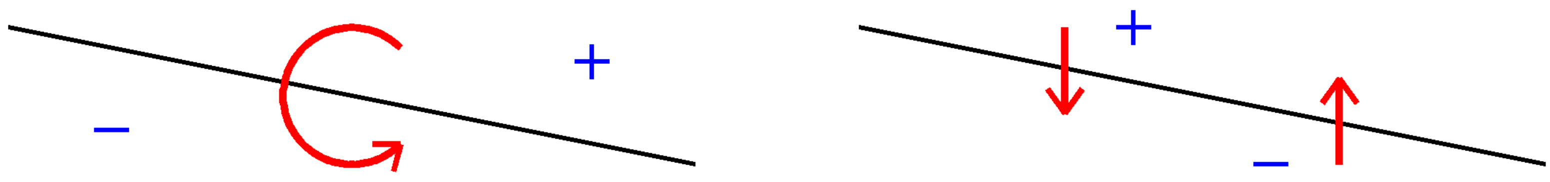
- ▷ Dave Muraki & Youngsuk Lee, Simon Fraser University

# Equations for a Stratified Fluid

## Vorticity/Buoyancy Dynamics

- 2D Euler fluid with Boussinesq buoyancy & constant stratification (stable) $\longrightarrow$ oscillations

$$\frac{D\eta}{Dt} = b_x \qquad\qquad \frac{Db}{Dt} = -w$$




## Streamfunction Formulation

- $\psi(x,z)$, incompressible streamfunction: $u = \psi_z \;\; ; \;\; w = -\psi_x \;\; ; \;\; \eta = -\nabla^2\psi$
- uniform wind & hydrostatic scaling: $\eta \longrightarrow -\psi_{zz}$

$$\begin{aligned} \psi_{zzt} \; + \psi_{zzx} \; + b_x \; + J(\psi_{zz}, \psi) &= 0 \\ b_t \; + b_x \; - \psi_x \; + J(b, \psi) &= 0 \end{aligned}$$

- nonlinearity via 2D streamfunction advection: Jacobian determinant

$$J(f,\psi) = \begin{vmatrix} f_x & \psi_x \\ f_z & \psi_z \end{vmatrix} = \begin{vmatrix} f_x & -w \\ f_z & u \end{vmatrix} = u f_x + w f_z$$

## An Exact Wave Solution

$$\boxed{\begin{array}{lllll} \psi_{zzt} & + \psi_{zzx} & + b_x & + \ J(\psi_{zz}, \psi) & = \ 0 \\ b_t & + b_x & - \psi_x & + \ J(b, \psi) & = \ 0 \end{array}}$$

Fourier Modes: $e^{i(kx+mz-\omega t)}$

- linear dispersion relation (slow/fast) for buoyancy-gravity waves

$$\omega(k,m) = k \mp \frac{k}{|m|} \quad ; \quad \vec{c}_g(k,m) = \left(1 \mp \frac{1}{|m|} \ , \ \pm \frac{km}{|m|^3}\right)$$

- steady wave: $k = m = 1$, $\omega = 0$ (slow wave with upward group velocity)

$$\begin{pmatrix} \psi \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} 2\epsilon \ \sin(x+z)$$

  - Jacobians are zero $\Rightarrow$ exact nonlinear solution!

Goal: to characterize the linear stability of this simple nonlinear wave

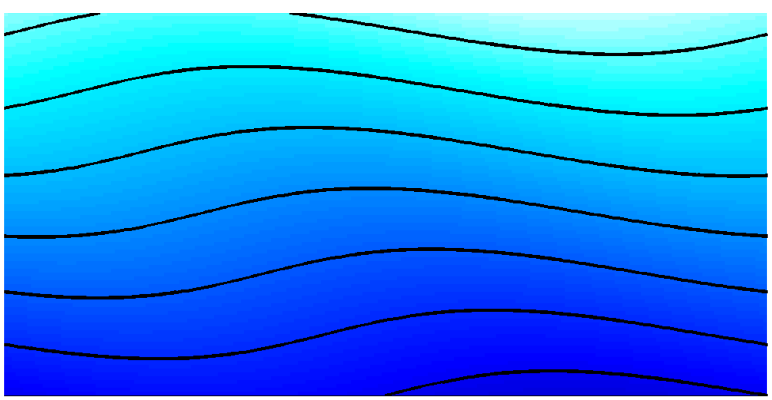
- to understand context for mountain flow instability (Youngsuk Lee, 2$^{\text{nd}}$ talk)
- instability: Mied 1976, Drazin 1977, Klostermeyer 1982, Sonmor & Klaassen 1997

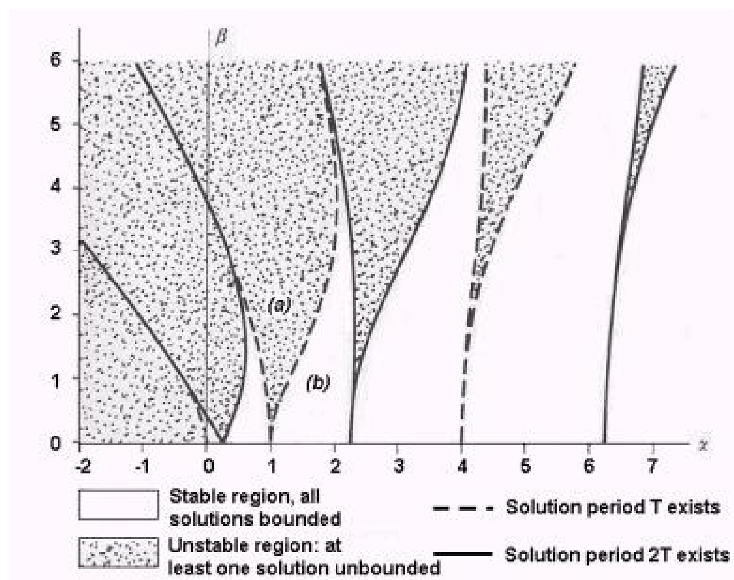
# Linearized Stability

$$
\boxed{
\begin{array}{llllcl}
\tilde{\psi}_{zzt} & + \tilde{\psi}_{zzx} & + \tilde{b}_x & + \ \epsilon \, J\left(\tilde{\psi}_{zz} + \tilde{\psi} \ , \ 2\sin(x+z)\right) & = & 0 \\
\tilde{b}_t & + \tilde{b}_x & - \tilde{\psi}_x & + \ \epsilon \, J\left(\tilde{b} - \tilde{\psi} \ , \ 2\sin(x+z)\right) & = & 0
\end{array}
}
$$

## A Problem for Floquet Theory

- linear PDE with periodic, non-constant coefficients
- instability via parametric resonance (as for the Mathieu ODE)





# Floquet Approach

$$\begin{array}{llllll} \tilde{\psi}_{zzt} & + \tilde{\psi}_{zzx} & + \tilde{b}_x & - \; i\epsilon \, J\left(\tilde{\psi}_{zz} + \tilde{\psi} \, , \, e^{i(x+z)} - e^{-i(x+z)}\right) & = & 0 \\ \tilde{b}_t & + \tilde{b}_x & - \tilde{\psi}_x & - \; i\epsilon \, J\left(\tilde{b} - \tilde{\psi} \quad , \, e^{i(x+z)} - e^{-i(x+z)}\right) & = & 0 \end{array}$$

## Floquet, Fourier & Hill

- product of Floquet exponential & co-periodic Fourier series

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = e^{i(kx+mz-\Omega t)} \left\{ \sum_{-\infty}^{+\infty} \vec{v}_n \, e^{in(x+z)} \right\}$$

  - perturbation wavevector, $\vec{K} = (k, m)$ & Floquet eigenvalue, $\mathrm{Im}(\Omega) > 0 \Rightarrow$ instability

- Hill's infinite matrix

$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix} - \Omega \begin{bmatrix} \ddots & & & \\ & \mathbf{\Lambda}_0 & & \\ & & \mathbf{\Lambda}_1 & \\ & & & \ddots \end{bmatrix}$$

  - $2 \times 2$ real blocks: $\mathbf{S}_n(k, m)$, symmetric; $\mathbf{\Lambda}_n(m)$, diagonal; $\mathbf{M}_n(k, m)$

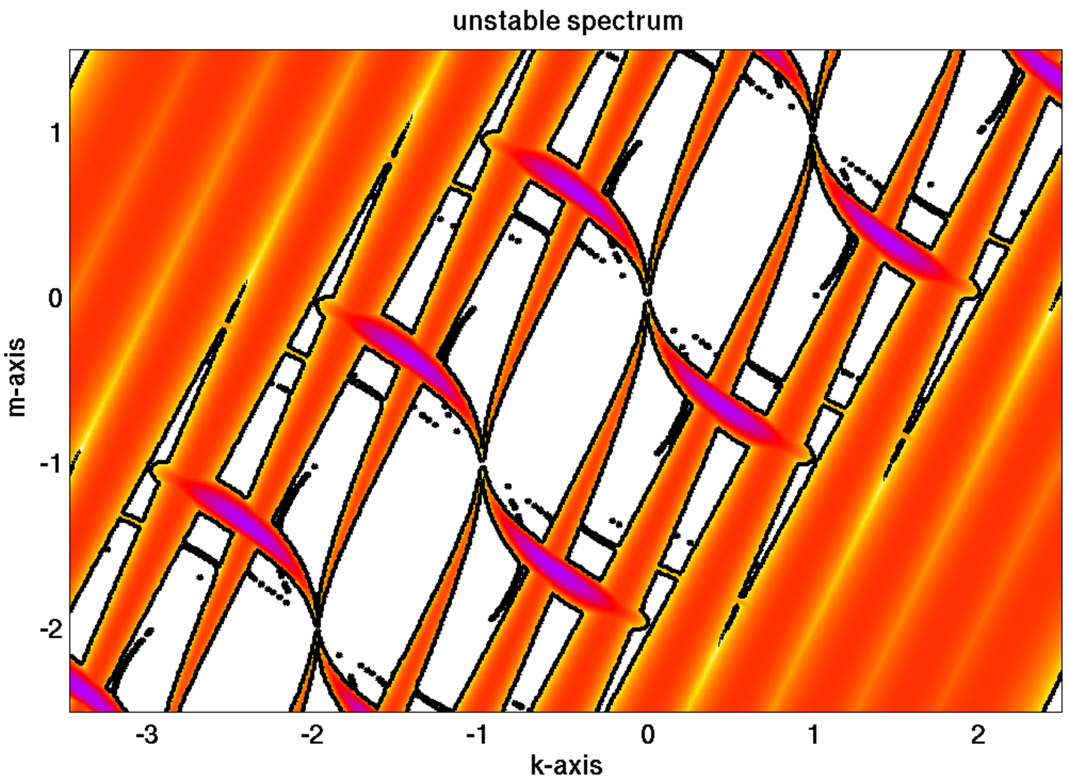
- truncate to $-N \leq n \leq N+1$ & compute eigenvalues: $\{\Omega(k, m)\}$

## Unstable Floquet Spectrum

### Maximum Growth Rate vs $\vec{K}$, $\epsilon = 0.1$

- natural periodicity due to non-uniqueness of series indexing

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = e^{i((k+q)x+(m+q)z-\Omega t)} \left\{ \sum_{-\infty}^{+\infty} \vec{v}_{n+q}\, e^{in(x+z)} \right\}$$

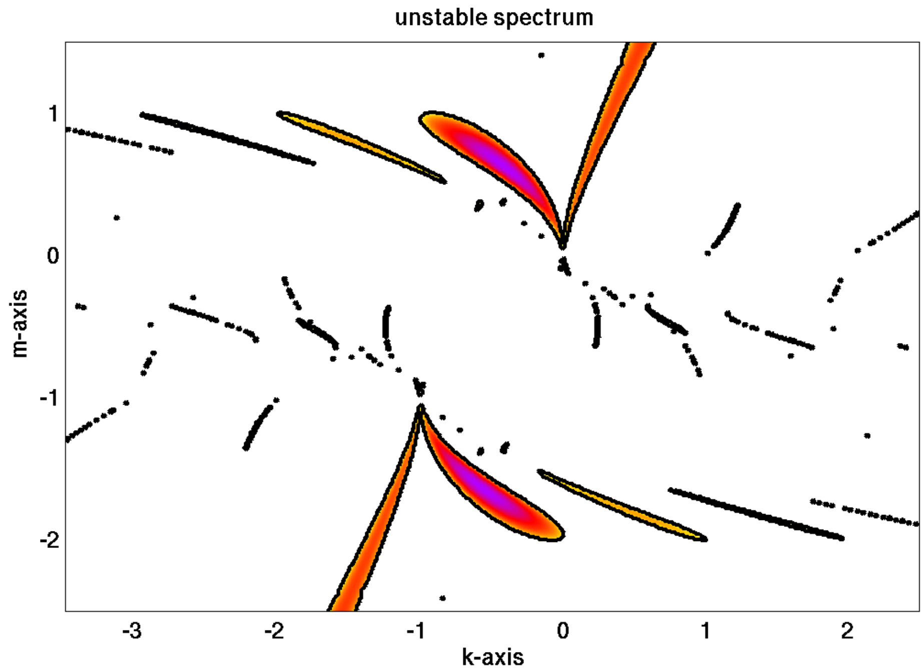


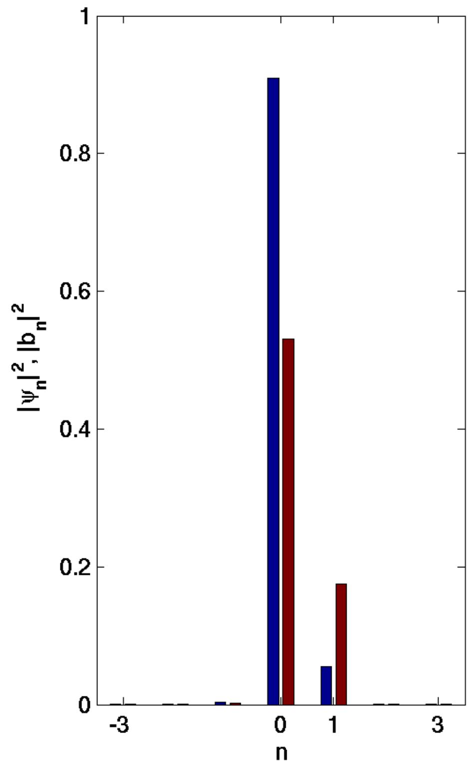
## Floquet Spectrum Unwrapped

### Maximum Growth Rate vs $\vec{K}$, $\epsilon = 0.1$

- *center-of-mass* criterion; preserves notion of central wavevector in $(k, m)$-space

$$0 \;\leq\; \frac{\sum_n n|\tilde{\psi}_n|^2}{\sum_n |\tilde{\psi}_n|^2} \;<\; 1$$




- where do the complex eigenvalues come from?

## Eigenvalue Degeneracy

$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix} - \Omega \begin{bmatrix} \ddots & & & \\ & \mathbf{\Lambda}_0 & & \\ & & \mathbf{\Lambda}_1 & \\ & & & \ddots \end{bmatrix}$$

### Instability of Small Amplitude Waves ($\epsilon \ll 1$)

- $\epsilon = 0$, linear dispersion relation $\Rightarrow$ real eigenvalues, $\Omega = \omega(k, m)$
- $\epsilon \neq 0$, characteristic polynomial is real
- for $0 < \epsilon \ll 1$, complex conjugate $\Omega$'s appear from multiple eigenvalues at $\epsilon = 0$

### Double Root in a 2-Mode Truncation

- adjacent $(n = 0, 1)$ Fourier modes $\Rightarrow$ $k_0 + 1 = k_1 \quad ; \quad m_0 + 1 = m_1$

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = \vec{v}_0 \, e^{i(k_0 x + m_0 z - \Omega t)} + \vec{v}_1 \, e^{i(k_1 x + m_1 z - \Omega t)}$$

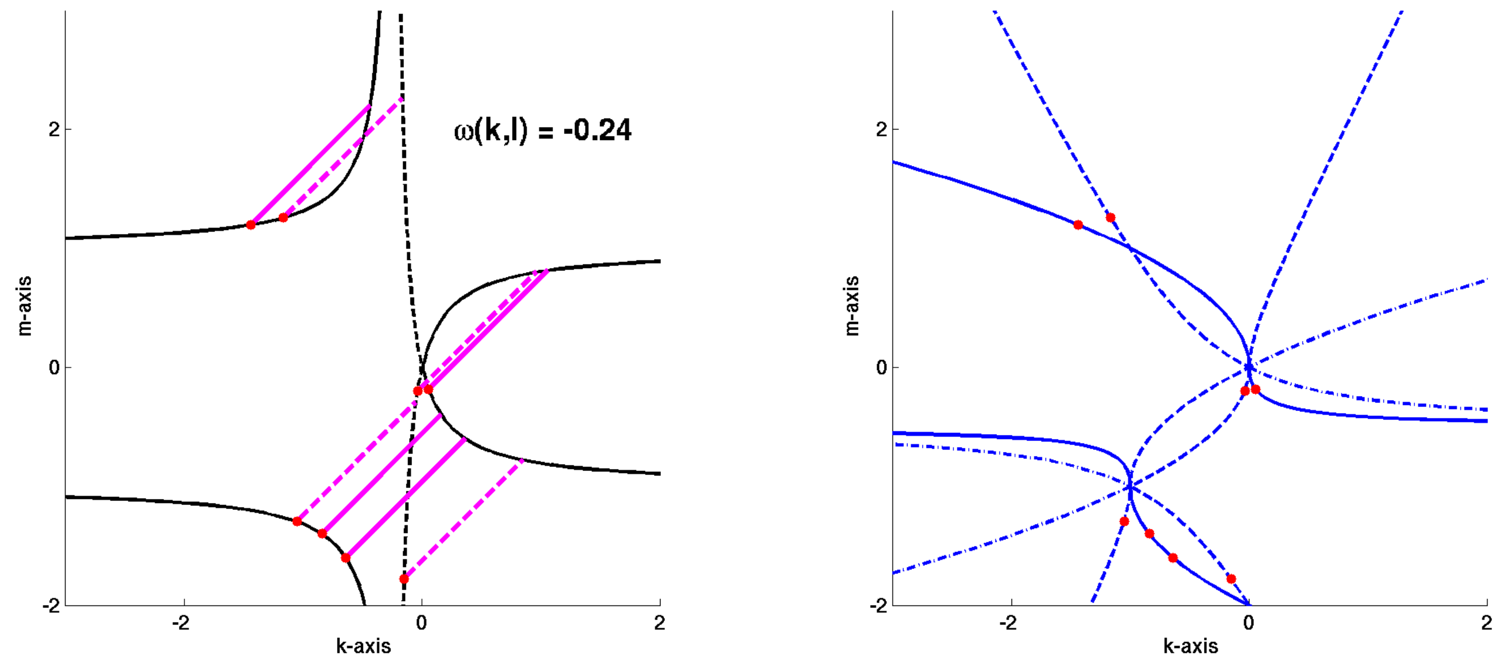
- at $\epsilon = 0$, if double root $\Longrightarrow$ $\omega_0 + 0 = \omega_1$ $\rightarrow$ triad resonance

# Triad Resonances

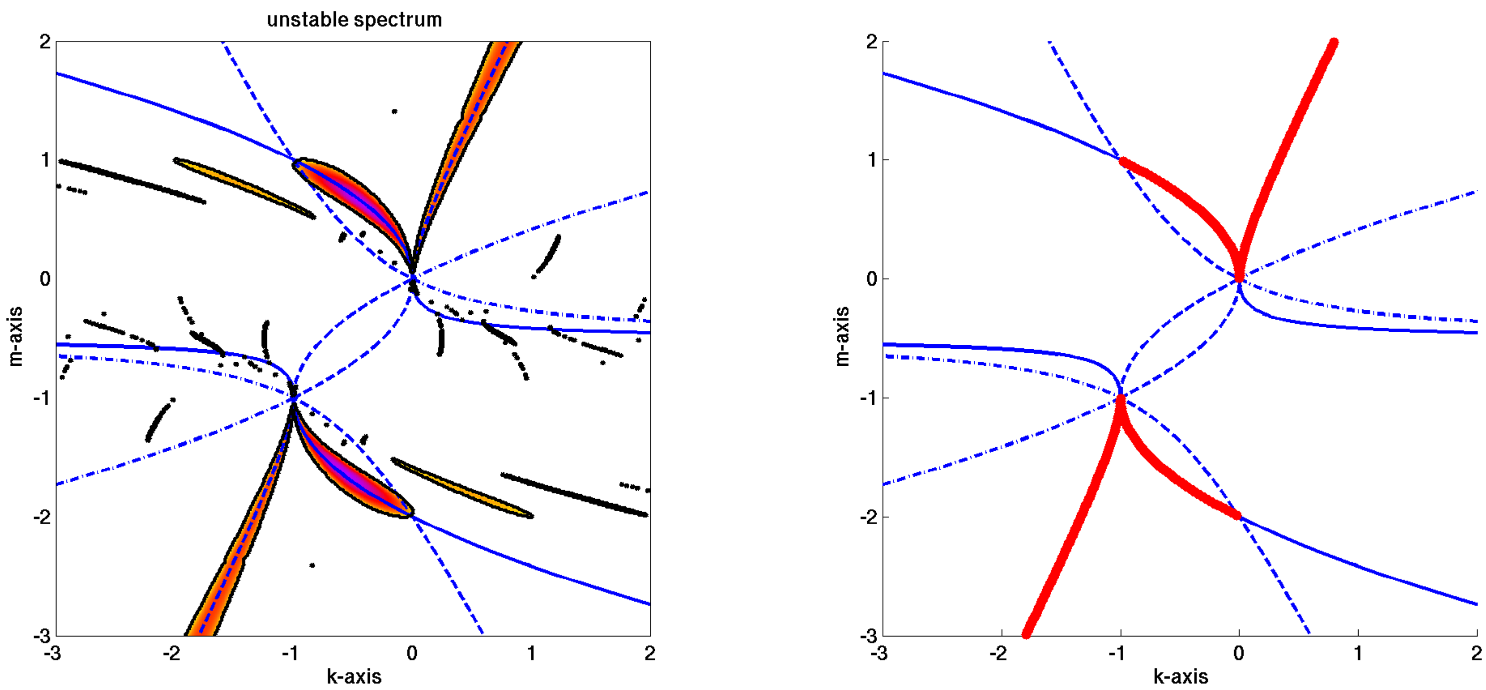
$$\boxed{\vec{k}_0 + \vec{k}_s = \vec{k}_1 \quad ; \quad \omega(\vec{k}_0) + \omega(\vec{k}_s) = \omega(\vec{k}_1)}$$

## Resonant Trace

- ▷ resonances identified as $\vec{k}_s$-connections between $\omega_0$ & $\omega_1$ dispersion curves




- ▷ curves of all $\vec{k}_0$ generating a triad (double root) ⟶ resonant trace




## Weakly Nonlinear Analysis ($\epsilon \ll 1$)

- ▷ double root only a necessary condition for small $\epsilon$ appearance of complex eigenvalues
- ▷ bifurcation analysis via eigenvalue perturbation: $\Omega(\vec{k}_0; \epsilon) = \omega_0 + \epsilon\, \Omega_1$

## Mountain Wave Instability

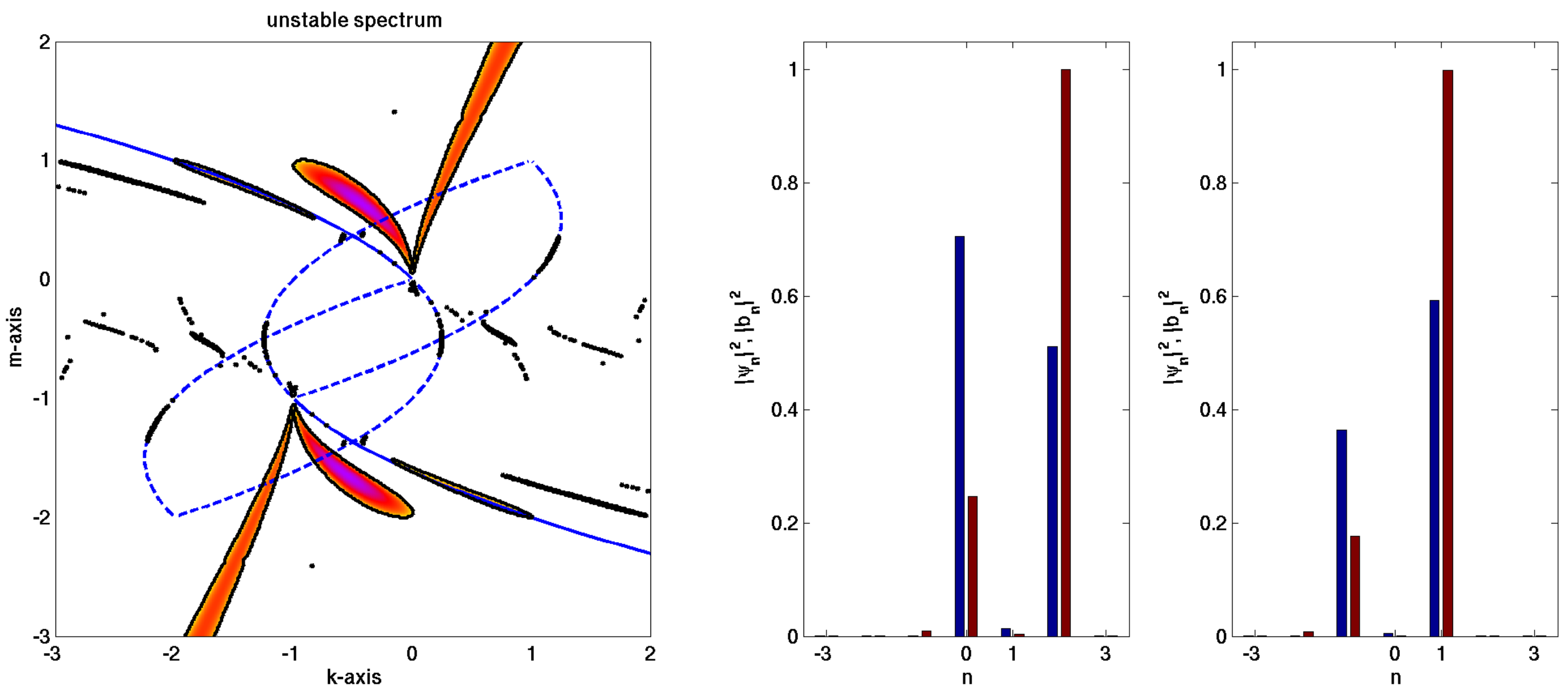
- ▷ only slow-slow resonance has counter-propagating group velocity

# "Quartet" Instability

$$\boxed{\vec{k}_0 + 2\,\vec{k}_s = \vec{k}_2 \quad ; \quad \omega(\vec{k}_0) + 2\,\omega(\vec{k}_s) = \omega(\vec{k}_2)}$$

## Next-to-Adjacent ($n = 0, 2$) Fourier Modes

- analogous to the 2nd Mathieu instability: $\Omega(\vec{k}_0; \epsilon) = \omega_0 + \epsilon^2\,\Omega_2$
- $n = 1$ mode plays a crucial role as a "catalyst" (since nearest-neighbor coupling only)



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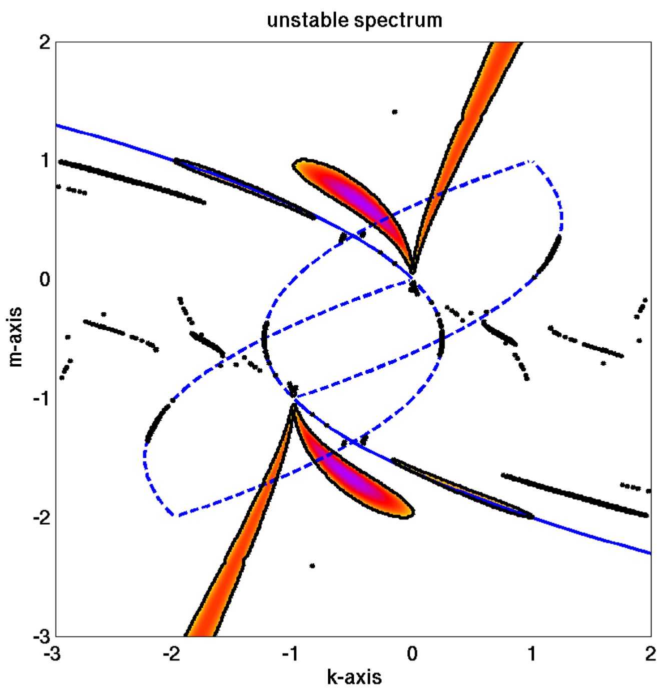
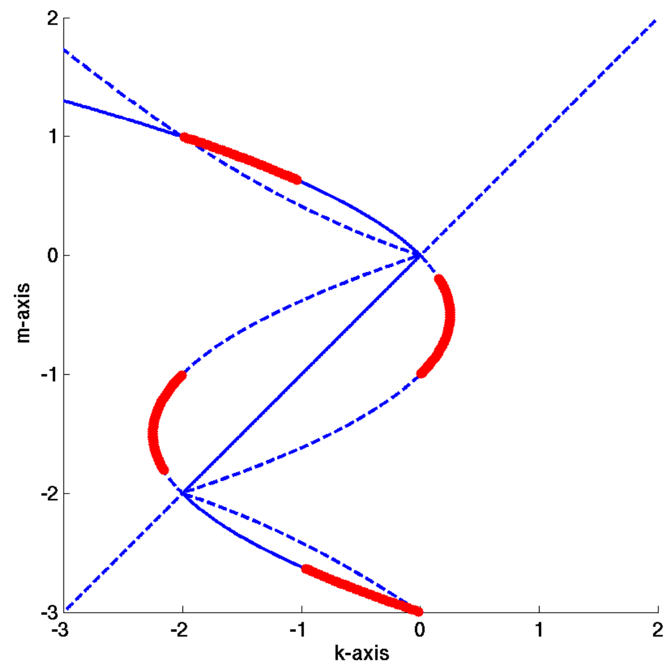


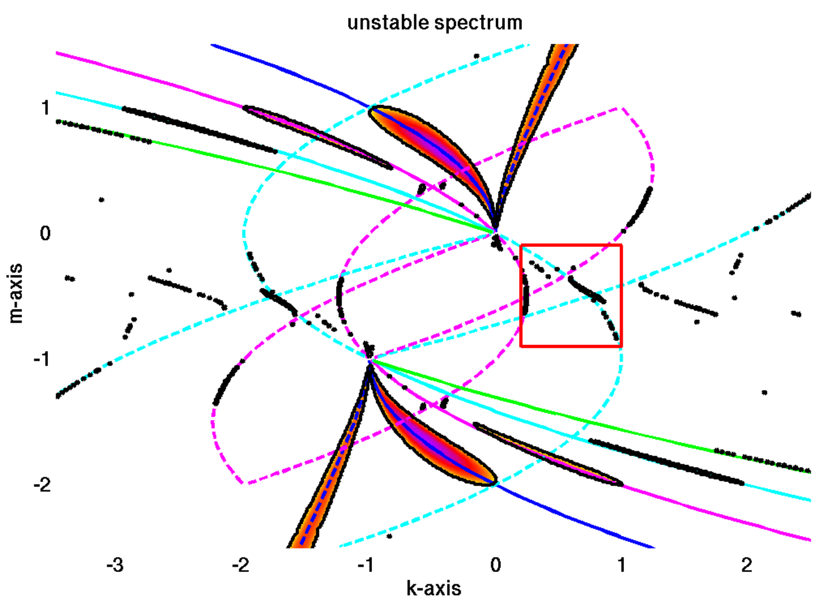



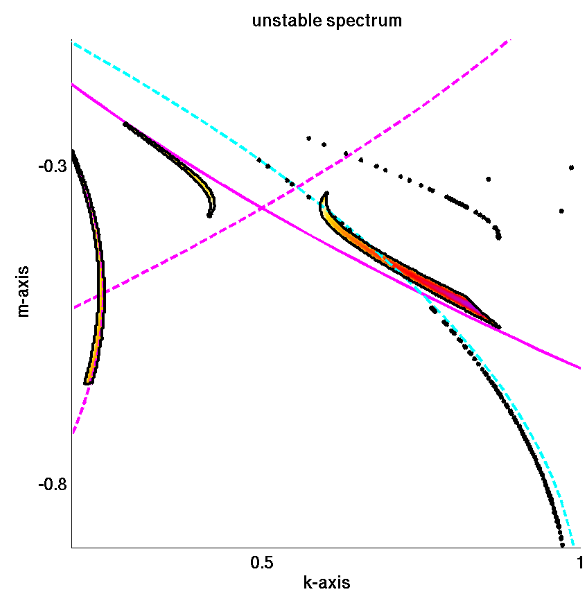
## Floquet Theory & Resonant Waves

$$\vec{k}_0 + j\,\vec{k}_s = \vec{k}_j \quad ; \quad \omega(\vec{k}_0) + j\,\omega(\vec{k}_s) = \omega(\vec{k}_j)$$

### 2D Map of Instabilities

- Floquet theory: Fourier series $\longrightarrow$ linear eigenvalue problem
- resonant waves: Fourier resonances $\longrightarrow$ eigenvalue degeneracies
  - are all instabilities born out of degeneracy?



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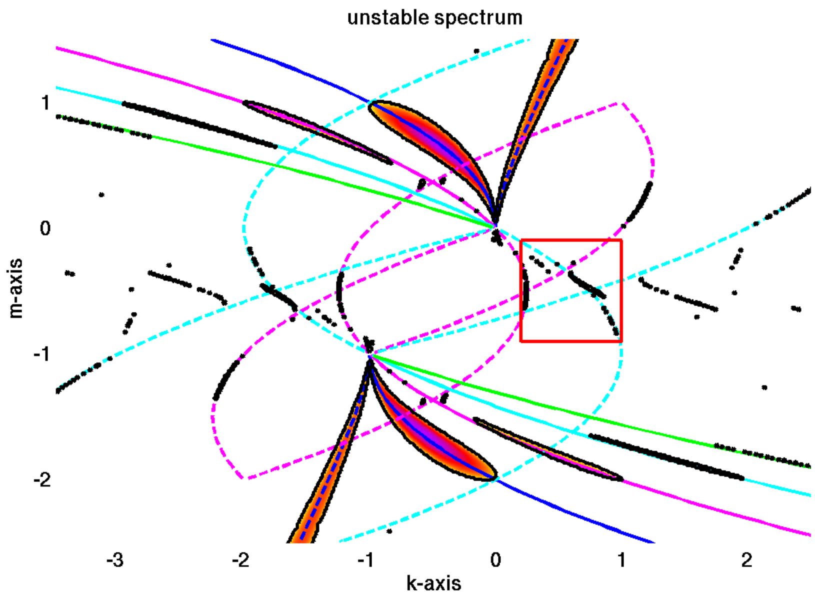
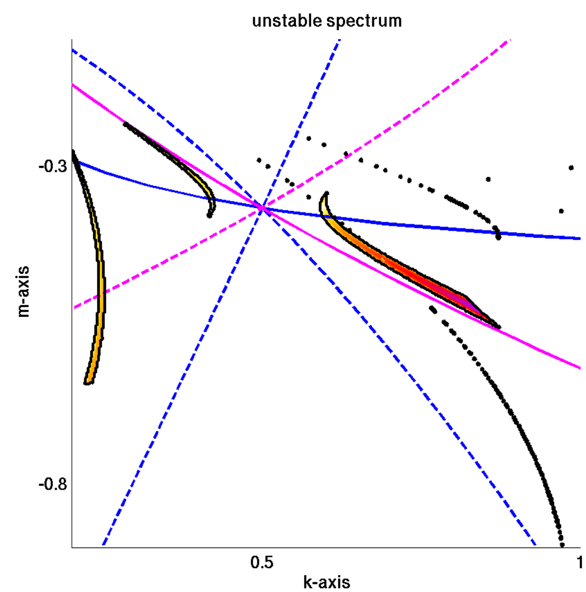


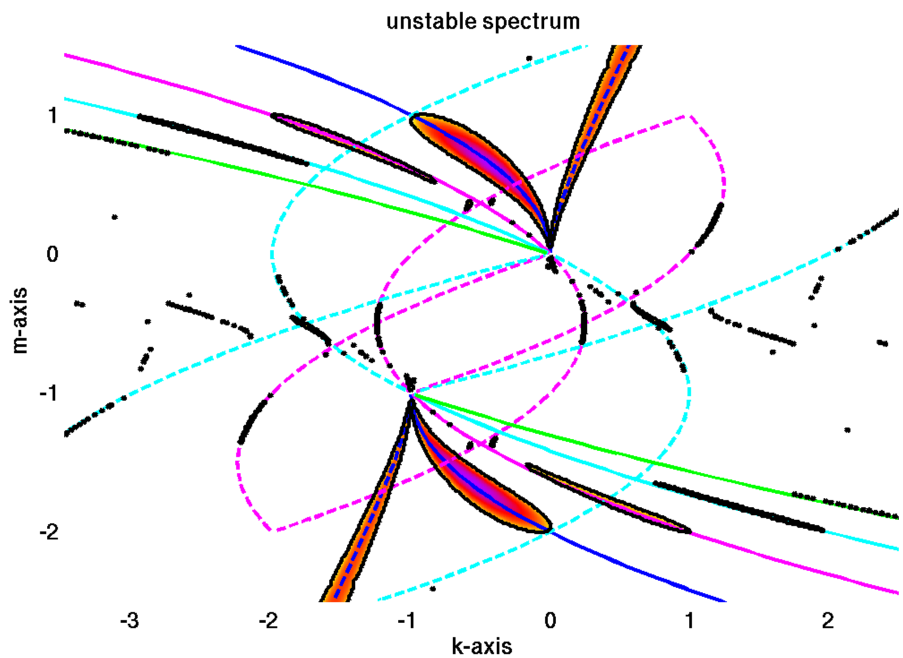



## In Closing

### Linear Stability of a Plane Gravity Wave

- ▷ clear characterization of Floquet instabilities by wave resonances
  - ▹ neutral curve, multiple-wave stability & nonhydrostatic flow
- ▷ application of weak turbulence ideas to linear stability
  - ▹ implications for atmospheric wave turbulence?

$\epsilon = 0.1$



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$\epsilon = 0.2$