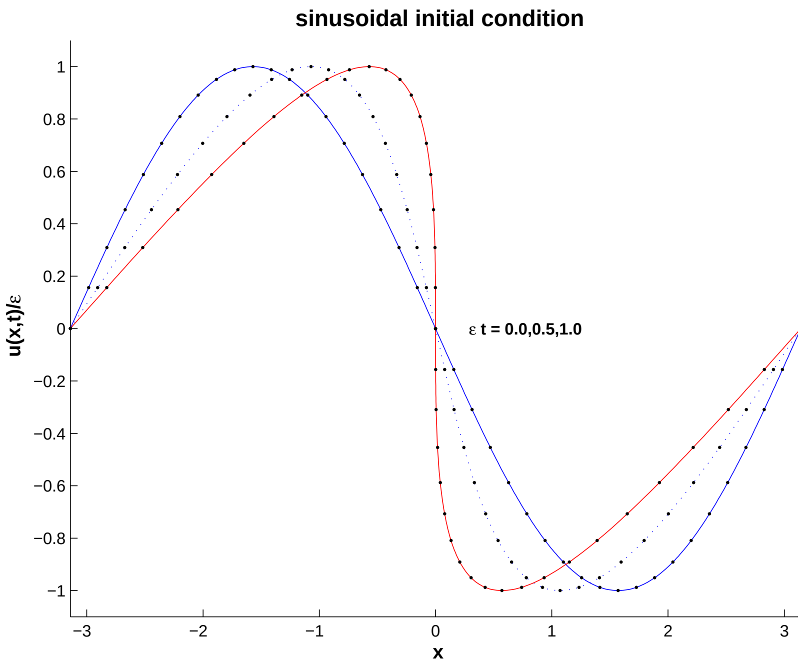
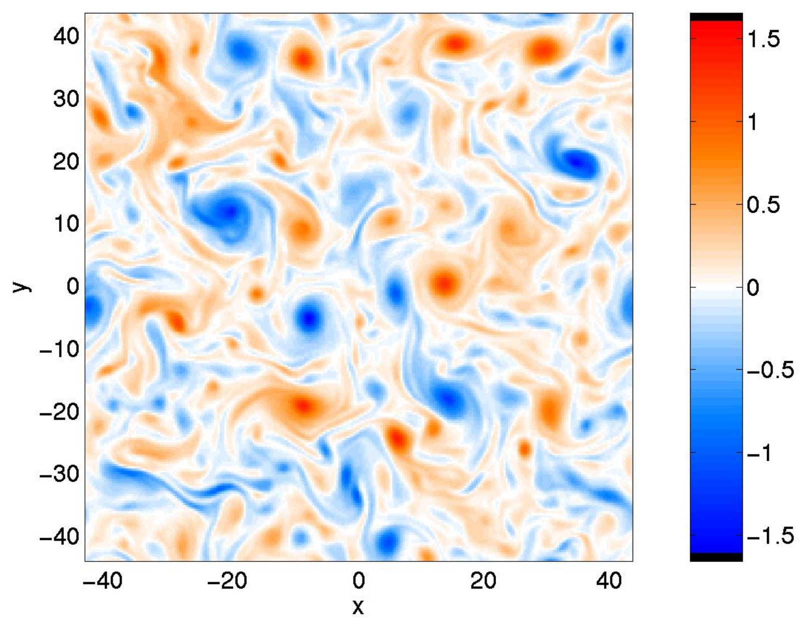
Tales of the Nonlinear:
a simple illustration of a spectral cascade

- ▷ Fourier properties of a simple nonlinear PDE
- ▷ whirlwind tour of applied mathematical methods





“Cascade” for a Nonlinear Eigenfunction

Linear Eigenvalue Problem

- linear boundary value problem

$$y'' + \lambda^2 y = 0 \quad ; \quad y(0) = y(\pi) = 0$$

- sinusoidal eigenfunctions, $y(x)$, for integers n

$$y(x) = A \sin nx \quad ; \quad \lambda^2 = n^2$$

Weakly Nonlinear Eigenvalue Problem

- nonlinear boundary value problem

$$y'' + \lambda^2 y = \epsilon y^3 \quad ; \quad y(0) = y(\pi) = 0$$

- exact solutions via elliptic functions
- perturbed eigenfunctions, $y(x)$, for $\epsilon A^2 \ll 1$

$$y(x) \sim A \sin nx + \frac{\epsilon A^2}{32 n^2}\left(1 - \frac{3\epsilon A^2}{16 n^2}\right) A \sin 3nx + \frac{3\epsilon^2 A^4}{512 n^4} A \sin 5nx + \dots$$

$$\lambda^2/n^2 \sim 1 + \frac{3\epsilon A^2}{4n^2} - \frac{9\epsilon^2 A^4}{64 n^4} + \dots$$

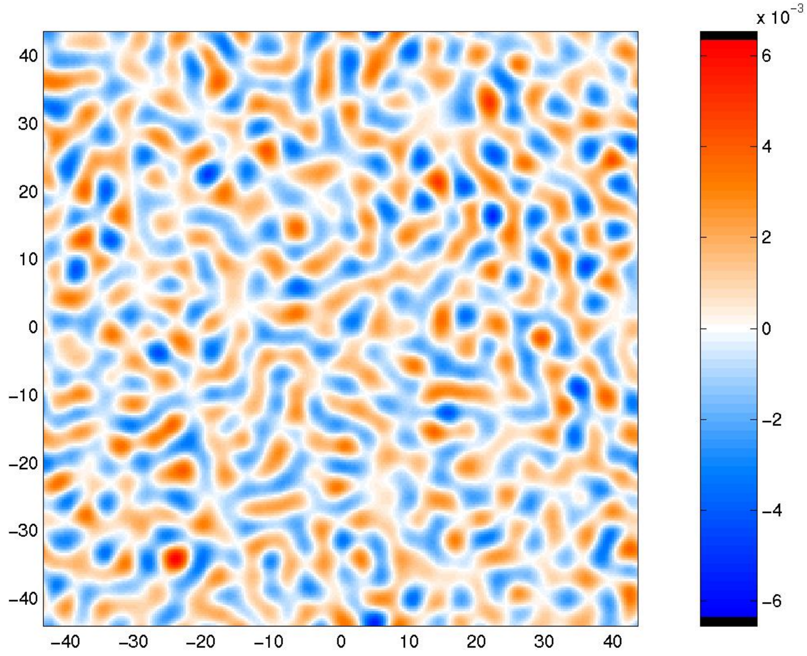
- nonlinear eigenfunction displays a cascade to full Fourier series for $\epsilon A^2 \neq 0$

Atmospheric Tropopause Model

- 2D fluid advection with hyperdiffusion & random forcing for potential temperature $\theta(x, y, t)$

$$\theta_t + u\theta_x + v\theta_y + \nu\nabla^8\theta = F(\vec{x}, t)$$

- quadratic nonlinearity: u and v linearly determined from θ (via 3D Laplace solve)



Fourier Spectra: Forcing & Response

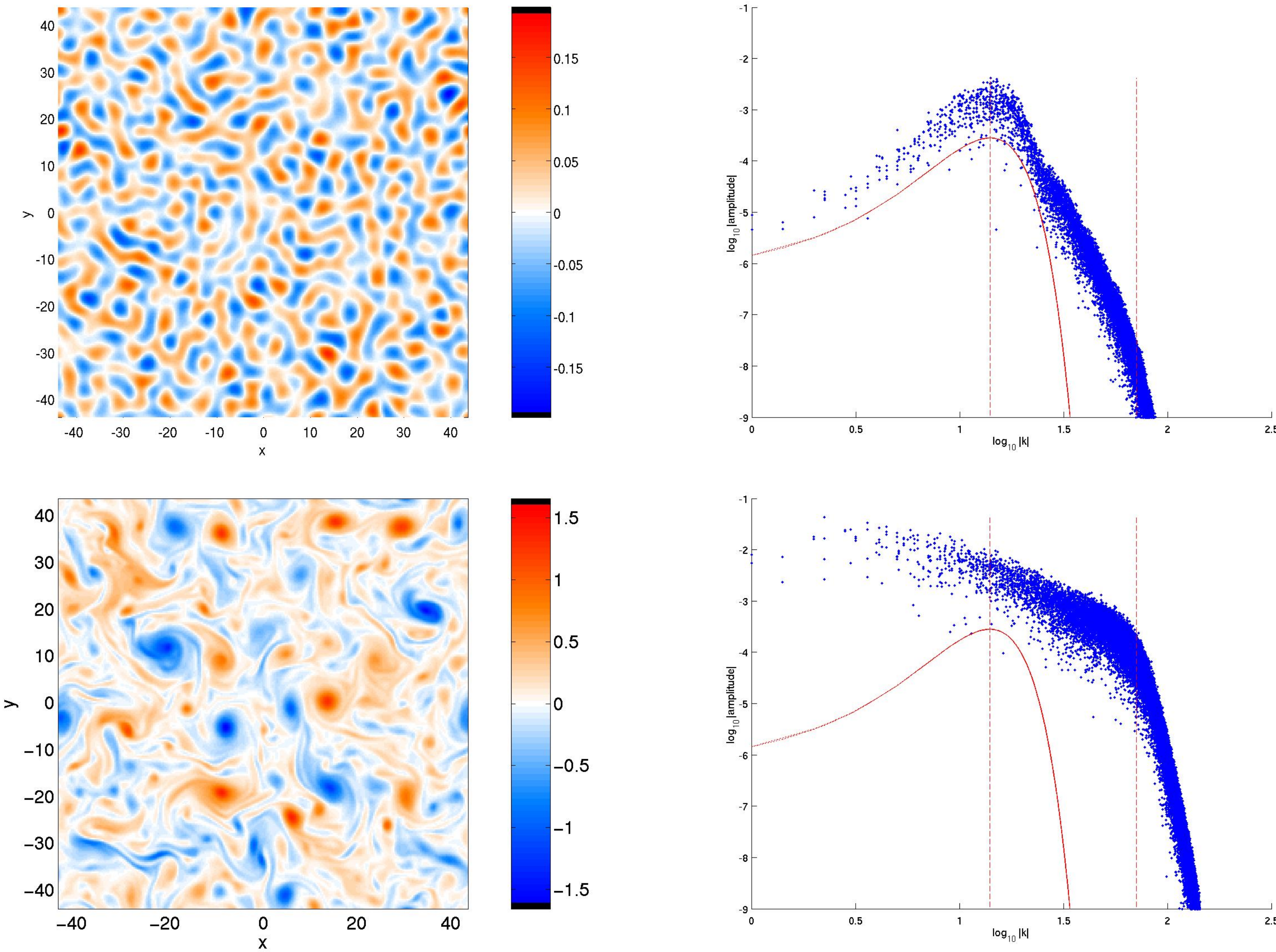
- $|\hat{F}(\vec{k}, t)|$ centred at intermediate scale k_0 with random phase

$$F(\vec{x}, t) = \iint \hat{F}(\vec{k}, t)\; e^{i\vec{k}\cdot\vec{x}}\; d\vec{k} \quad ; \quad \theta(\vec{x}, t) = \iint \hat{\theta}(\vec{k}, t)\; e^{i\vec{k}\cdot\vec{x}}\; d\vec{k}$$

Cascade in a Turbulent Fluid

Computational Results (Roy Wilds, 2003)

▷ early & late stages of vortex organization: $\hat{\theta}(\vec{k}, t)$ cascades to large & small scales



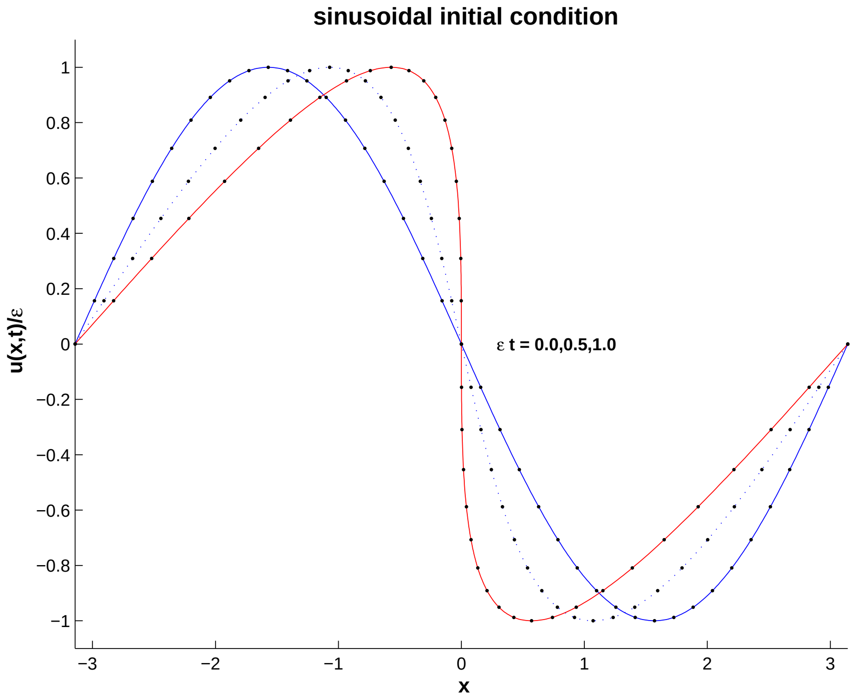
Simple Kinematic Wave Equation

Textbook Nonlinear PDE (Inviscid Burgers Equation)

- PDE of hyperbolic type, initial value problem for $u(x,t)$

$$u_t + u\,u_x = 0 \quad ; \quad u(x,0) = f(x)$$

- $exact$ solution by method of characteristics
- example of wave steepening & finite-time wavebreaking
- propagation of discontinuities determined by Rankine-Hugoniot conditions



- also embodies a simple one-dimensional cascade

An Exact Solution

Characteristic ODEs

- define characteristics as parametric curves $(x(s), t(s))$ originating from $(x_0, 0)$

$$\frac{dx}{ds} = u \quad ; \quad x(0) = x_0 \quad \longrightarrow \quad x = x_0 + s\, f(x_0)$$

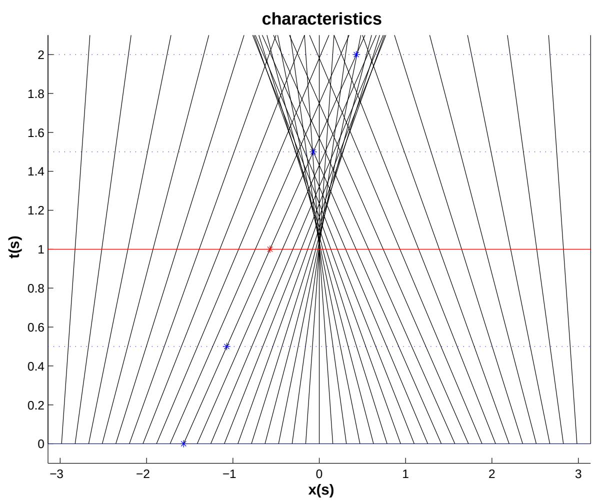
$$\frac{dt}{ds} = 1 \quad ; \quad t(0) = 0 \quad \longrightarrow \quad t = s$$

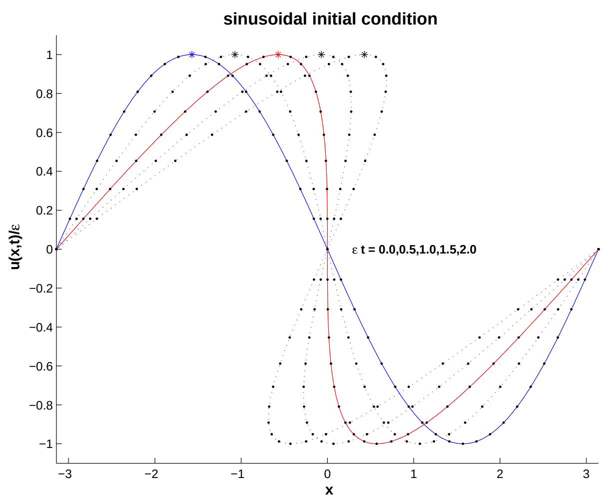
- PDE becomes ODE for $u(s)$ along each characteristic

$$\frac{du}{ds} = 0 \quad ; \quad u(0) = f(x_0) \quad \longrightarrow \quad u = f(x_0)$$

- parametric solution in terms of x_0 and t

$$u = f(x_0) \quad ; \quad x = x_0 + t\, f(x_0)$$





Series Representation

$$u(x,t) \quad = \quad a_0 + \sum_1^\infty \{a_n(t) \cos nx + b_n(t) \sin nx\}$$

Fourier Coefficients

- assume $u(x,t)$ continuous, prior to wavebreaking
- period integral of $u(x,t)$ is conserved $\longrightarrow$ time-independent mean

$$a_0 \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x_0)\ dx_0$$

- sine coefficient & integration by parts $(n \neq 0)$

$$b_n(t) \quad = \quad \frac{1}{\pi} \int_{-\pi}^{+\pi} u(x,t)\ \sin nx\ dx \quad = \quad \frac{1}{\pi n} \int_{-\pi}^{+\pi} u_x(x,t)\ \cos nx\ dx$$

- u_x via implicit differentiation of $x = x_0 + t\,u$

$$b_n(t) \quad = \quad \frac{1}{\pi n t} \int_{-\pi}^{+\pi} \left(1 - \frac{dx_0}{dx}\right) \cos nx\ dx \quad = \quad -\frac{1}{\pi n t} \int_{-\pi}^{+\pi} \cos nx\ dx_0$$

- introduce IVs via parametric solution $x = x_0 + t\, f(x_0)$

$$b_n(t) \quad = \quad -\frac{1}{\pi n t} \int_{-\pi}^{+\pi} \cos\left[n x_0 + n t\, f(x_0)\right]\ dx_0$$

$$a_n(t) \quad = \quad \frac{1}{\pi n t} \int_{-\pi}^{+\pi} \sin\left[n x_0 + n t\, f(x_0)\right]\ dx_0$$

Platzman's 1964 Solution

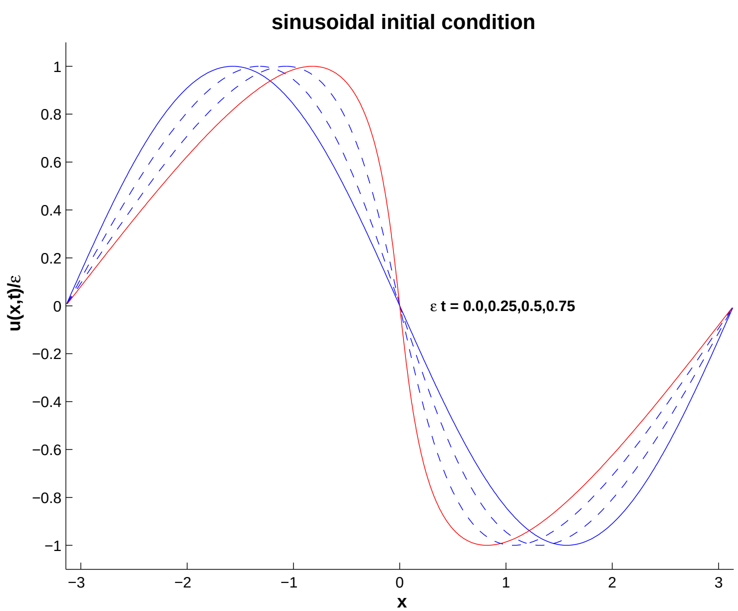
Sinusoidal Initial Condition: $f(x) = -\epsilon \sin x$

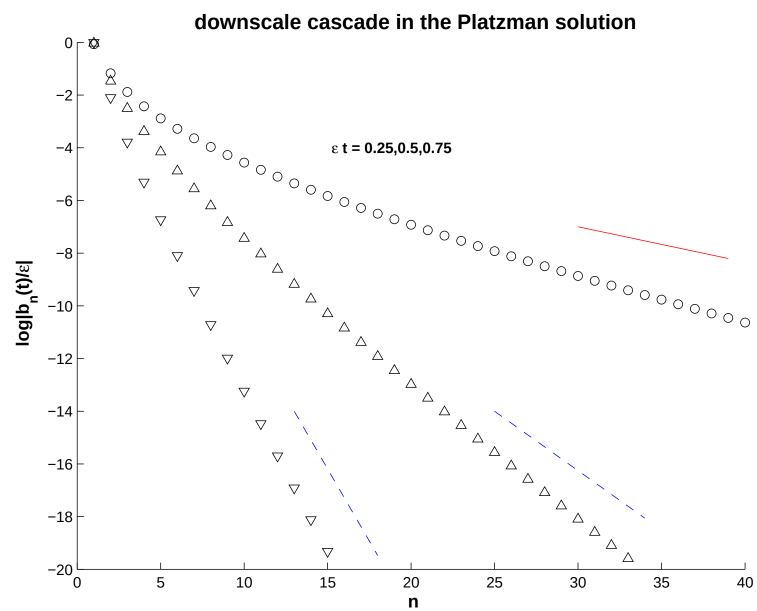
- solution remains a sine series for $t > 0$: $a_n(t) \equiv 0$
- sine coefficient is integral representation for $J_n(\cdot)$ Bessel function

$$b_n(t) \quad = \quad -\frac{1}{\pi n t}\int_{-\pi}^{+\pi} \cos\left[n x_0 - n\epsilon t \, \sin x_0\right] \, dx_0 \quad = \quad -2\,\frac{J_n(n\epsilon t)}{nt}$$

- exact series solution with time-dependent fourier coefficients $\longrightarrow$ cascade

$$u(x,t) \quad = \quad \sum_1^{\infty} -2\,\frac{J_n(n\epsilon t)}{nt} \, \sin nx$$





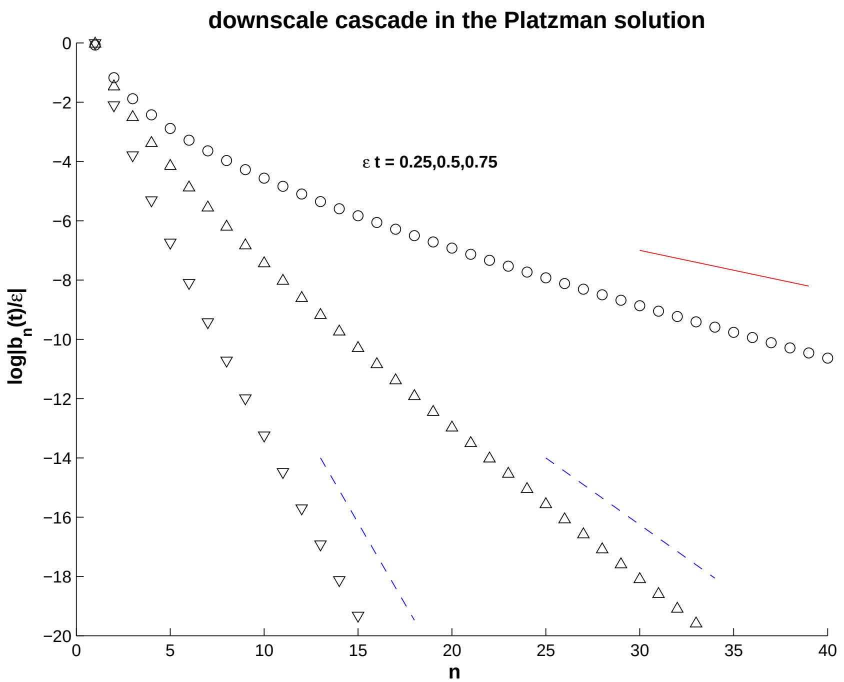
Large Wavenumber Asymptotics

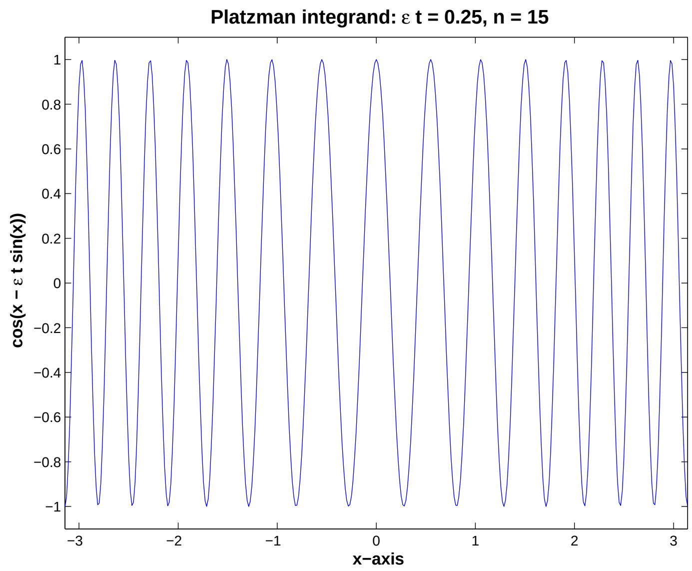
- Bessel function asymptotics for large index & argument $(n \longrightarrow \infty, \epsilon t < 1)$

$$|b_n(t)| \sim \sqrt{\frac{2}{\pi \tanh \alpha}}\; t^{-1}\; n^{-3/2}\; e^{n(-\alpha + \tanh \alpha)} \quad ; \quad \cosh \alpha = \frac{1}{\epsilon t}$$

- spectral slope, $n \longrightarrow \infty$

$$\frac{\ln |b_n(t)|}{n} \sim \ln\left(\frac{\epsilon t}{2}\right) + \sqrt{1-\epsilon^2 t^2} - \ln\left(\frac{1+\sqrt{1-\epsilon^2 t^2}}{2}\right)$$





Highly-Oscillatory Integrand

- ▷ can we obtain spectral slope without Platzman's series solution?
- ▷ sine coefficient in complex exponential form

$$b_n(t) \quad = \quad -\frac{1}{\pi n t}\,\mathrm{Re}\left\{\int_{-\pi}^{+\pi} e^{in(x_0-\epsilon t\,\sin x_0)}\,dx_0\right\}$$

- ▷ large n integral asymptotics

  $\longrightarrow$ no stationary-phase point for $\epsilon t < 1$

  $\longrightarrow$ periodicity neutralizes integration by parts

  $\longrightarrow$ complex analysis & <u>steepest descent methods</u>

Path Deformation & Complex Phase

▷ consider integral in complex z-plane by analytic continuation

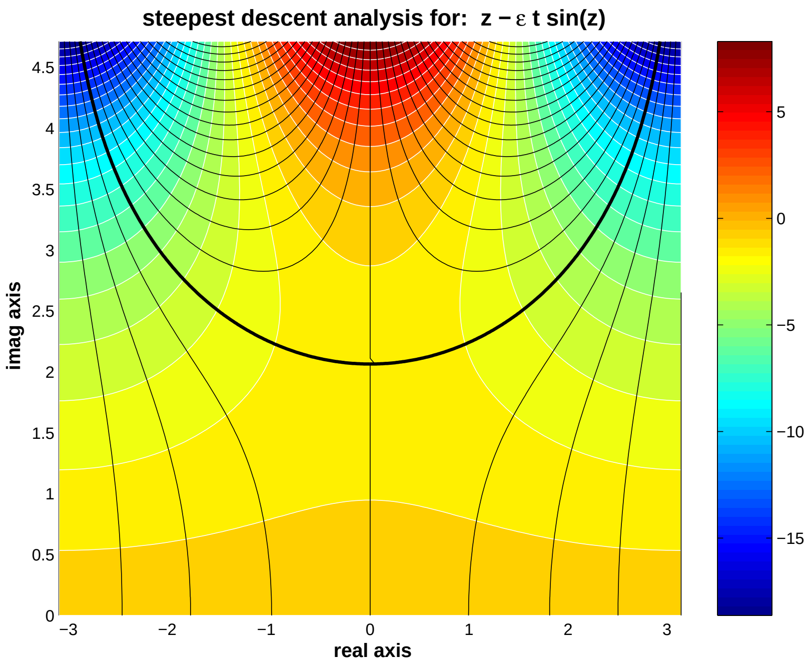
$$b_n(t) = -\frac{1}{\pi n t} \, \mathrm{Re} \left\{ \int_{\mathcal{C}} e^{in(z-\epsilon t \sin z)} \, dz \right\}$$

▷ integrand is 2π-periodic & has no singularities

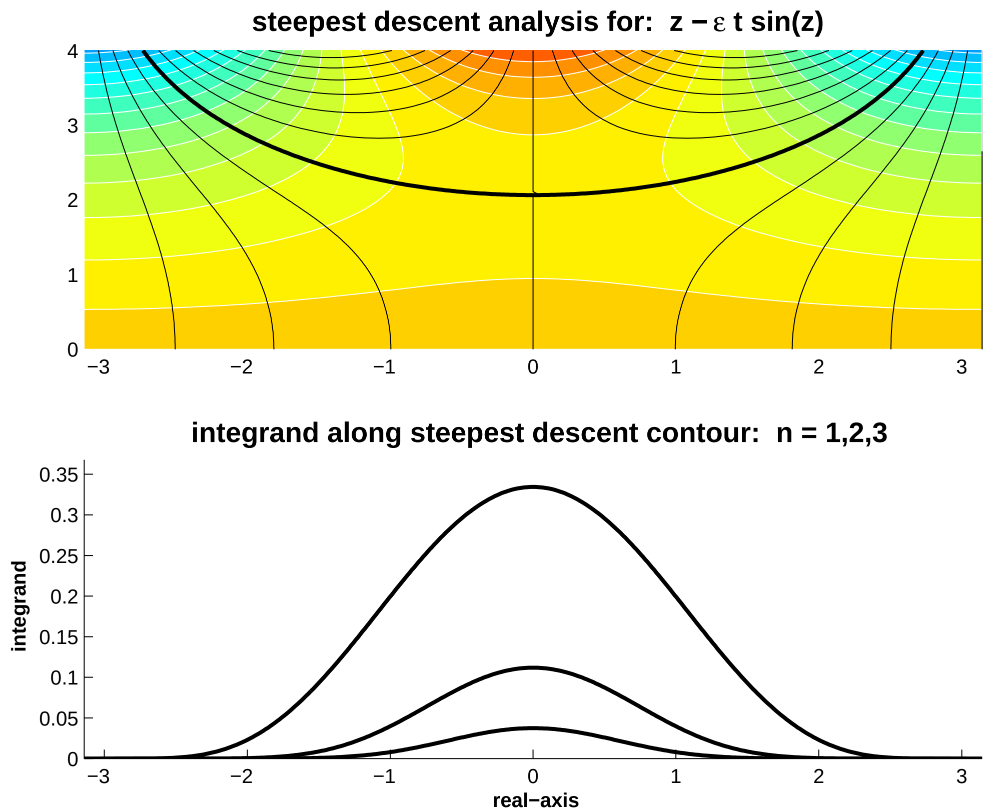
▷ complex analysis of phase function, $\phi(z) = i(z - \epsilon t \sin z)$, for $\epsilon t = 0.25$

$\longrightarrow$ blue indicates regions of negative Re(ϕ): exponentially small integrand

$\longrightarrow$ black contours are curves of constant Im(ϕ): paths of non-oscillation



Method of Steepest Descent



Saddlepoint Contribution

- ▷ maximum of integrand occurs at $z = i\alpha$ where $\cosh\alpha = \dfrac{1}{\epsilon t}$
- ▷ for $n \longrightarrow \infty$, integrand localizes near saddlepoint like a gaussian

$$b_n(t) \quad \sim \quad -\frac{1}{\pi n t}\, e^{n(-\alpha + \tanh\alpha)} \int_{-\infty}^{+\infty} e^{-(n \tanh\alpha) x^2/2}\, dx$$

Other Initial Conditions?

$$b_n(t) - i\,a_n(t) \;=\; -\frac{1}{\pi n t}\int_{\mathcal{C}} e^{in(z+tf(z))}\,dz$$

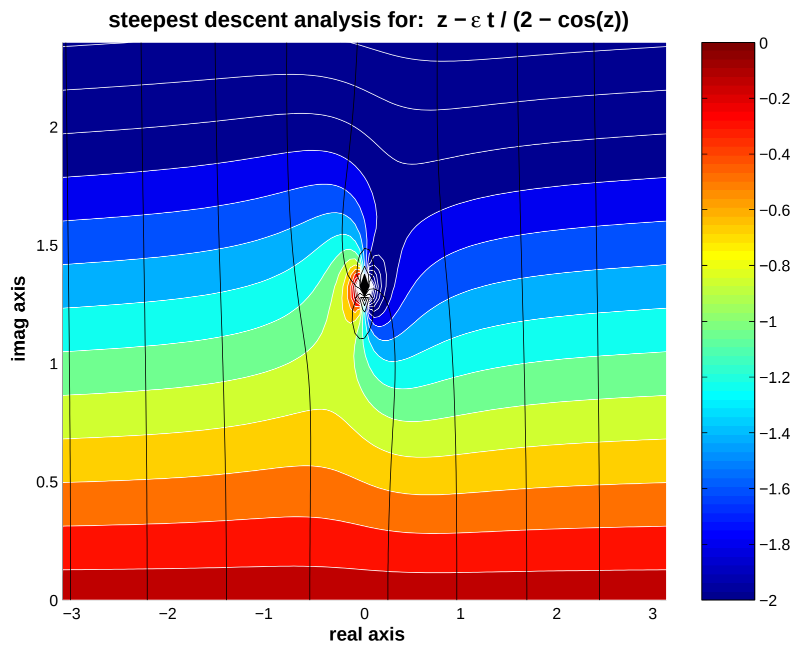
Saddlepoint & Non-Saddlepoint Contributions

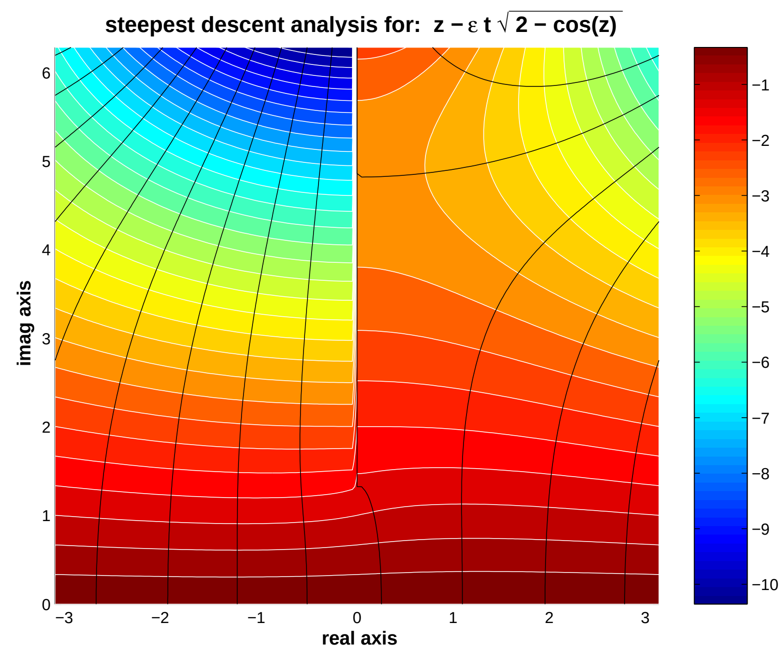
- pole from essential singularity?

$$f(x) = -\,\frac{\epsilon}{2-\cos x}$$

branch point singularity?

$$f(x) = -\,\epsilon\sqrt{2-\cos x}$$





- are there any other special cases where spectral slopes can be obtained?

An Odd Quadrature Solution

When is an Integral not an Integral?

- substitute integral coefficients into series, interchange summation & integration

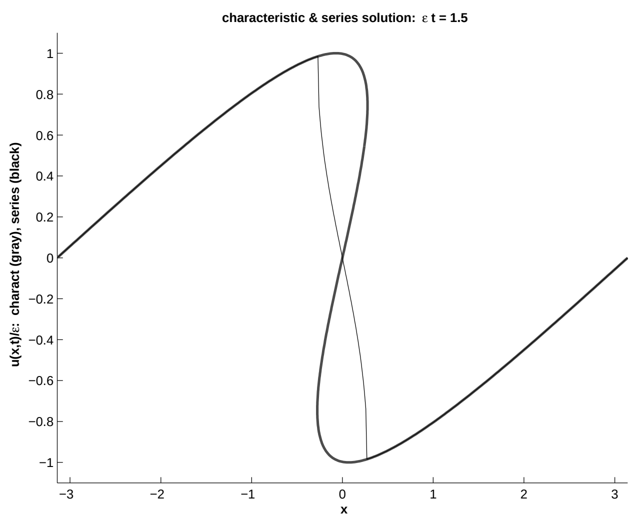
$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \frac{1}{\pi n t} \int_{-\pi}^{+\pi} \sin n[x - x_0 - t\, f(x_0)]\; dx_0$$

$$= a_0 + \frac{1}{t} \int_{-\pi}^{+\pi} \left\{ \sum_{n=1}^{\infty} \frac{\sin n[x - x_0 - t\, f(x_0)]}{\pi n} \right\} dx_0$$

- series can be summed

$$u(x,t) = a_0 + \frac{1}{t} \int_{-\pi}^{+\pi} \left\{ \left(\frac{x - x_0 - t\, f(x_0)}{2\pi} \bmod 1 \right) - \frac{1}{2} \right\}\; dx_0$$

- paradox: characteristic solution is local, solution cannot depend globally on initial function



Spectral Dynamics

$$u(x,t) \;=\; \sum_{1}^{\infty} b_n(t)\sin nx$$

Direct Substitution into $u_t + uu_x = 0$

- identify all terms which produce $\sin nx$

$$\begin{aligned}
\ldots\; b_n' \sin nx + \;\ldots\; + \sum_{1}^{n-1} &\quad k\, b_k\, b_{n-k}\;\cos kx\;\sin(n-k)x \quad + \;\ldots \\
\ldots + \sum_{1}^{\infty} &\quad k\, b_k\, b_{n+k}\;\cos kx\;\sin(n+k)x \quad + \;\ldots \\
\ldots + \sum_{1}^{\infty} &\quad (n+k)\, b_{n+k}\, b_k\;\cos(n+k)x\;\sin kx \quad + \;\ldots \;=\; 0
\end{aligned}$$

- spectral dynamics ODEs: <u>triad resonances</u>

$$b_n' \;=\; -\,\frac{n}{4}\sum_{1}^{n-1} b_k\, b_{n-k} \;+\; \frac{n}{2}\sum_{1}^{n-1} b_k\, b_{n+k} \;+\; \frac{n}{2}\sum_{n}^{\infty} b_k\, b_{n+k}$$

$\longrightarrow$ 1^{st}-sum: downscale transfer from smaller wavenumber, long waves

$\longrightarrow$ 2^{nd}-sum: mixing transfer from straddling wavenumbers

$\longrightarrow$ 3^{rd}-sum: upscale transfer from larger wavenumber, short waves

Spectral ODEs

$$b'_n \;=\; -\frac{n}{4}\sum_{1}^{n-1} b_k\, b_{n-k} \;+\; \frac{n}{2}\sum_{1}^{n-1} b_k\, b_{n+k} \;+\; \frac{n}{2}\sum_{n}^{\infty} b_k\, b_{n+k}$$

Solution Strategies

- Platzman solution is exact for initial conditions, $\{b_n(0)\} = \{-\epsilon, 0, 0, \ldots\}$

$$\text{slope} \;=\; \ln\left(\frac{\epsilon t}{2}\right) \;+\; \sqrt{1-\epsilon^2 t^2} \;-\; \ln\left(\frac{1+\sqrt{1-\epsilon^2 t^2}}{2}\right)$$

- downscale transfer only solution, asymptotically valid for $0 \le \epsilon t \ll 1$

$$b_n(t) \;\sim\; -\epsilon\,\frac{n^{n-1}}{n!}\left(\frac{\epsilon t}{2}\right)^{n-1} \qquad \longrightarrow \qquad \text{slope} \;\approx\; \ln\left(\frac{\epsilon t}{2}\right) \;+\; 1$$

$\longrightarrow$ first Taylor term of Platzman solution for small ϵt & Stirling approximation

- small ϵt perturbation series

$$\left.\begin{aligned}
b_1(t) &\sim -\epsilon + \epsilon\left(\frac{\epsilon t}{2}\right)^2 + \ldots \\
b_2(t) &\sim -\epsilon\left(\frac{\epsilon t}{2}\right) + \ldots \\
b_3(t) &\sim -\epsilon\,\frac{3}{2}\left(\frac{\epsilon t}{2}\right)^2 + \ldots
\end{aligned}\right\} \;\longrightarrow\; \text{slope} \;\approx\; \ln\left(\frac{\epsilon t}{2}\right)$$

Cascade Solutions

- is there a general approach for constructing approximate solutions that embody spectral cascade?

Perturbation Expansion for $f(x) = O(\epsilon)$

- PDE for disturbance about initial condition: $u(x,t) = f(x) + \hat{u}(x,t)$

$$\hat{u}_t = -ff_x - (f\hat{u})_x - \hat{u}\hat{u}_x \quad ; \quad \hat{u}(x,0) = 0$$

- simple iterative construction $\hat{u} \sim \hat{u}_2(x,t) + \hat{u}_3(x,t) + \hat{u}_4(x,t) + \ldots$

$$\begin{aligned} (\hat{u}_2)_t &= -ff_x \\ (\hat{u}_3)_t &= -(f\hat{u}_2)_x \\ (\hat{u}_4)_t &= -(f\hat{u}_3)_x - (\hat{u}_2)(\hat{u}_2)_x \end{aligned}$$

$\longrightarrow$ generates polynomial-in-ϵt solutions; $\hat{u}_2$-error is $O(\epsilon^3)$

$\longrightarrow$ solution up to $\hat{u}_k$ contains wavenumbers up to k — partial cascade only

- linearizing truncation

$$(\hat{u}_2)_t + (f\hat{u}_2)_x = -ff_x$$

$\longrightarrow$ non-constant coefficient PDE; $\hat{u}_2$-error is $O(\epsilon^4)$

$\longrightarrow$ solution by characteristics

Linearizing Truncation

Characteristic ODEs

- define characteristics as parametric curves $(x(s), t(s))$ originating from $(x_0, 0)$

$$\frac{dx}{ds} = f(x) \qquad ; \qquad x(0) = x_0$$

$$\frac{dt}{ds} = 1 \qquad ; \qquad t(0) = 0$$

- f times PDE becomes ODE for $f\hat{u}_2$ along each characteristic

$$\frac{d}{ds}(f\hat{u}_2) = -\frac{1}{2}f(f^2)_x = -\frac{1}{2}\frac{d}{ds}(f^2) \qquad ; \qquad u(0) = 0$$

- can integrate to obtain $\hat{u}_2$

$$\hat{u}_2 = \frac{1}{2}\,\frac{f^2(x_0) - f^2(x)}{f(x)}$$

$\longrightarrow$ but relation determining $x_0(x,t)$ requires solution to nonlinear ODE

Platzman to the Rescue (again)

- exact characteristics for $f(x) = -\,\epsilon \sin x$

$$\frac{dx}{dt} = -\epsilon \sin x \quad ; \quad x(0) = x_0 \qquad \longrightarrow \qquad \sin x_0 = \frac{\operatorname{sech} \epsilon t}{1 - \tanh \epsilon t\ \cos x} \sin x$$

Approximate Cascade Solution

Two Small ϵt Asymptotic Miracles

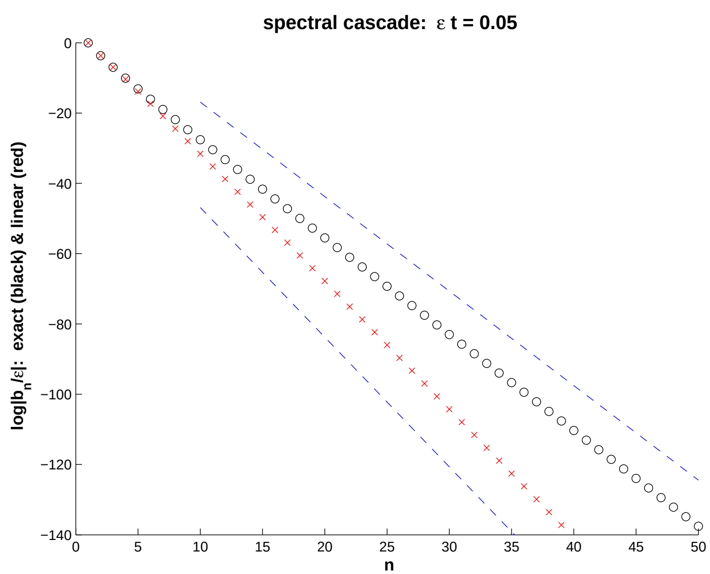
- formally $O(\epsilon^3)$-accurate solution from linearizing truncation

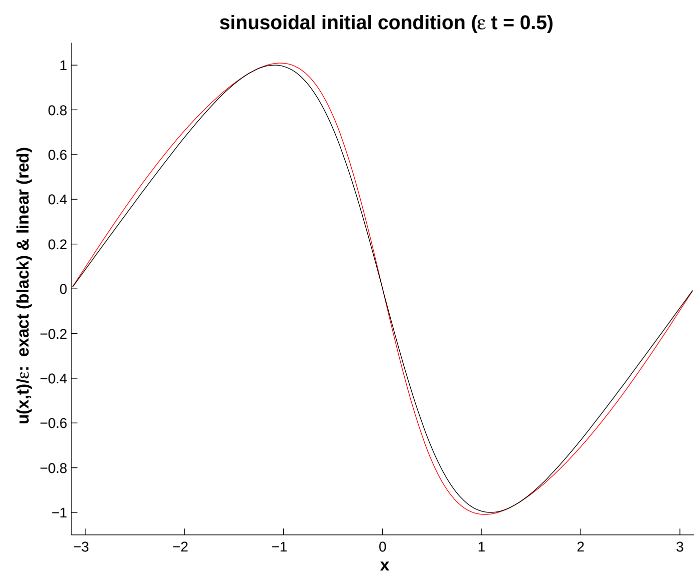
$$u(x,t) \sim -\epsilon \sin x + \frac{\epsilon}{2}\left(1 - \frac{\operatorname{sech}^2 \epsilon t}{(1-\tanh \epsilon t\ \cos x)^2}\right) \sin x$$

- need Fourier series representation

$$u(x,t) \sim -\frac{\epsilon}{2}\left\{ \sin x + \operatorname{sech}^2\left(\frac{\epsilon t}{2}\right) \sum_1^\infty n \tanh^{n-1}\left(\frac{\epsilon t}{2}\right) \sin nx \right\}$$

$\longrightarrow$ contains a cascade with spectral slope $= \ln\left(\tanh \frac{\epsilon t}{2}\right) \sim \ln\left(\frac{\epsilon t}{2}\right)$ for $0 < \epsilon t \ll 1$

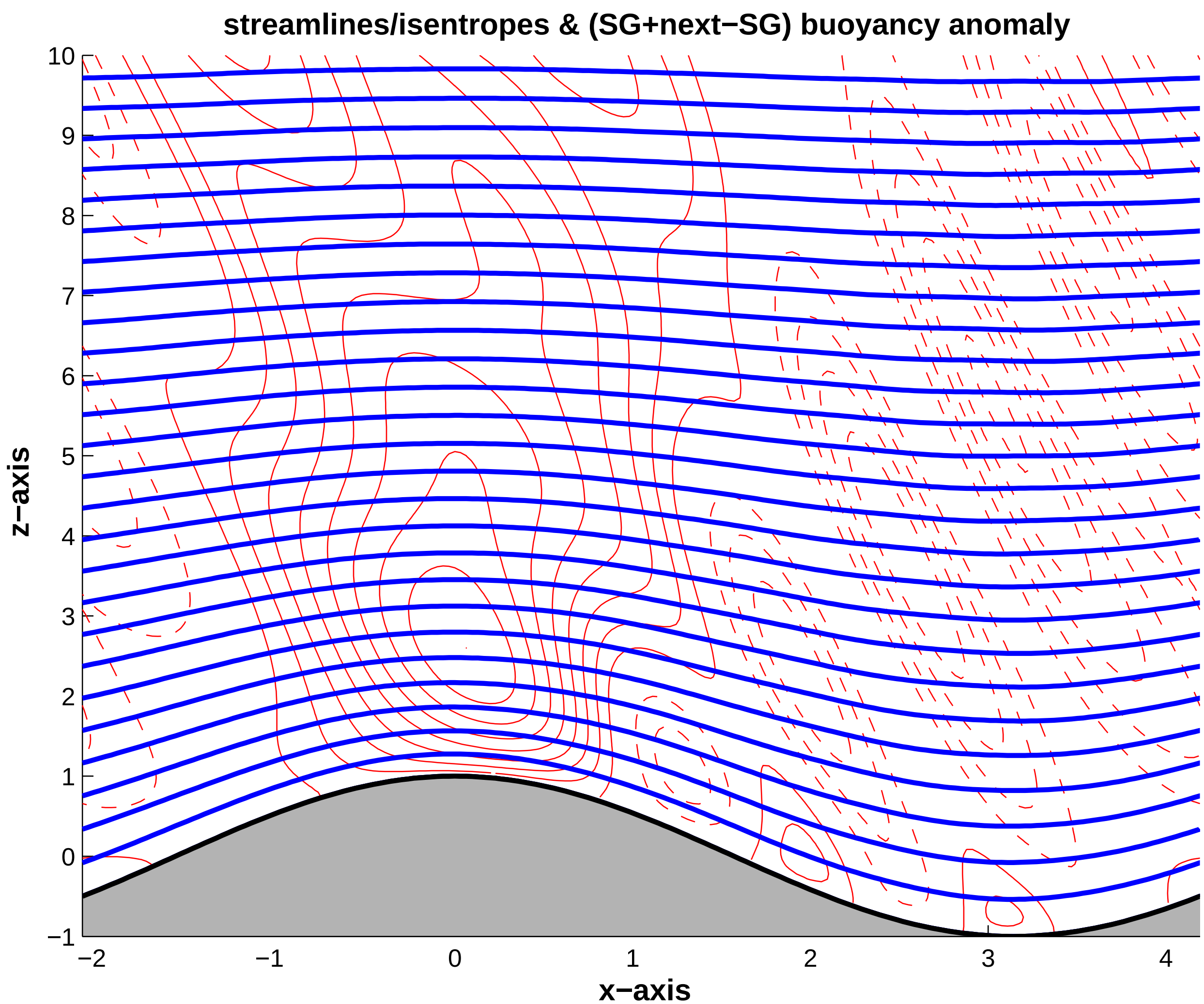




2D Flow Over Sinusoidal Topography

Small Rossby Number Limit ($\mathcal{R} = 1/4$) & Large Amplitude Topography

▷ no visible waves in streamlines, but $n = 4$ short waves are present via weak nonlinear cascade!



Quantifying the Weak Cascade

Simple Quantitative Illustration of a Spectral Cascade

- ▷ perturbative construction of a solution containing approximate cascade
- ▷ use Platzman solution for kinematic wave equation as benchmark
- ▷ linearizing truncation uses non-constant coefficient in PDE to generate cascade
  - → robust methodology may be adapted to more difficult problems
  - → formal accuracy is not improved
  - → yet, full spectral content with leading behaviour of spectral slope
- ▷ likely new general results on an old problem: $u_t + uu_x = 0$
  - → fourier series solution for continuous evolutions
  - → exact characteristic solution of linearizing truncation about initial conditions

Emerging Applications to Atmospheric Fluid Dynamics

- ▷ generation of short-scale inertia-gravity waves by large-scale flows