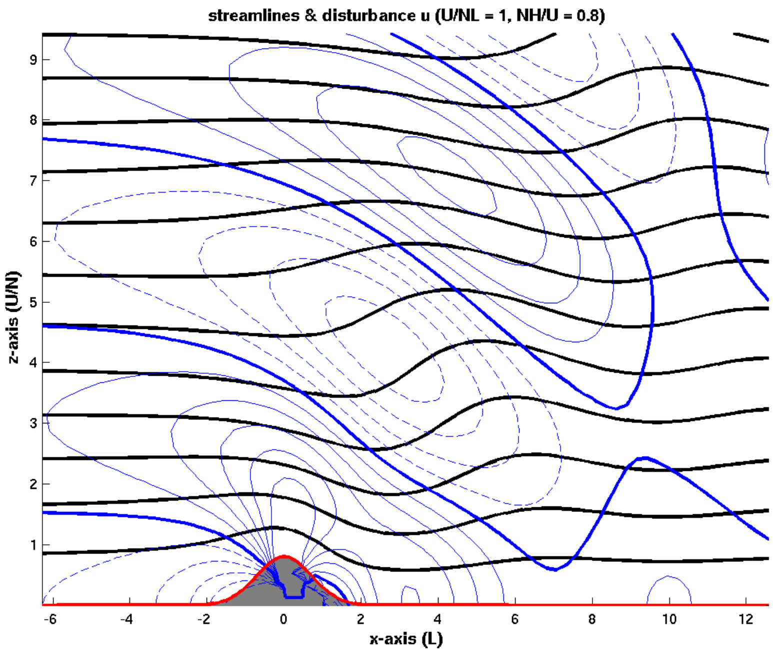
Waves Generated by Airflow over a Mountain Ridge

- a Helmholtz theory for steady waves in density-stratified flow
- linear instability & a critical resonant triad



http://www.fridgeproductions.pwp.blueyonder.co.uk/

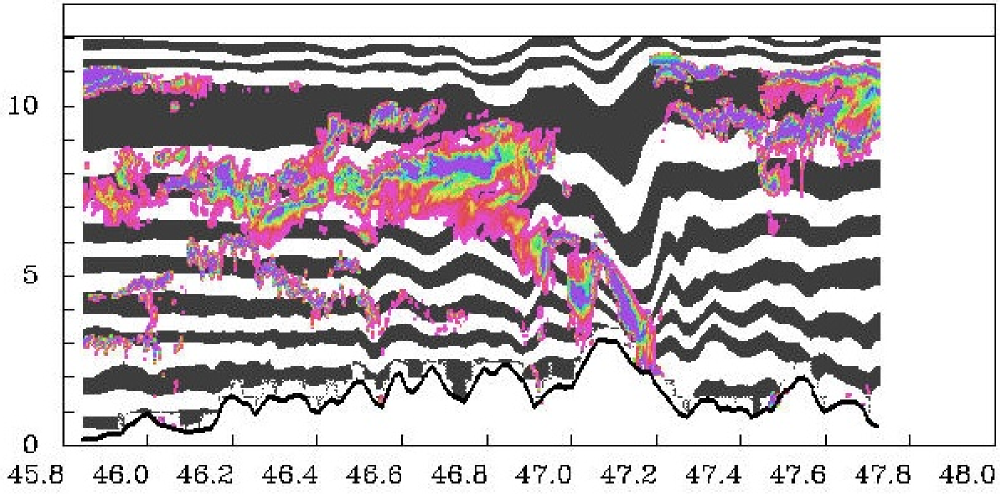
- Dave Muraki, Simon Fraser University
- Youngsuk Lee & David Alexander, SFU
- Craig Epifanio, Texas A&M

Topographic Gravity Waves

Atmospheric Concerns

http://www.casa.gov.au/avreg/fsa/download/02jan/ATSB.pdf

http://users.snowcrest.net/weshawk/LayeredLentic.jpeg



Volkert, et.al. 2002

- ▷ mathematical story: idealized steady 2D flows & their stability

Fluid Dynamics & Thermodynamics

- incompressible 2D Euler equations with Boussinesq buoyancy

$$\begin{aligned} u_x + w_z &= 0 \\ \frac{Du}{Dt} &= -\phi_x \\ \frac{Dw}{Dt} - B &= -\phi_z \\ \frac{DB}{Dt} &= 0 \end{aligned}$$

- adiabatic buoyancy, B (buoyant $\leftrightarrow$ light) & geopotential, ϕ (pressure)
- 2D advection: $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\,\dfrac{\partial}{\partial x} + w\,\dfrac{\partial}{\partial z}$

Streamfunction & Vorticity

- streamfunction, $\Psi \quad \longrightarrow \quad u = \Psi_z \quad ; \quad w = -\Psi_x$
- vorticity, $\eta \quad \longrightarrow \quad \eta = u_z - w_x = \nabla^2\Psi$

Stratified Potential Flow

Vorticity/Buoyancy Formulation

- vorticity inversion: $\nabla^2\Psi = \eta$

$$\begin{aligned}\frac{D\eta}{Dt} + B_x &= \eta_t + J(\eta,\Psi) + B_x = 0 \\ \frac{DB}{Dt} &= B_t + J(B,\Psi) = 0\end{aligned}$$

- 2D streamfunction advection $\longrightarrow$ Jacobian determinant

$$J(f,\Psi) = \begin{vmatrix} f_x & \Psi_x \\ f_z & \Psi_z \end{vmatrix} = \begin{vmatrix} f_x & -w \\ f_z & u \end{vmatrix} = u f_x + w f_z$$

Steady Flow

- zero Jacobian condition: $J(B,\Psi) = 0 \longrightarrow B$ is constant along streamlines
- upstream/mean conditions (uniform wind & constant stratification):

$$\left.\begin{aligned}\Psi &= \mathcal{U}\, z + \psi \\ B &= \mathcal{N}^2 z + b\end{aligned}\right\} \longrightarrow \quad B = \frac{\mathcal{N}^2}{\mathcal{U}}\,\Psi$$

- zero Jacobian condition: localized disturbance streamfunction, $\psi(x,z)$

$$J(\eta,\Psi) + \frac{\mathcal{N}^2}{\mathcal{U}}\,\Psi_x = J\left(\nabla^2\psi + \left(\frac{\mathcal{N}}{\mathcal{U}}\right)^2\psi \;,\; \Psi\right) = 0$$

Long's Theory (1953)

Helmholtz Equation

- linear Helmholtz equation for steady 2D streamfunction, $\psi(x, z)$

$$\nabla^2\psi + \left(\frac{\mathcal{N}}{\mathcal{U}}\right)^2 \psi = 0$$

- special nonlinear solutions for constant stratification & uniform incident wind

Scales

- simple topographic case: three length scales

$\mathcal{U}/\mathcal{N}$ = wave scale ; H = mountain height ; L = mountain width

- two dimensionless parameters

$\mathcal{A} \equiv \frac{\mathcal{N}H}{\mathcal{U}}$, height parameter ; $\sigma \equiv \frac{\mathcal{U}}{\mathcal{N}L}$, nonhydrostatic parameter

Nondimensionalized Problem

- Helmholtz equation ($\sigma \longrightarrow 0$, hydrostatic case)

$$\sigma^2\psi_{xx} + \psi_{zz} + \psi = 0$$

- zero surface streamfunction: $\Psi(x, \mathcal{A}h(x)) = \mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = 0$

A Fourier Approach

Fourier Modes, $e^{i(kx+mz)}$

- Helmholtz dispersion relation: $m^2 = 1 - \sigma^2 k^2$
- sign choice $\longrightarrow$ far-field conditions: upward group velocity or decay (Queney, 1948)

$$m(k) = \begin{cases} \mathrm{sign}(k)\sqrt{1-\sigma^2 k^2} & \text{for } |\sigma k| \le 1 \text{ (long scale radiation)} \\ i\sqrt{\sigma^2 k^2 - 1} & \text{for } |\sigma k| \ge 1 \text{ (short scale decay)} \end{cases}$$

General Helmholtz Solution

- Fourier integral representation with far-field conditions

$$\psi(x,z) = -\mathcal{A} \int_{-\infty}^{+\infty} \hat{c}(k)\, e^{i(kx+m(k)z)}\, dk$$

- $z = \mathcal{A}h(x)$ surface condition: $\mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = 0$

$$h(x) - \int_{-\infty}^{+\infty} \hat{c}(k)\, e^{i(kx+m(k)\mathcal{A}h(x))}\, dk = 0$$

- linear integral operator on $\hat{c}(k)$ $\longrightarrow$ Fredholm integral equation of first-kind
- numerically equivalent to a matrix inversion

Direct Steady Solve

$$h(x) \;-\; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)\mathcal{A}h(x))}\; dk \;=\; 0$$

Numerical Discretization

- collocation points: $\{x_1 \;\ldots\; x_\alpha \;\ldots\; x_N\}$ & N knowns: $h_\alpha = h(x_\alpha)$
- wavenumbers: $\{k_1 \;\ldots\; k_\beta \;\ldots\; k_N\}$ & N unknowns: $\hat{c}_\beta \approx \hat{c}(k_\beta)$
- approximate integral for each x_α by trapezoidal rule over $\beta = 1 \ldots N$

$$h_\alpha \;-\; \sum_{\beta=1}^{N} \hat{c}_\beta\; \underbrace{e^{i(k_\beta x_\alpha+m(k_\beta)\mathcal{A}h(x_\alpha))}\; w_\beta\, \Delta k}_{\mathbf{K}_{\alpha,\beta}} \;=\; 0$$

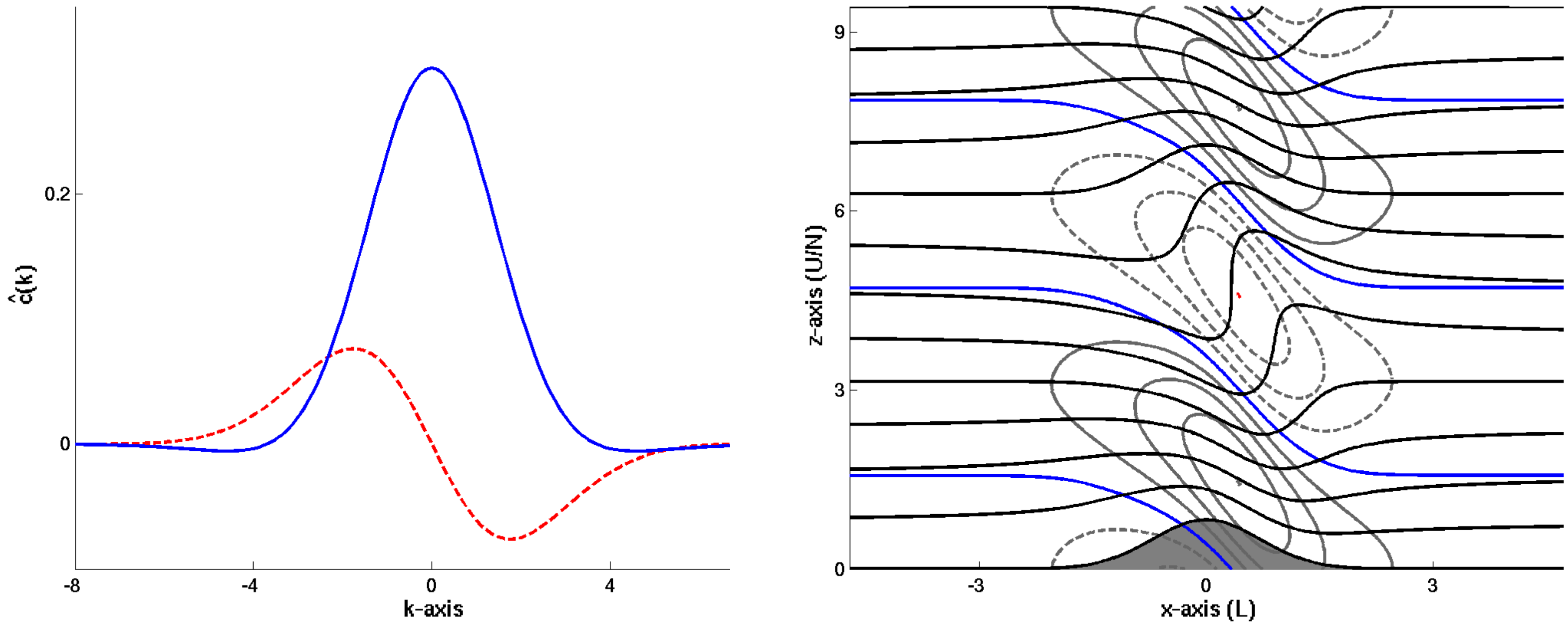
Matrix Inversion

- N linear equations in N unknowns: $(\vec{h_\alpha}) = [\mathbf{K}_{\alpha,\beta}]\;(\vec{c_\beta})$
- $m(k)$ is discontinuous at $k=0 \longrightarrow$ half-line integrals
- full matrix $\mathbf{K}$ can be ill-conditioned $\longrightarrow$ catastrophic loss of precision as N increases

Numerical Implementation

Fourier Conditioning

- for $\mathcal{A} = 0$ linear theory, discrete Fourier transform is well-conditioned
- equi-spaced discretizations with $\Delta k \, \Delta x = 2\pi/N$ is essential

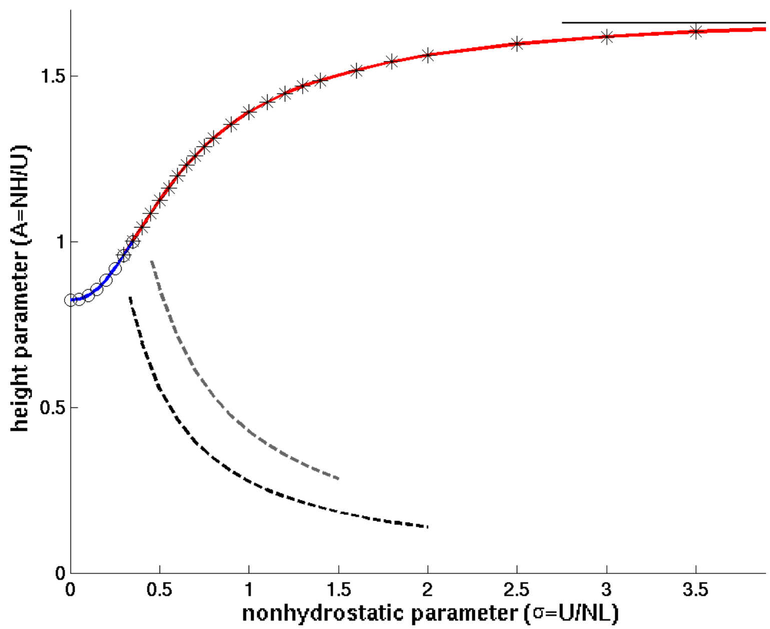


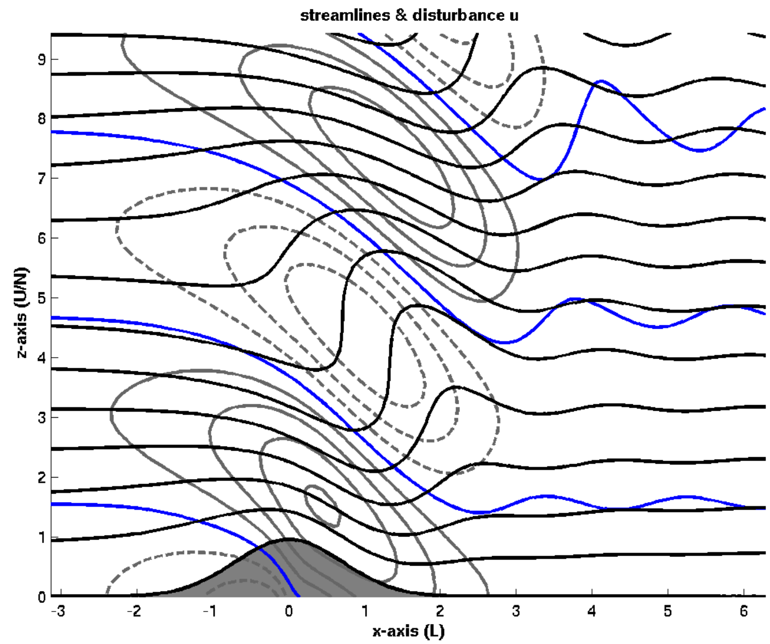
- hydrostatic ($\sigma = 0$), critical overturning ($\mathcal{A}_c = 0.82$) case for gaussian topography
  - $N = 256, x_\infty = 8\pi$: log-condition number = 2.85
- Fourier representation allows periodic wraparound $\longrightarrow$ large computational domains

Critical Overturning I

Gaussian Topography

- critical overturning height $\mathcal{A}_c(\sigma)$ as a function of nonhydrostatic parameter σ
- wavebreaking limit for static stability of density-stratified flow



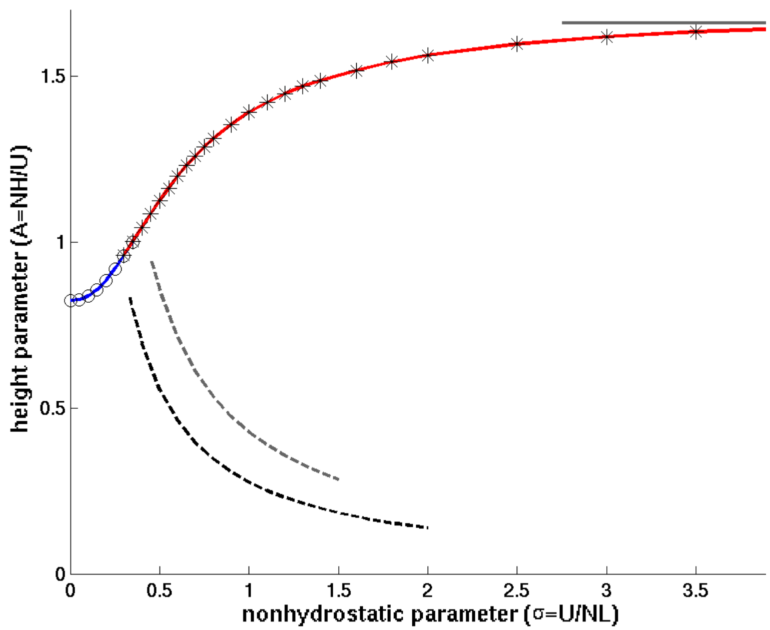


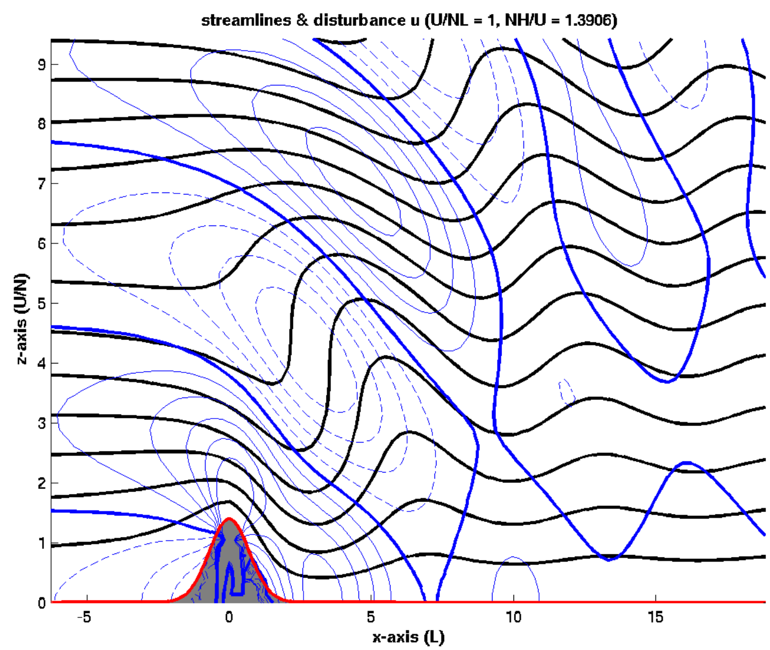
- Fourier formulation (o) limited by large condition numbers
  - ill-conditioning edge: 7 & 9 digits lost (- - -)
  - $\sigma = 0.35$ & $\mathcal{A}_c = 1.00$ shown above

Critical Overturning II

A Boundary Integral Method Talk Goes Here . . .

- for strongly nonlinear & nonhydrostatic flows ($\sigma \geq 0.3$)
- boundary integral method ($*$) remains well-conditioned ($\sigma = 1.0$ & $\mathcal{A}_c = 1.39$ below)





- second-kind Fredholm integral equation & non-standard Green's function $\mathcal{G}(\vec{x}_s, \vec{\xi})$ (Lyra, 1943)

$$h(\vec{x}_s) = \mu(\vec{x}_s) - 2 \int_S \mu(s) \frac{\partial \mathcal{G}}{\partial n}(\vec{x}_s - \vec{\xi}(s)) \, ds$$

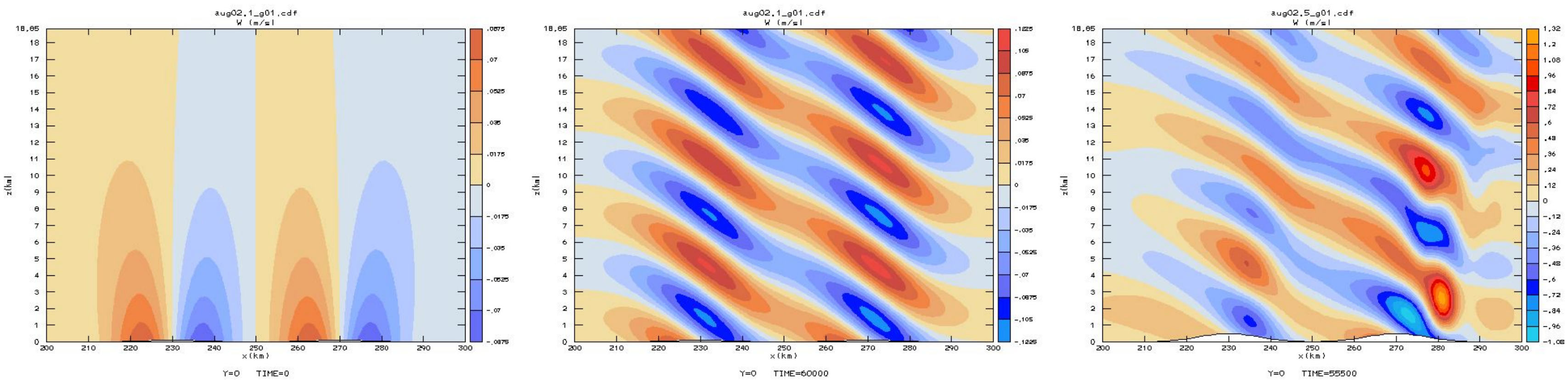
Question of Stability

Gravity Wave Instability

- ▷ Mied (1976), plane gravity waves are parametrically unstable
- ▷ Lilly/Klemp (1979), instability observed for sinusoidal topography
- ▷ Scinocca/Peltier (1994), unstable dynamics near critical overturning

Time-Dependent Simulations (Craig Epifanio, Texas A&M)

- ▷ twin peaks: hydrostatic ($\sigma = 0$), vertical motion w plots
  - ▷ initialized from potential flow
  - ▷ small height $\longrightarrow$ Long's steady solution is stable
  - ▷ medium height $\longrightarrow$ oscillatory instability to blow-up ($\mathcal{A} \approx 0.5$)



Linear Stability of Long's Steady Solutions

Hydrostatic ($\sigma = 0$) Disturbance Equations (David Alexander & Youngsuk Lee, SFU)

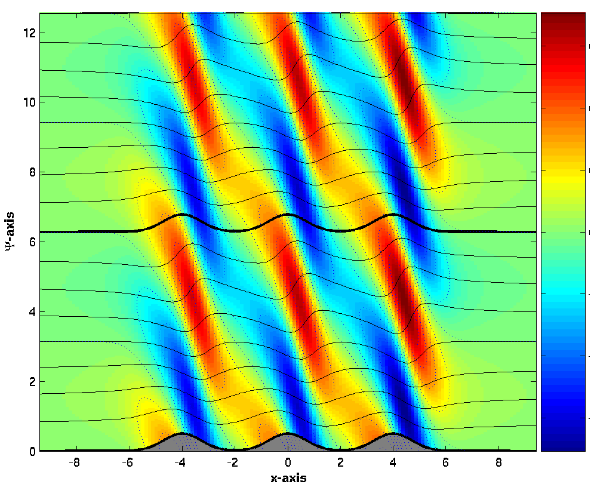
- non-constant coefficients from Long's streamfunction $\Psi(x,z)$

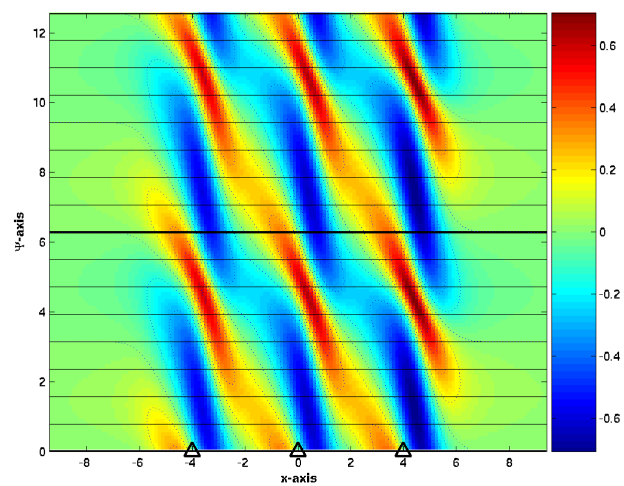
$$\begin{aligned}\tilde{\psi}_{zzt} &+ J(\tilde{\psi}_{zz} + \tilde{\psi}, \Psi) + (\tilde{b} - \tilde{\psi})_x = 0 \\ \tilde{b}_t &+ J(\tilde{b} - \tilde{\psi}, \Psi) = 0\end{aligned}$$

- 2D PDE eigenvalue problem for $\tilde{\psi} \longrightarrow \tilde{\psi}(x,z)e^{\lambda t}$ & $\tilde{b} \longrightarrow \tilde{b}(x,z)e^{\lambda t}$

Numerical Linear Algebra

- steady streamline coordinates $(x, \Psi(x,z)) \longrightarrow$ lower boundary at $\Psi = 0$
- self-adjoint formulation $\longrightarrow$ Arnoldi iterative search for eigenvalues (large & sparse)

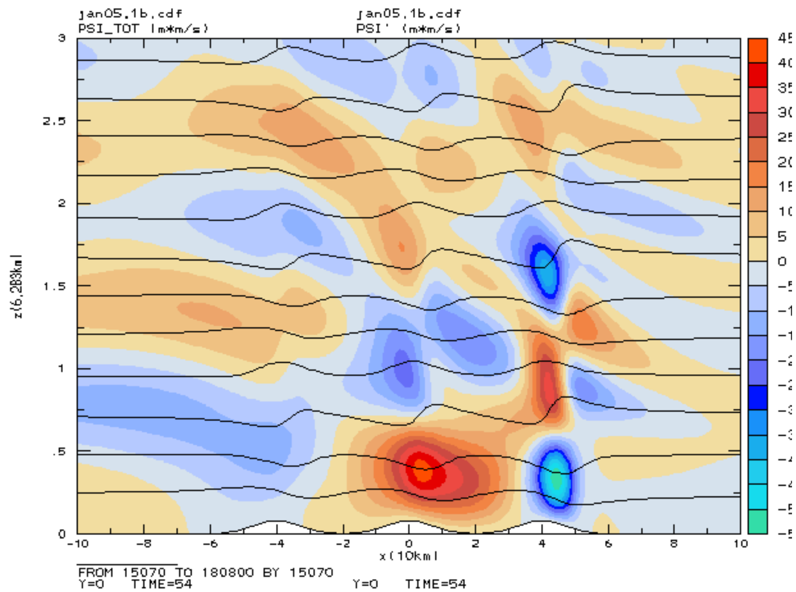


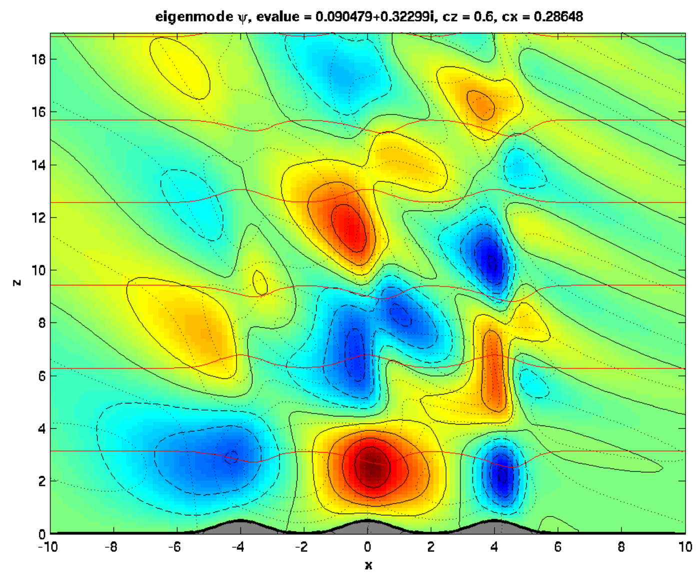


A Search for Eigenvalues

Simulated Instability vs Unstable Eigenfunction

▷ 3-peaks: a rough comparison of $\tilde{\psi}(x, z)$ . . .





Observations & Results

▷ growth rate (≈ 0.05) & frequency (≈ 0.29) $\leftrightarrow$ most unstable $\lambda = 0.09 + 0.32i$

▷ drift of cells upwind & upward from 3$^{\text{rd}}$ ridge

▷ sharp node line running upward from 3$^{\text{rd}}$ ridge

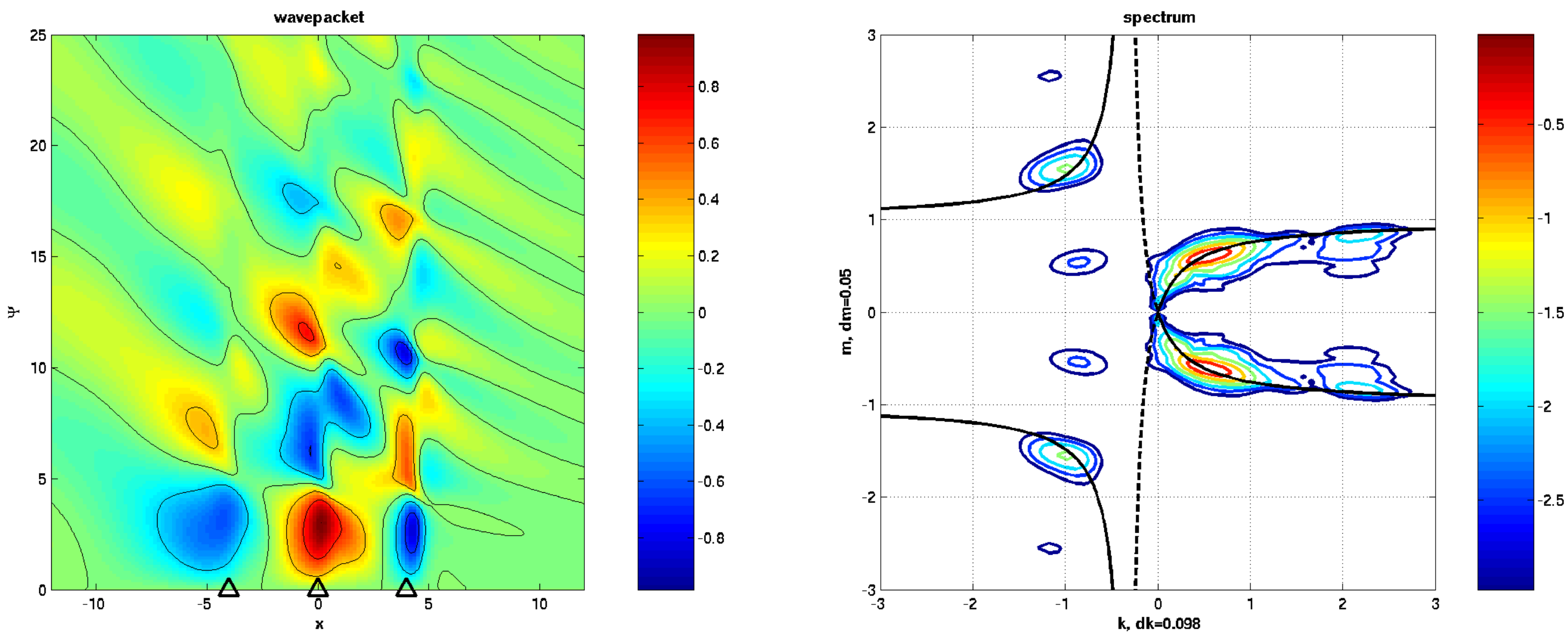
▷ cellular pattern above 1$^{\text{st}}$ ridge

▷ plane waves far upstream & downstream

An Idea from Turbulence Thinking

Look at Fourier Spectrum

- eigenmode of a non-constant coefficient PDE in a perturbed 1/2-space ($\lambda = 0.09 + 0.32i$)
- transform with odd extension (to $\Psi < 0$) in streamfunction coordinates



Linear Waves

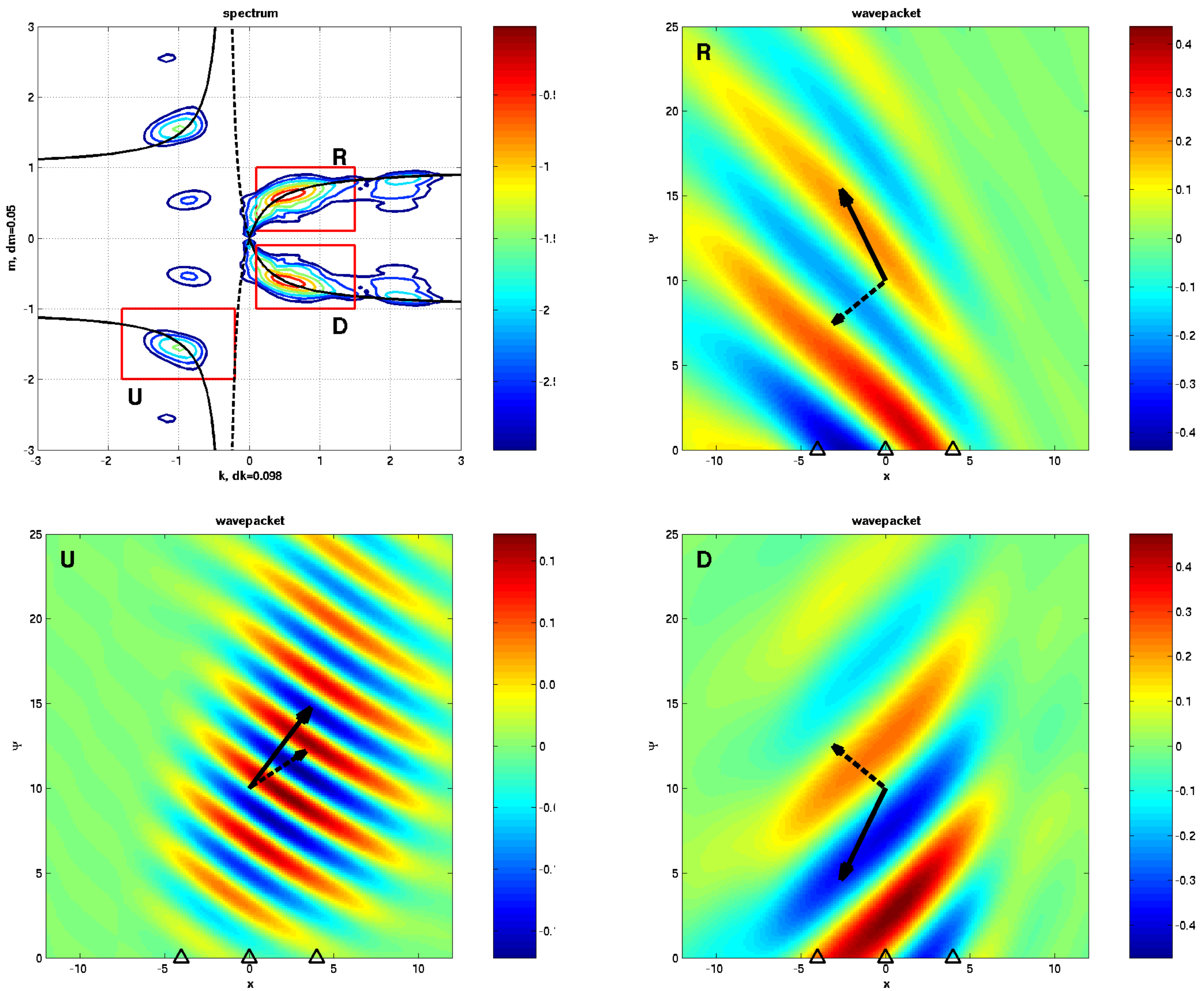
- Fourier spectrum concentrated on (undisturbed flow) dispersion relation: $\omega(k,m) = -0.32$

$$\omega(k,m) = k \mp \frac{k}{|m|} \qquad ; \qquad \vec{c}_g(k,m) = \left(1 \mp \frac{1}{|m|} \;,\; \frac{k\,|m|}{m^2}\right)$$

- eigenmode is primarily a superposition of linear waves!

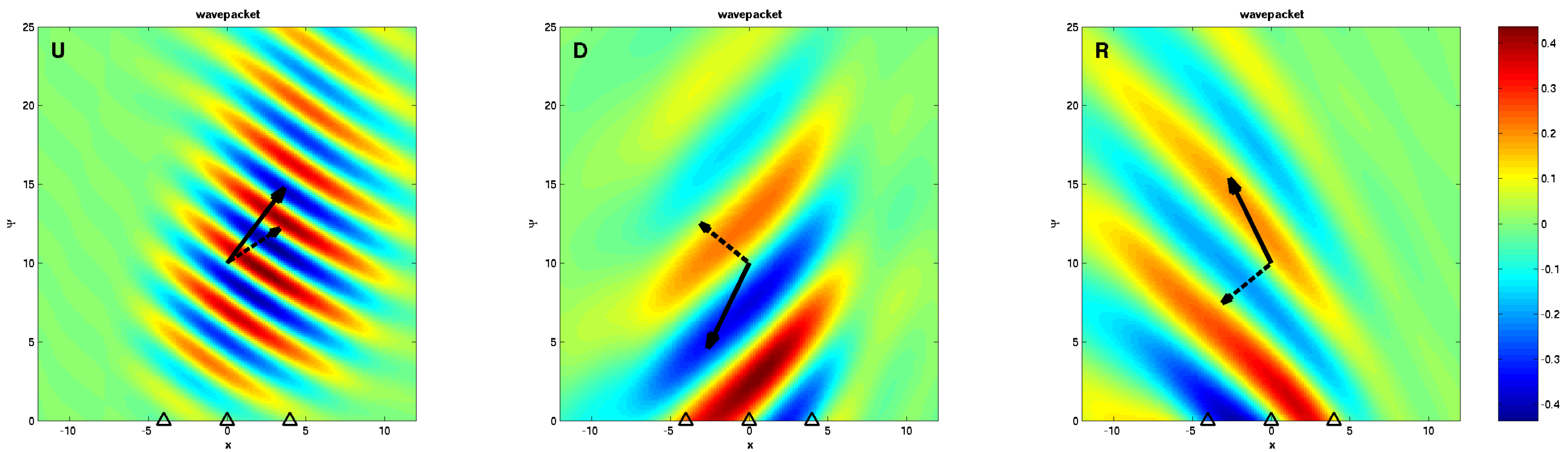
Spectral Wavepackets

Inversion of Spectral Peaks



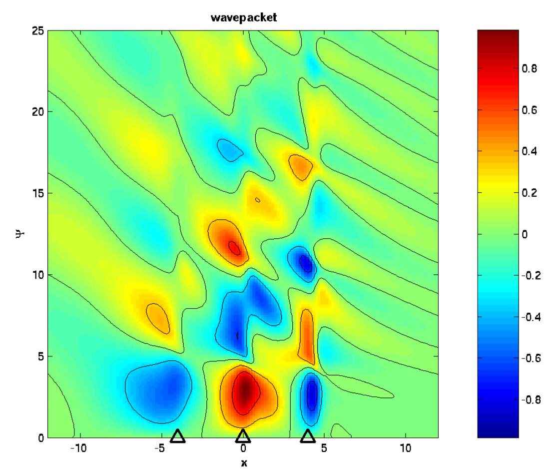
Wavepacket Interference

Phase (- -) & Group (—) Velocity Dynamics



Observations Again

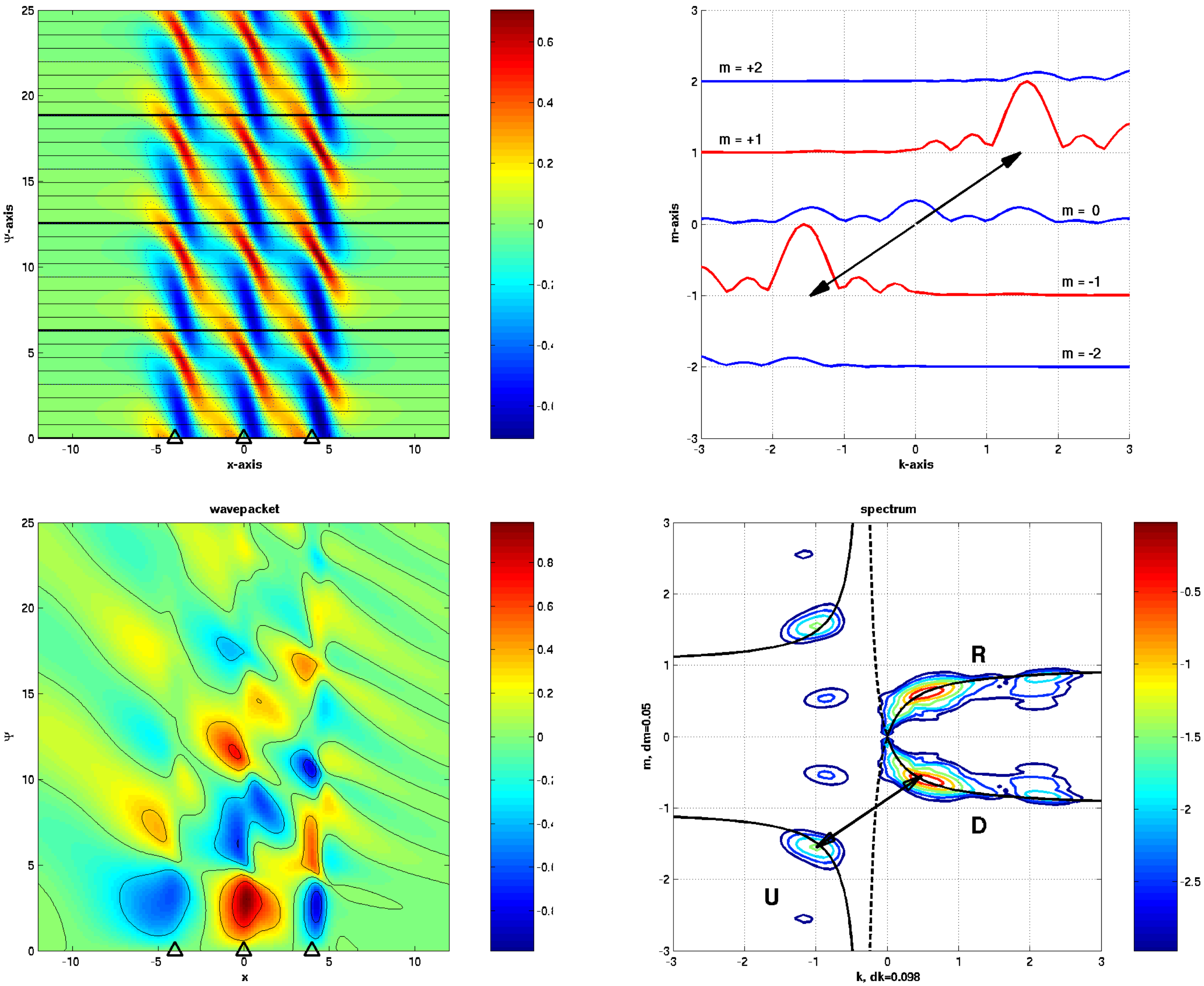
- ▷ wavepackets satisfy $\omega(k, m) = -\mathrm{Im}(\lambda) = -0.32$
- ▷ drift of cells upwind & upward from 3^{rd} ridge
- ▷ sharp node line running upward from 3^{rd} ridge
- ▷ cellular pattern above 1^{st} ridge
- ▷ plane waves far upstream & downstream



What Mechanism Generates the U & D Wavepackets?

A Resonant Triad

Fourier Wavevectors: steady flow & eigenfunction ($\vec{k}_U + \vec{k}_s = \vec{k}_D$)

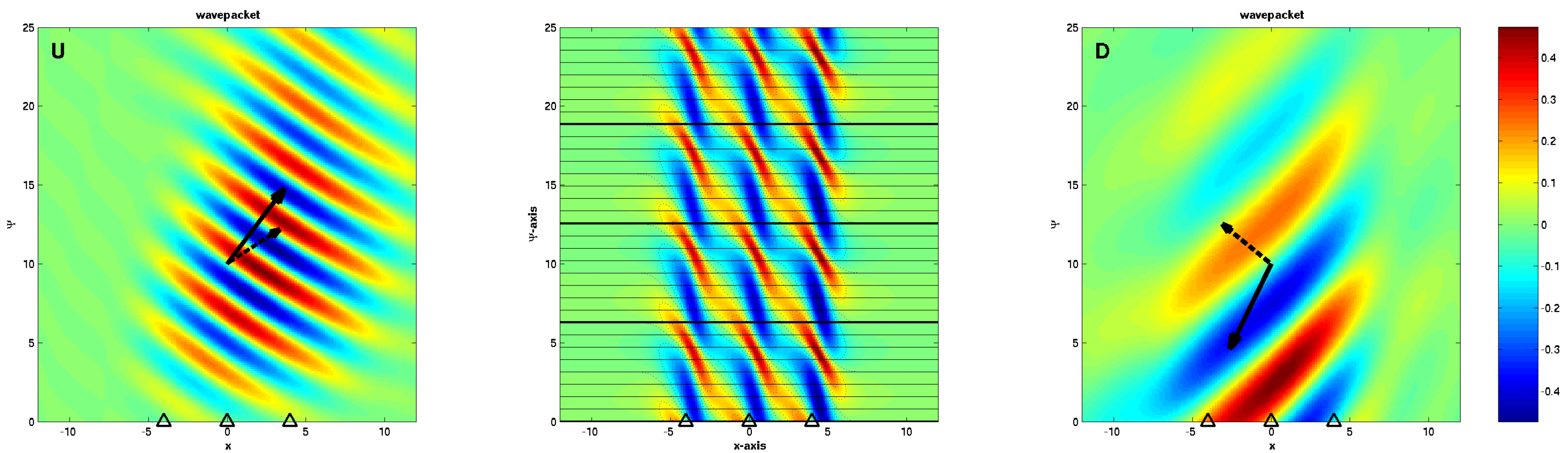


Instability via Triad Resonance

4-Wave Interaction

- $u(x, \Psi)$ & $w(x, \Psi)$ are non-constant coefficients for linear disturbances $\tilde{\psi}(x, \Psi)$ & $\tilde{\theta}(x, \Psi)$
- multiplication of Fourier modes $\leftrightarrow$ addition of wavevectors

U-wavepacket	$\times$	steady flow	$\longrightarrow$	D-wavepacket
U-wavepacket	$\longleftarrow$	steady flow	$\times$	D-wavepacket

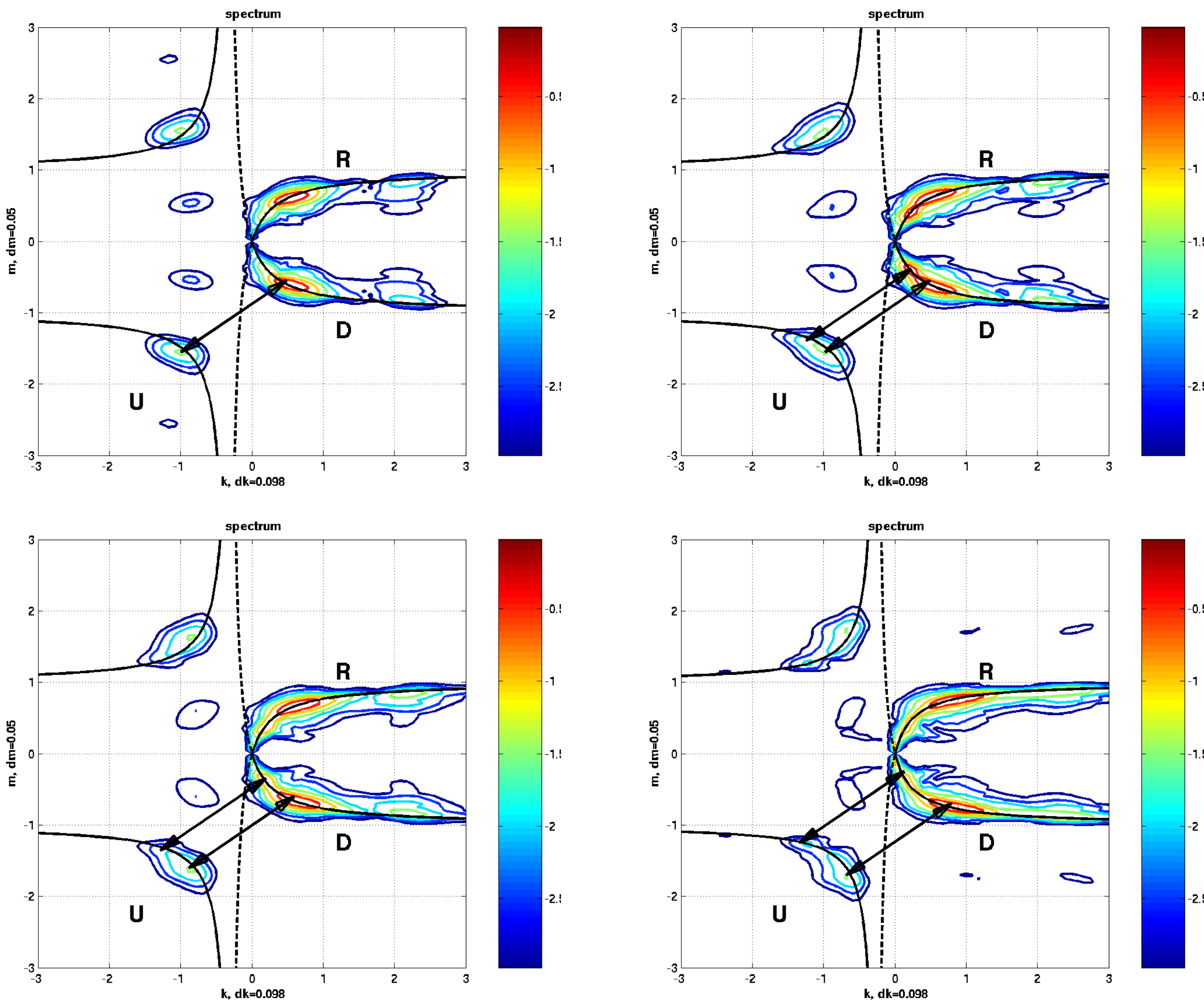


Resonant Instability

- occurs if wave generation leads to positive feedback by constructive interference (phase matters)
- projection onto U- and D-wavepackets alone gives estimate of λ (within 15%)
- depends on height of topography ($\mathcal{A} > 0.35$?)

Multiple Triads

Spectra for 4 Fastest Growing Modes (of 6 unstable computed)



Critical Resonant Triad

Triad Resonance as Function of ω_0

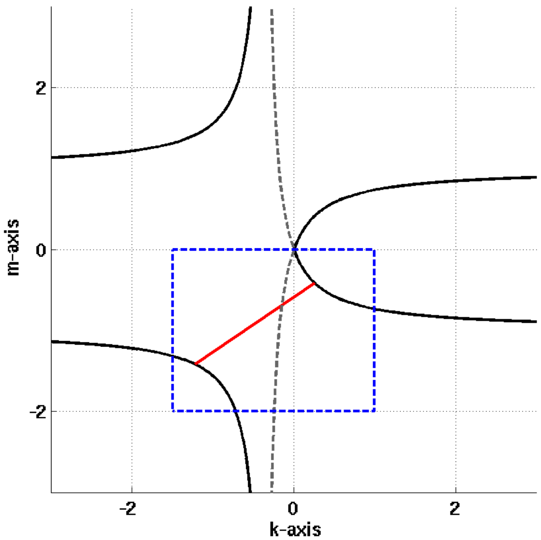
- triad resonance condition for (k_U, m_U)

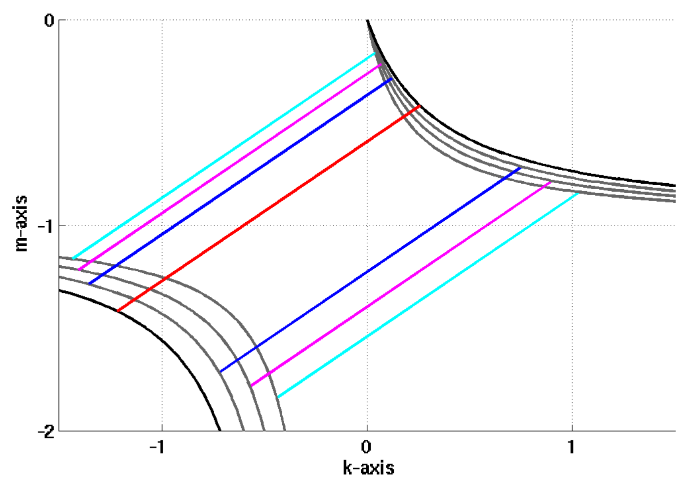
$$\omega_0 = \omega(k_U, m_U) = \omega(k_U + k_s, m_U + 1) = \omega(k_D, m_D)$$

- generically 2 solutions of U-D type $\longrightarrow$ critical triad occurs for double root!

$$\omega_c = -\frac{k_s}{4} \quad ; \quad k_U = -3k_D = -\frac{3k_s}{4} \quad ; \quad m_U = 3m_D = -\frac{3}{2}$$

  - triad resonances only occur for $|\omega_0| < |\omega_c| \quad \longrightarrow \quad$ maximum frequency





- is the critical resonant triad responsible for the most unstable mode?

In Closing

Direct Steady 2D Solve

- ▷ non-iterative formulations for exact topographic surface condtion
  - ▹ Fourier-based 1^{st}-kind solver: near-hydrostatic regime $(0 \leq \sigma < 0.5)$
  - ▹ Green's function-based 2^{nd}-kind solver: hydrostatic regime $(0.3 \leq \sigma < 4^{+})$
- ▷ overturning criterion to strongly nonhydrostatic regime
- ▷ accurate solutions for linear stability analysis

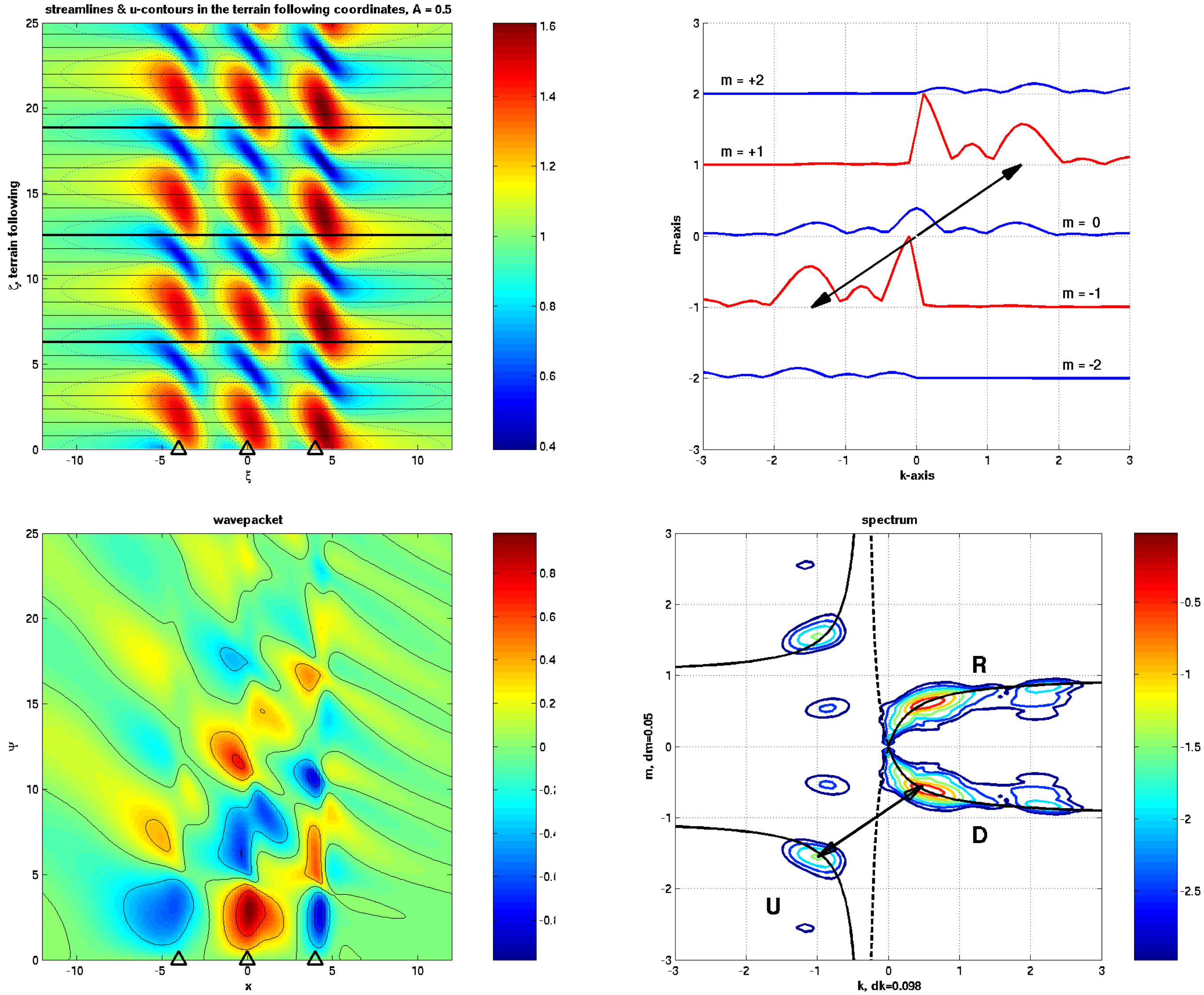
2D Linear Stability

- ▷ identification of linear instabilities for multiply-peaked terrain
  - ▹ benchmark against time-dependent simulations
  - ▹ triad resonance mechanism & critical triad conjecture
  - ▹ height & separation criterion for instability

- ▷ implications for atmospheric wave drag estimates/parametrizations?

A Resonant Triad

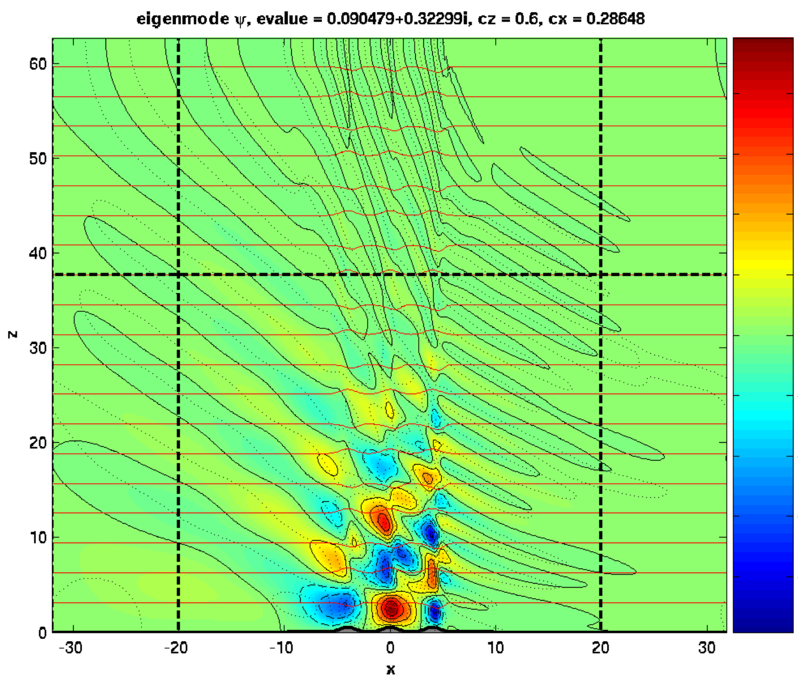
Fourier Wavevectors: Steady Flow & Eigenfunction ($\vec{k}_U + \vec{k}_s = \vec{k}_D$)

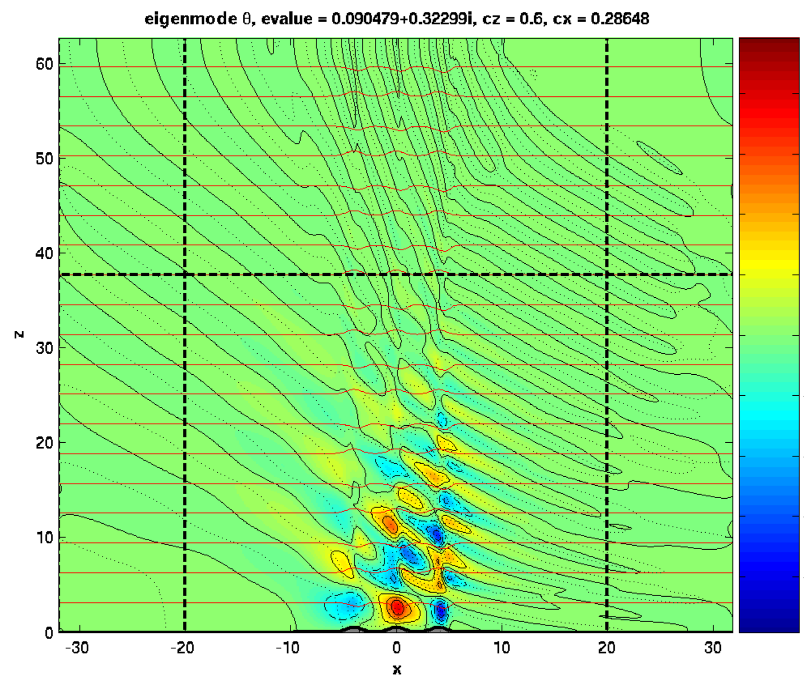


Most Unstable Mode

Computational Details

- $\tilde{\psi}(x, \Psi)$ & $\tilde{\theta}(x, \Psi)$ on 384×480 grid
- 2$^{\text{nd}}$-order finite differences: $\Delta x = 1/6 = 0.17$ & $\Delta\Psi = \pi/24 = 0.13$
- zero on top/bottom, horizontally periodic & damping layers
- sparse matrix dimension = 367,872; Krylov subspace dimension = 10

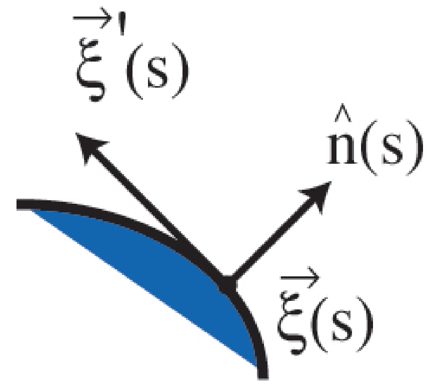




$$\mathcal{G}_{xx} + \mathcal{G}_{zz} + \mathcal{G} = \delta(\vec{x} - \vec{\xi})$$

Helmholtz Free-Space Green's Function ($\sigma = 1$)

- radiating solution for a delta-function source at $\vec{\xi}$: $\mathcal{G}(\vec{x} - \vec{\xi})$
- classical, time-harmonic scattering problem in electromagnetics/acoustics
  - delta-function response in 2D involves Hankel functions: $J_0(r) \pm i\, Y_0(r)$
  - sign choice determined by far-field radiation condition (implied by time-harmonicity)



Boundary Integral Method

- $\mu(s)$, weighted surface distribution of Green's functions
- $\vec{\xi}(s)$, parametrization of surface boundary (clockwise)

$$\psi(\vec{x}) = -\mathcal{A} \int_{\mathcal{S}} \mu(s)\ \ 2\, \frac{\partial \mathcal{G}}{\partial n}(\vec{x} - \vec{\xi}(s))\ ds$$

- need topographic Green's function $\mathcal{G}(\vec{x} - \vec{\xi})$ & weights $\mu(s)$

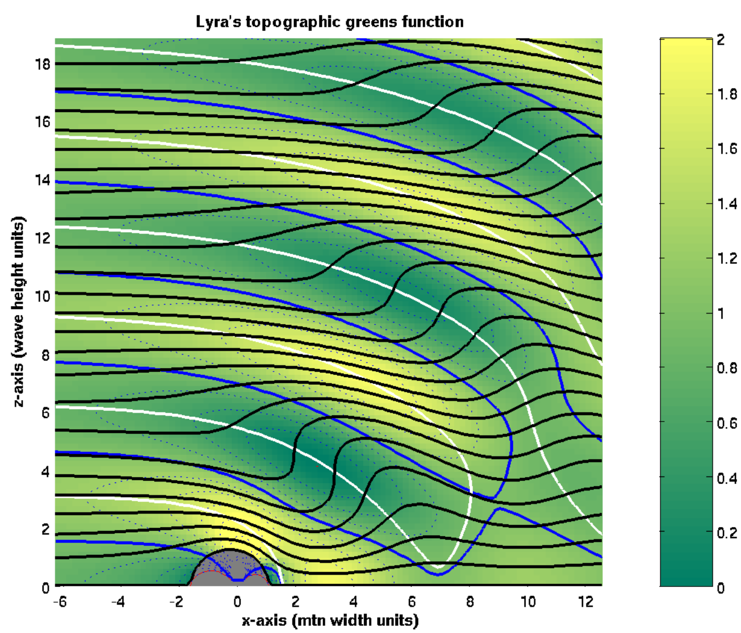
Lyra's Topographic Green's Function

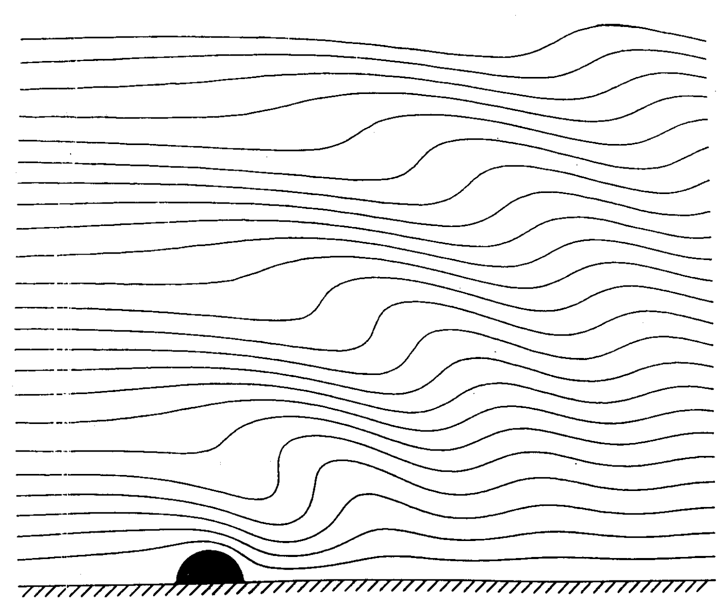
Delta-Function Topography (linear theory)

- from Lyra 1940 & 1943 (via Alaka 1960) for $\sigma = 1$ as Bessel series

$$\mathcal{G}_z(r,\theta) \;=\; \frac{1}{2}\,Y_1(r)\sin\theta \;+\; \frac{1}{\pi}\sum_{1}^{\infty}\frac{4n}{4n^2-1}\,J_{2n}(r)\sin 2n\theta$$

- Lyra's critical overturning solution: $\Psi = z + 4.06\;\mathcal{G}_z(r,\theta)$





Miles/Huppert 1968

- left/right asymmetric Greens function: waves must be downstream (Miles/Huppert 1968)

Fredholm Integral Equation of Second-Kind

Singular Integral Representation

- ▷ Plemelj formula for surface values, $\vec{x}_s$

$$\psi(\vec{x}_s) = -\mathcal{A}\ \mu(\vec{x}_s) - \mathcal{A} \int_{\mathcal{S}} \mu(s)\ \ 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x}_s - \vec{\xi}\,(s))\ ds$$

- ▷ surface boundary condition $\longrightarrow$ second-kind integral equation for $\mu(\vec{x}_s)$

$$\boxed{\mu(\vec{x}_s) + \int_{\mathcal{S}} \mu(s)\ \ 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x}_s - \vec{\xi}\,(s))\ ds \quad = \quad h(\vec{x}_s)}$$

- ▷ kernel function is continuous at $\vec{x}_s = \vec{\xi}\,(s)$ $\longrightarrow$ curvature contribution
- ▷ discretized quadrature gives diagonally-dominant matrix $\longrightarrow$ well-conditioned inversion

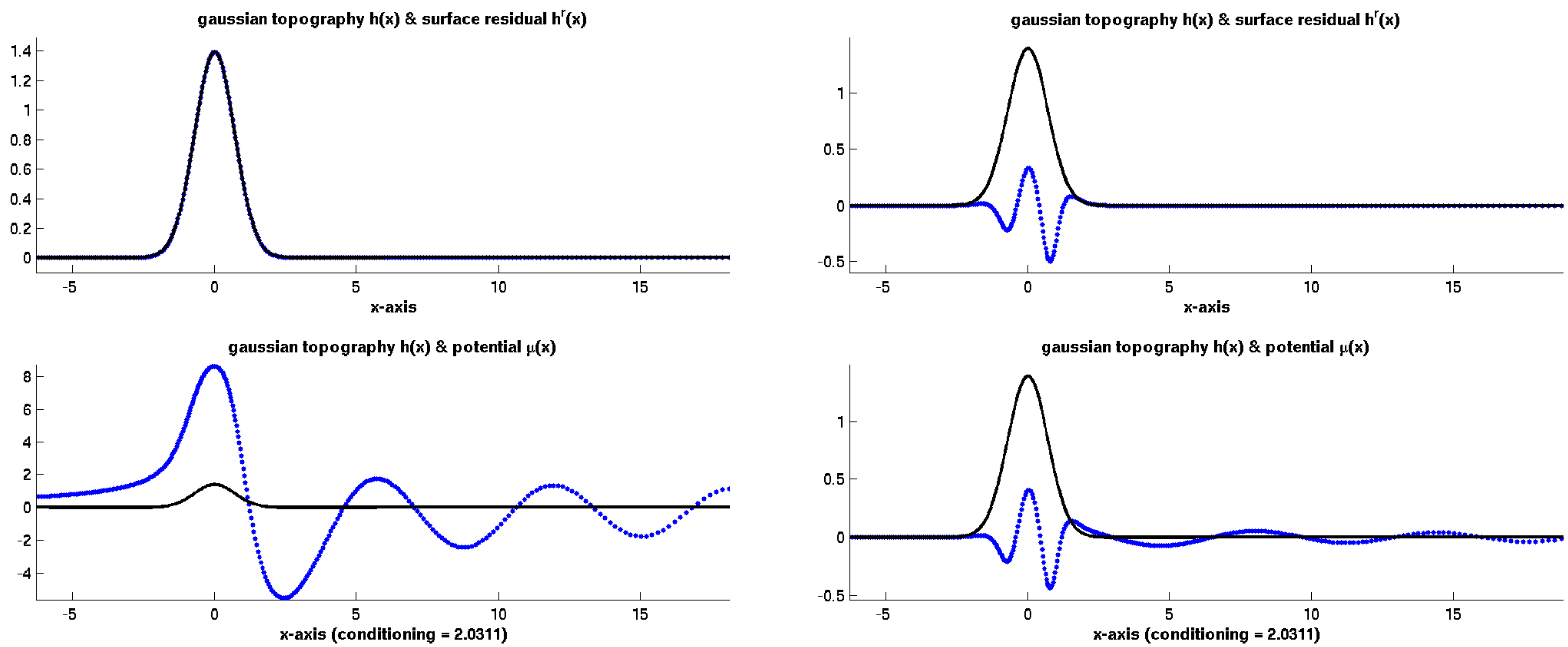
- ▷ amplitude parameter, $\mathcal{A}$, enters through surface parametrization: $\vec{\xi}\,(s) = \begin{pmatrix} x(s) \\ \mathcal{A}h(x(s)) \end{pmatrix}$
- ▷ small $\mathcal{A}$ limit: $\mu(\vec{x}_s) \longrightarrow h(\vec{x}_s)$

- ▷ nonhydrostatic parameter, σ, handled by rescaling in x (singular as $\sigma \longrightarrow 0$)

Large Amplitude Solutions

Slow Decay

- ▷ boundary integral method limited by downstream wake in $\mu(x)$



- ▷ use Lyra's analytical solution as first guess

$$\psi(\vec{x}) = \Lambda\, \mathcal{G}_z(\vec{x}) - \mathcal{A} \int_{\mathcal{S}} \mu(s) \;\; 2\, \frac{\partial \mathcal{G}}{\partial n}(\vec{x} - \vec{\xi}\,(s))\; ds$$

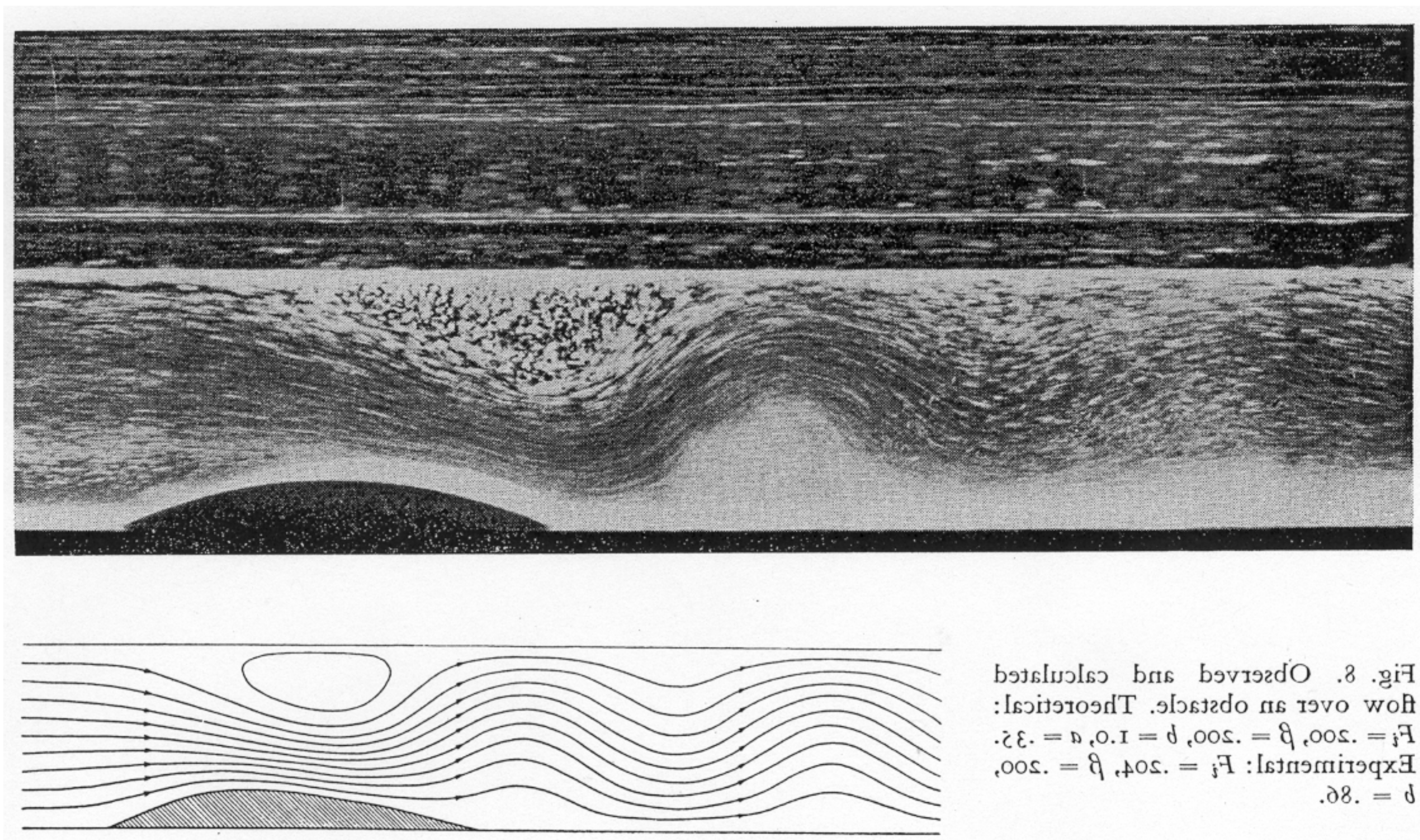
  - ▷ accurate computation based on surface residual: $h^r(x) = h(x) + \Lambda\, \mathcal{G}_z(x, h(x))$
  - ▷ Λ obtained by good guesswork (4.06 for critical overturning)

Long 1955: Theory & Experiment

$$\sigma^2 \psi_{xx} + \psi_{zz} + \psi = 0$$

Finite Amplitude Topography

- on streamline boundaries: $\psi = \mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) =$ constant



Long 1953