On the Circuit Diameter Conjecture

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Abstract. From the point of view of optimization, a critical issue is relating the combinatorial diameter of a polyhedron to its number of facets f and dimension d. In the seminal paper of Klee and Walkup [KW67], the Hirsch conjecture of an upper bound of $f - d$ was shown to be equivalent to several seemingly simpler statements, and was disproved for unbounded polyhedra through the construction of a particular 4-dimensional polyhedron U_4 with 8 facets. The Hirsch bound for bounded polyhedra was only recently disproved by Santos [San12].

We consider analogous properties for a variant of the combinatorial diameter called the *circuit* diameter. In this variant, the walks are built from the circuit directions of the polyhedron, which are the minimal non-trivial solutions to the system defining the polyhedron. We are able to recover the equivalence results that hold in the combinatorial case and provide an additional variant. Further, we show the circuit analogue of the non-revisiting conjecture implies a linear bound on the circuit diameter of all unbounded polyhedra – in contrast to what is known for the combinatorial diameter. Finally, we give two proofs of a circuit version of the 4-step conjecture. These results offer some hope that the circuit version of the Hirsch conjecture may hold, even for unbounded polyhedra.

A challenge in the circuit setting is that different realizations of polyhedra with the same combinatorial structure may have different diameters. Among other things, we adapt the notion of simplicity to work with circuits in the form of *C-simple* and *wedge-simple* polyhedra. We show that it suffices to consider such polyhedra for studying circuit analogues of the Hirsch conjecture, and use this to prove the equivalences of the different variants.

1   Introduction

The *combinatorial diameter* of a polyhedron is the maximum number of edges that is necessary to connect any pair of vertices by a walk. It has been studied extensively due to its intimate connection to the simplex algorithm for linear programming: it is a lower bound on the best-case performance of the simplex algorithm, independent of the pivot rule used. This motivates the question: *What is the largest possible combinatorial diameter of a d-dimensional convex polyhedron with a given number f of facets?* In particular, if there is a family of polyhedra with combinatorial diameter that is exponential in f and d, then a polynomial pivot rule cannot exist.

The quantity $f - d$, conjectured by Hirsch in the late 1950s [Dan63] as a prospective upper bound for the combinatorial diameter, remains a landmark in discussion of the combinatorial diameter. It is tight for the key special cases of d-cubes and d-simplices. It is known to hold for important classes of polyhedra including 0/1-polytopes [Nad89] and network flow polytopes [BLF16a], but does not hold in general. For unbounded polyhedra, there is a 4-dimensional counterexample [KW67] and for bounded polytopes a 20-dimensional counterexample [MSW15] (the original counterexample was of dimension 43 [San12]). These counterexamples can be used to generate families of counterexamples in higher dimension, but it is otherwise difficult to generate exceptions. Indeed, the known counterexamples only give rise to a violation of the stated bound by a factor of at most $\frac{5}{4}$ for unbounded polyhedra and $\frac{21}{20}$ for bounded polytopes. It is open whether these bounds can be exceeded. See [KS10] for the state-of-the-art and [BDHS13] for some recent progress for low values of f and d.

Recent avenues of research consider alternative models in an effort to understand *why* the combinatorial diameter violates the Hirsch bound. These include working with combinatorial

abstractions of polytopes, see e.g. [EHRR10,LMS15], and augmented pivoting procedures that are not limited to the vertices and edges of the polytope, see e.g. [BFH15,BLF16b,BLFM16,DLHL15]. We follow the latter path, focusing on the model of *circuit walks*, arguably the most natural of the augmented procedures.

1.1 Circuit walks

Let a polyhedron P be given by

$$P = \{\, \mathbf{z} \in \mathbb{R}^d : \ A^1\mathbf{z} = \mathbf{b}^1, A^2\mathbf{z} \geq \mathbf{b}^2 \,\}$$

for matrices $A^i \in \mathbb{Q}^{f_i \times d}$ and vectors $\mathbf{b}^i \in \mathbb{Q}^{f_i}$, $i = 1, 2$. The *circuits* $\mathcal{C}(A^1, A^2)$ of A^1 and A^2 are those vectors $\mathbf{g} \in \ker(A^1) \backslash \{\, \mathbf{0} \,\}$, for which $A^2\mathbf{g}$ is support-minimal in $\{A^2\mathbf{x} : \ \mathbf{x} \in \ker\left(A^1\right) \backslash \{0\}\}$, where $\mathbf{g}$ is normalized to coprime integer components. (For $f_1 = 0$, we assume $\ker(A^1) = \mathbb{R}^d$.) Note that circuits correspond to the so-called elementary vectors as introduced in [Roc69]. Clearly, the set of circuits contains the actual edges of the polyhedron P. We use $\mathcal{C}(P)$ to refer to the set of circuits of P without explicit consideration of the underlying matrices.

Following Borgwardt et al. [BFH15], for two vertices $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ of P, we call a sequence $\mathbf{v}^{(1)} = \mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(k)} = \mathbf{v}^{(2)}$ a *circuit walk of length k* if for all $i = 0, \ldots, k-1$ we have

1. $\mathbf{y}^{(i)} \in P$,
2. $\mathbf{y}^{(i+1)} - \mathbf{y}^{(i)} = \alpha_i\mathbf{g}^{(i)}$ for some $\mathbf{g}^{(i)} \in \mathcal{C}(A^1, A^2)$ and $\alpha_i > 0$, and
3. $\mathbf{y}^{(i)} + \alpha\mathbf{g}^{(i)}$ is infeasible for all $\alpha > \alpha_i$.

Informally, a circuit walk takes steps of maximal length along circuits of P. The above properties are also well-defined when $\mathbf{v}^{(1)} = \mathbf{y}^{(0)}$ is not a vertex or when $\mathbf{v}^{(2)} = \mathbf{y}^{(k)}$ is not a vertex. In fact, we will sometimes use the terms in this more general sense. If this is the case, we will explicitly state that the walks at hand may start or end at a non-vertex. Note that properties 2. and 3. rule out the use of a circuit in the unbounded cone of P, which essentially would give an unbounded step.

The *circuit distance* $\mathrm{dist}_\mathcal{C}(\mathbf{v}^{(1)}, \mathbf{v}^{(2)})$ from $\mathbf{v}^{(1)}$ to $\mathbf{v}^{(2)}$ then is the minimum length of a circuit walk from $\mathbf{v}^{(1)}$ to $\mathbf{v}^{(2)}$. The *circuit diameter* $\Delta_\mathcal{C}(P)$ of P is the maximum circuit distance between any two vertices of P. We denote the maximum circuit diameter that is realizable in the set of d-dimensional polyhedra with f facets as $\Delta_\mathcal{C}(f, d)$. For the maximum combinatorial diameter in this class of polyhedra, we use $\Delta_\mathcal{E}(f, d)$. Additionally, we use the terms $\mathrm{dist}_f(\mathbf{v}^{(1)}, \mathbf{v}^{(2)})$, $\Delta_f(P)$, and $\Delta_f(f, d)$ for the corresponding notions for the so-called *feasible circuit walks* [BLF16b], where the walk does not have to take steps of maximal length, but is only required to stay feasible. To distinguish these walks from the original circuit walks, the original ones are sometimes called *maximal circuit walks*.

In [BLF16b], several concepts of walking along circuits with respect to different restrictions are compared, giving a hierarchy of circuit distances as well as diameters. We would particularly like to understand the part of this hierarchy involving $\Delta_\mathcal{E}(P)$, $\Delta_\mathcal{C}(P)$, and $\Delta_f(P)$. Note that all edge walks are special (maximal) circuit walks; the restriction is to use only actual edges instead of any circuits for the directions of the steps. Further all maximal circuit walks are feasible walks; the restriction is to only use maximal step lengths. This gives the relation

$$\Delta_\mathcal{E}(f, d) \geq \Delta_\mathcal{C}(f, d) \geq \Delta_f(f, d).$$

Recall that the Hirsch conjecture (for the combinatorial diameter) is false, i.e. $\Delta_\mathcal{E}(f, d)$ does violate the Hirsch bound $f - d$ [KW67,San12]. In contrast, it is possible to show $\Delta_f(f, d) \leq f - d$ for all f, d [BLF16b]. For the middle part of this hierarchy, i.e. the (maximal) circuit walks themselves, the corresponding claim is open for bounded and for unbounded polyhedra:

Conjecture 1 (Circuit Diameter Conjecture – Original formulation [BFH15]).
For any d-dimensional polyhedron with f facets the circuit diameter is bounded above by $f - d$.

This means that either between $\Delta_\mathcal{E}(f, d)$ and $\Delta_\mathcal{C}(f, d)$, or between $\Delta_\mathcal{C}(f, d)$ and $\Delta_f(f, d)$ we lose validity of the bound $f - d$. Suppose Conjecture 1 is true – then there is a significant conceptual difference between walking along edges and walking along any circuits. If Conjecture 1

is not true, there is a significant conceptual difference between taking steps of maximal length and just staying feasible. In fact, the distinction may be even finer, for example it is possible to have $\Delta_{\mathcal{E}}(f,d) > \Delta_{\mathcal{C}}(f,d) > \Delta_f(f,d)$.

This is one of the many incentives to study Conjecture 1 and leads to an interpretation as investigating why the Hirsch bound is violated for the combinatorial diameter.

1.2 Our contributions

In their seminal paper [KW67], Klee and Walkup gave several insights related to the Hirsch bound for polyhedral diameter. They showed that it is enough to work with simple polyhedra, and showed that the general Hirsch conjecture is equivalent to three more restricted statements. These include the *d-step conjecture*, which is the special case where $f = 2d$, and the *non-revisiting conjecture*, that any two vertices can be joined by a walk that visits each facet at most once. They then exhibited an unbounded 4-dimensional polyhedron U_4 with 8 facets that has combinatorial diameter 5, disproving these results for unbounded polyhedra. However, restricting attention to bounded polytopes, the equivalences again hold, and here they show that, in contrast to the unbounded case, the d-step conjecture *does* hold at $d = 4$ and $d = 5$. Polyhedra based on U_4 remained the only known non-Hirsch polyhedra for more than four decades until Santos [San12] found counterexamples to the bounded versions of the conjectures, refuting d-step at $d = 43$ and subsequently [MSW15] at $d = 20$, the current record.

In this paper, we study the analogous questions in the context of circuit diameter.

One of the main challenges we face is that in the circuit context, different geometric realizations of polyhedra with the same combinatorial structure may have different circuit diameters; see e.g. the example in [BFH15]. Thus we need to introduce a notion of simplicity which depends on the geometry of the problem, including the circuits. We call polyhedra which satisfy this property $\mathcal{C}$-simple, and show:

Lemma 2. *Let P be a polyhedron. Then there is a $\mathcal{C}$-simple polyhedron P' in the same dimension and with the same number of facets with $\Delta_{\mathcal{C}}(P) \leq \Delta_{\mathcal{C}}(P')$.*

Indeed, we use a slightly stronger property that also requires P to remain $\mathcal{C}$-simple under the wedge operation; we call this *k-wedge-simplicity* and show that Lemma 2 holds for k-wedge-simplicity as well. This then allows us to show the equivalence of several variants of Conjecture 1 for circuit diameter. We are able to recover generalizations of all of the results that hold in the combinatorial case:

Theorem 3. *The following statements are equivalent:*

1. *Let u, v be two vertices of a k-wedge-simple polyhedron P for $k \geq f$. Then there is a non-revisiting circuit walk from u to v.*
2. $\Delta_{\mathcal{C}}(f,d) \leq f - d$ *for all $f \geq d$*
3. $\Delta_{\mathcal{C}}(2d,d) \leq d$ *for all d*
4. *For all d-dimensional Dantzig figures (P, u, v), the circuit distance of u and v is at most d.*

We state these conjectures for all polyhedra, both unbounded and bounded. Moreover, for circuit diameters, we find an intimate connection of the diameters of unbounded polyhedra and bounded polytopes. In particular, we show that the existence of a non-revisiting circuit walk between any pair of vertices in a $\mathcal{C}$-simple bounded polytope guarantees a linear bound on the diameter of all $\mathcal{C}$-simple unbounded polyhedra. To distinguish between bounded and unbounded quantities, we use superscripts: $\Delta_{\mathcal{C}}^u(f,d)$ denotes the maximal circuit diameter of an unbounded d-dimensional polyhedron with f facets, while $\Delta_{\mathcal{C}}^b(f,d)$ denotes its counterpart for bounded d-dimensional polytopes. We expect that $\Delta_{\mathcal{C}}(f,d)$ is attained at an unbounded polytope, and thus $\Delta_{\mathcal{C}}(f,d) = \Delta_{\mathcal{C}}^u(f,d)$, but the proof for the combinatorial case, see below, does not easily carry over.

Theorem 4. *If all $\mathcal{C}$-simple bounded (f', d')-polytopes with $f' \leq f + d - 1$ and $d' \leq d$ satisfy the non-revisiting conjecture (Conjecture 18), then $\Delta_{\mathcal{C}}^u(f,d) \leq f - 1$.*

We would like to stress that Theorem 4 gives a connection between the bounded and unbounded diameters of a type that is not known for the combinatorial diameter. In fact, for the combinatorial case it can be shown that $\Delta_{\mathcal{E}}^b(f, d) \leq \Delta_{\mathcal{E}}^u(f, d) = \Delta_{\mathcal{E}}(f, d)$ for all $f > 2d$. This is derived using a series of inequalities and reductions (see e.g. [KW67]): For $f > 2d$, $\Delta_{\mathcal{E}}^b(f, d)$ is achieved by a path in a simple polytope; $\Delta_{\mathcal{E}}^b(f, d) \leq \Delta_{\mathcal{E}}^u(f - 1, d)$ by projecting to infinity a facet not containing the endpoints of this path; $\Delta_{\mathcal{E}}^u(f - 1, d) < \Delta_{\mathcal{E}}^u(f, d)$ by truncating at one of the endpoints of the long path. This gives a *lower bound* on the unbounded combinatorial diameter based on the bounded diameter:

$$\Delta_{\mathcal{E}}^b(f, d) < \Delta_{\mathcal{E}}^u(f, d), \text{ for } f > 2d.$$

For $f = 2d$, Klee and Walkup also prove $\Delta_{\mathcal{E}}^b(2d, d) \leq \Delta_{\mathcal{E}}^u(2d - 2, d - 1) + 1$.

In contrast, in case the non-revisiting conjecture is valid for circuit walks, Theorem 4 would give an *upper bound* of the form

$$\Delta_{\mathcal{C}}^b(f, d) + d - 1 \geq \Delta_{\mathcal{C}}^u(f, d).$$

Recall that for the combinatorial diameter, a disproof of the Hirsch conjecture was much easier for the unbounded case, with a counterexample U_4 in dimension 4 [KW67]. We prove that U_4, in contrast to the combinatorial case, *does* satisfy the Hirsch bound in the circuit setting, independent of realization.[1] Indeed, we show that for circuits, the 4-step conjecture holds even for unbounded polytopes:

Theorem 5 (Circuit 4-step). $\Delta_{\mathcal{C}}(8, 4) = 4$.

We give two proofs of this fact. The first uses the uniqueness of U_4, and the second strengthens the result of Santos et al. [SST12] to get a u-v walk in an arbitrary 4-spindle where a v-facet is entered at each step.

In summary, the results in this paper include:

- Adapting the concept of simple polyhedra to the circuit context via $\mathcal{C}$-simple, wedge-simple and k-wedge-simple polyhedra. *(Section 2)*
- Using this to prove the equivalence of various conjectures on circuit diameter, including a circuit equivalent of the Hirsch conjecture and its circuit d-step and non-revisiting variants (Theorem 3). We also show the circuit diameter conjecture for bounded polytopes implies a linear bound on the circuit diameter of unbounded polytopes (Theorem 4). *(Section 3)*
- Two proofs of the circuit 4-step conjecture (Theorem 5), even for unbounded polyhedra. The first one is based on showing that the circuit diameter conjecture holds in the most prominent case where the combinatorial Hirsch conjecture fails (Theorem 26). The second proof extends a result of Santos et al. [SST12]. *(Section 4)*
- A brief discussion of related open questions. *(Section 5)*

2 $\mathcal{C}$-Simplicity and Wedge-Simplicity

When considering bounds for the combinatorial diameter, we can restrict to simple polyhedra. This is because for any d-dimensional non-simple polyhedron with f facets, a mild perturbation of the right-hand sides gives a simple polyhedron with the same number of facets and at least the same diameter [YKK84].

The key advantage of a simple polyhedron in the studies of the combinatorial diameter is that each step along an edge leaves exactly one facet and enters exactly one other facet. This makes an analysis significantly easier. To take advantage of non-edge circuits, walks will leave more than one facet at a time. However, we can require that circuit steps do not arrive at more than one facet at a time. The benefit of such walks has been observed in a special case before, see the last page of [BFH15].

In Section 2.1, we introduce such a property ($\mathcal{C}$-simplicity) and prove that $\Delta_{\mathcal{C}}(f, d)$ is realized by a $\mathcal{C}$-simple polyhedron. In Section 2.2 we discuss a further specialization to polyhedra for which there is a wedge over each of its facets that still satisfies this property, and prove that $\Delta_{\mathcal{C}}(f, d)$ is also always realized in this class of polyhedra.

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[1] A preliminary version of this result appeared in the proceedings of Eurocomb 2015 [SY15b].

2.1 $\mathcal{C}$-Simplicity

In the following, let $P = \{\mathbf{z} \in \mathbb{R}^d : A^1\mathbf{z} = \mathbf{b}^1, A^2\mathbf{z} \geq \mathbf{b}^2\}$ be a d-dimensional polyhedron with $f = f_1 + f_2$ facets. To keep the arguments straightforward, we may assume that P has an irredundant representation as full-dimensional polyhedron in $\mathbb{R}^d$. This implies $f_1 = 0$ and $f_2 = f$, i.e. we do not have to consider equality constraints and use $\mathbf{b}^2 \in \mathbb{Q}^f$. This gives us a simpler representation $P = \{\mathbf{z} \in \mathbb{R}^d : A\mathbf{z} \geq \mathbf{b}\}$.

For $\mathbf{y}^{(i)} \in P$, let $H^{(i)}$ denote the set of facets of P that are incident to $\mathbf{y}^{(i)}$. First, let us introduce some terminology for circuit walks, in which we enter only one new facet in each step.

Definition 6 (Simple walks). *Let P be a polyhedron. A circuit walk $\mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(k)}$ in P is simple if $|H^{(i+1)} \backslash H^{(i)}| = 1$ for $i = 1, 2, \ldots, k$, where $H^{(i)}$ denotes the set of facets incident to $\mathbf{y}^{(i)}$. Walks that violate this condition are called* non-simple.

We are particularly interested in polyhedra for which it suffices to only consider simple circuit walks. As the combinatorial diameter is an upper bound on the circuit diameter of a polyhedron, for the study of Conjecture 1, it suffices to consider circuit walks of length at most $\Delta_{\mathcal{E}}(f, d)$, which is bounded above for example by $f^{\log d + 2}$ [KK92]. This leads to the following definition.

Definition 7 ($\mathcal{C}$-simple). *Let P be a polyhedron. We say P is $\mathcal{C}$-simple if all circuit walks of length at most $\Delta_{\mathcal{E}}(f, d)$ are simple.*

Let M be a finite set of points in P that includes the set of vertices. If all circuit walks starting at any point in M and of length at most $\Delta_{\mathcal{E}}(f, d) + d$ are simple, then we say P is $\mathcal{C}$-simple with respect to M.

Note that $\mathcal{C}$-simplicity is a strictly stronger condition than simplicity of a polyhedron, as edge walks are a special type of circuit walks. Further, in our studies non-simple circuit walks will only appear in polyhedra that are not $\mathcal{C}$-simple.

The goal for this section is to prove that it suffices to consider $\mathcal{C}$-simple polyhedra for the study of Conjecture 1, which leads to the following variant of the conjecture.

Conjecture 8 ($\mathcal{C}$-Simplicity). For any $\mathcal{C}$-simple d-dimensional polyhedron with f facets the circuit diameter is bounded above by $f - d$.

We prove the equivalence of Conjecture 8 and Conjecture 1 by showing that for fixed f and d, $\Delta_{\mathcal{C}}(f, d)$ can be realized by a $\mathcal{C}$-simple polyhedron. We do so by describing a perturbation of a polyhedron P such that the perturbed polyhedron P' is $\mathcal{C}$-simple and has at least the same circuit diameter as P.

The perturbations we consider are to the right hand sides of the defining equations, and thus do not change the set of circuits, which depends only on A. That is, the right-hand side $\mathbf{b}$ is changed to $\mathbf{b} \to \mathbf{b}' = \mathbf{b} + \mathbf{p}$ for some vector $\mathbf{p}$ with $\|\mathbf{p}\| < \epsilon$ for a sufficiently small ϵ. We call such a perturbation a *mild perturbation*. The perturbed polyhedron is $P' = \{\mathbf{z} \in \mathbb{R}^d : A\mathbf{z} \geq \mathbf{b}'\}$. Note that for a mild perturbation, each facet remains a facet and the dimension does not change.

The challenge here lies in the fact that the circuit diameter of a polyhedron depends on its realization, and not only its combinatorial structure (see [BFH15] for examples). Hence the effect of a perturbation, in theory, might reduce the diameter. We have to carefully check that there is a mild perturbation for which this is not the case.

Lemma 2. *Let P be a polyhedron. Then there is a $\mathcal{C}$-simple polyhedron P' in the same dimension and with the same number of facets with $\Delta_{\mathcal{C}}(P) \leq \Delta_{\mathcal{C}}(P')$.*

Proof. Let P be a d-dimensional polyhedron with f facets and let P' denote a polyhedron derived by a mild perturbation. By definition of a mild perturbation, P and P' have the same number of f facets and dimension d. Thus it suffices to prove that this perturbation can be performed such that P' is $\mathcal{C}$-simple and has at least the same circuit diameter as P.

First, recall that P and P' share the same finite set of circuits. Further, observe that it suffices to only consider circuit walks of length at most $\Delta_{\mathcal{E}}(f, d)$ to validate that P' is $\mathcal{C}$-simple. There are a finite number of points $\mathbf{y} \in P'$ that may appear in such a walk. Hence the condition $|H^{(i+1)} \backslash H^{(i)}| = 1$ only has to be satisfied for a finite set of pairs $(\mathbf{y}^{(i)}, \mathbf{y}^{(i+1)})$. This implies that

(for fixed ϵ) the set of right-hand sides $\mathbf{b}' = \mathbf{b} + \mathbf{p}$ with $\|\mathbf{p}\| \leq \epsilon$ that do not give a $\mathcal{C}$-simple polyhedron P' is of volume 0. In turn, for any given ϵ there are infinitely many perturbations that yield a $\mathcal{C}$-simple polyhedron P'.

It remains to see that there is such a perturbation for which the circuit diameter of P' is at least the circuit diameter of P. Let Y_P denote the set of all points on circuit walks in P of length at most $\Delta_{\mathcal{E}}(f,d)$. A simple but important observation is that the points in Y_P are at least a certain fixed distance from each other. We say that a pair of points are *close* if they are less than half the minimum distance between pairs of points in Y_P. Consider then a perturbation that is small enough that basic solutions in P' remain close to basic solutions in P. Take $\mathbf{y} \in P$ and $\mathbf{z} \in P'$. Let $I(\mathbf{y})$ denote the *inner cone*, i.e. the cone of feasible directions, of $\mathbf{y}$ with respect to P and $I(\mathbf{z})$ denote the inner cone of $\mathbf{z}$ with respect to P'. Then we have the following:

1. There is a one-to-many correspondence between vertices $\mathbf{y}$ of P and vertices $\mathbf{z}$ of P' where each $\mathbf{z}$ is close to a unique $\mathbf{y}$ and at least one $\mathbf{z}$ is close to a given $\mathbf{y}$ (and possibly many are). In particular, each vertex $\mathbf{z}$ of P' is associated to precisely one close vertex $\mathbf{y}$ in P.
2. Let $\mathbf{y} \in Y_P$, $\mathbf{z} \in P'$ be close and let $\mathbf{c}$ be a circuit in $I(\mathbf{y}) \cap I(\mathbf{z})$. Then a step along $\mathbf{c}$ from $\mathbf{z} \in P'$ gives a $\mathbf{z}' \in P'$ that is close to precisely one $\mathbf{y}' \in Y_P$, which is derived from a step along $\mathbf{c}$ from $\mathbf{y} \in P$.
3. Let $\mathbf{y} \in Y_P$, $\mathbf{z} \in P'$ be close. Then a step along $\mathbf{c} \in I(\mathbf{z}) \backslash I(\mathbf{y})$ from $\mathbf{z}$ will give a $\mathbf{z}' \in P'$ that is also close to precisely $\mathbf{y}$.

Let us consider a circuit walk $\mathbf{z}^{(0)}, \ldots, \mathbf{z}^{(k')} \in P'$ for $k' \leq \Delta_{\mathcal{E}}(f,d)$. Informally, the above properties tell us that it starts close to a vertex of P (1.) and stays close to points in Y_P in each step (2., 3.). More precisely, each $\mathbf{z}^{(i)}$ is close to precisely one $\mathbf{y}^{(i)} \in Y_P$. If (2.) is valid for $\mathbf{y}^{(i)}, \mathbf{z}^{(i)}$, then $\mathbf{y}^{(i+1)} \neq \mathbf{y}^{(i)}$. Else if (3.) holds, then $\mathbf{y}^{(i+1)} = \mathbf{y}^{(i)}$. This implies that each $\mathbf{z}^{(0)}, \ldots, \mathbf{z}^{(k')}$ corresponds to a circuit walk $\mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(k)}$ in P with $k \leq k'$.

Let now $\Delta_{\mathcal{C}}(P) = k$ and let $\mathbf{y}^{(0)}, \ldots, \mathbf{y}^{(k)}$ be a walk in P realizing the diameter. Further, let $\mathbf{z}^{(0)}$ be a vertex of P' close to $\mathbf{y}^{(0)}$ and $\mathbf{z}^{(k')}$ be a vertex of P' close to $\mathbf{y}^{(k)}$. If $\text{dist}(\mathbf{z}^{(0)}, \mathbf{z}^{(k')}) < k$, then there is a circuit walk $\mathbf{z}^{(0)}, \ldots, \mathbf{z}^{(k')}$ and $i, i' \leq k$ such that $\mathbf{z}^{(i')}$ is close to $\mathbf{y}^{(i)}$ for $i' < i$. By the above, we then know that the walk $\mathbf{z}^{(0)}, \ldots, \mathbf{z}^{(i')}$ corresponds to a walk $\mathbf{y}^{(0)}, \mathbf{y}'^{(1)}, \ldots, \mathbf{y}'^{(i')} = \mathbf{y}^{(i)}$ of length $i' < i$. This implies $\text{dist}(\mathbf{y}^{(0)}, \mathbf{y}^{(k)}) < k$, a contradiction. Thus $\text{dist}(\mathbf{z}^{(0)}, \mathbf{z}^{(k')}) \geq k$, which proves the claim.

$\square$

Hence we have proven the following:

Corollary 9. *For any $f > d > 1$, $\Delta_{\mathcal{C}}(f,d)$ is attained by a $\mathcal{C}$-simple d-dimensional polyhedron with f facets.*

2.2 Wedge-Simplicity

One of our main tools will be the well-known wedge construction. Let us recall the formal definition.

Definition 10 (Wedge). *Let P be a d-dimensional polyhedron and let F be a facet of P. A wedge on P over F is a $(d+1)$-dimensional polyhedron $P' = H^{\leq} \cap (P \times L)$, where $P \times L$ denotes the product of P with $L = [0, \infty)$ and $H^{\leq} \subset \mathbb{R}^{d+1}$ is a closed halfspace with $P \times \{0\} \subset H^{\leq}$ that is defined by a hyperplane H that intersects the interior of $P \times L$ and satisfies $H \cap (P \times \{0\}) = F \times \{0\}$.*

Figure 1 depicts an example. Note that we only consider the wedging operation when it is done over a facet of P. The operation can be extended to faces of smaller dimension, but we don't use this here. Also, note that there is a distinction between 'the' wedge on P over F (the combinatorial class) and 'a' wedge on P over F (one realization in this combinatorial class; a different realization arises when choosing different a $H^{\leq}$). In what follows, which one we refer to will be clear from the context.

By construction, P' has $f + 1$ facets. The *lower base* $P_l = P \times \{0\}$ of P' and the *upper base* $P_u = H \cap (P \times L)$ of P' are facets of P', and both are isomorphic to the original polyhedron P. The remaining $f - 1$ facets of P' are contained in spaces of the form $G \times L$, where $G \neq F$ is

a facet of P; we call them the *sides* of the wedge. The lower base P_l lies in the affine subspace $\mathbb{R}^d \times \{0\}$ while the upper base P_u lies in the affine subspace H of dimension d. We use ϕ to denote the projection of a vector from $\mathbb{R}^d \times \{0\}$ to H corresponding to the product with L and ϕ^{-1} to denote the projection of a vector from H to $\mathbb{R}^d \times \{0\}$.
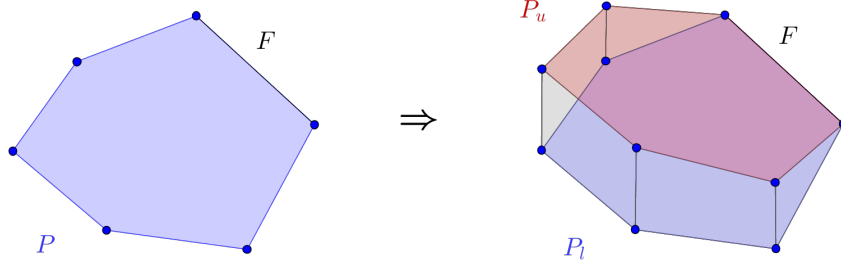

Fig. 1. The wedge P' on the hexagon P over facet F. Bases are P_l and P_u.

Let us take a look at the set of circuits of a wedge $\mathcal{C}(P')$. The set $\mathcal{C}(P')$ contains precisely the normalized vectors in the linear subspaces coming from the intersection of any subset of d facets. The following lemma characterizes this set in terms of $\mathcal{C}(P)$:

Lemma 11 (Circuits of a wedge). *Let $P \subseteq \mathbb{R}^d \times \{0\}$ be a d-dimensional polyhedron with set of circuits $\mathcal{C}(P)$ and let F be one of its facets. Then if P' is a wedge on P over F, the set of circuits $\mathcal{C}(P')$ is comprised of vectors of the form*

(i) $(0, 0, \ldots, 0, \pm 1)^T \in \mathbb{R}^{d+1}$
(ii) $(\pm c, 0)^T \in \mathbb{R}^{d+1}$*, where* $c \in \mathcal{C}(P) \subseteq \mathbb{R}^d$
(iii) $\phi\big((\pm c, 0)^T\big) \in \mathbb{R}^{d+1}$*, where* $c \in \mathcal{C}(P) \subseteq \mathbb{R}^d$

Proof. Each circuit direction of P' is defined by a selection of d facets with linearly independent outer normals.

(i) First, consider the intersection of d sides of P' and recall that they correspond to d facets $G_1, \ldots, G_d$ of P. $G_1 \cap \cdots \cap G_d = \{0\}$ in $\mathbb{R}^d$ and thus by construction $(0, \ldots, 0, \pm 1)^T \in \mathbb{R}^{d+1}$ are the corresponding circuits.

(ii) Next, let the lower base P_l be one of the d facets in the intersection. Then the other $d - 1$ facets again correspond to facets $G_1, \ldots, G_{d-1}$ of P: For the sides, these are the same facets as above; the facet corresponding to P_u is F. For this, the intersection of these d facets is in one-to-one correspondence with the intersection $G_1 \cap \cdots \cap G_{d-1}$, which gives two circuits $\pm c \in \mathcal{C}(P) \subseteq \mathbb{R}^d$. We obtain circuits $(\pm c, 0)^T \in \mathbb{R}^{d+1}$.

(iii) It remains to consider the intersection of the upper base P_u with $d - 1$ sides. (Note that the case with both P_u and P_l in the intersection is already covered above.) But by construction, the $d - 1$ sides intersect P_u in the facets $G_1, \ldots, G_{d-1}$ of P_u. Analogously, these facets correspond to facets $G'_1, \ldots, G'_{d-1}$ of P_l, which gives circuits $\pm c \in \mathbb{R}^d$. In this case, we obtain circuits $\phi((\pm c, 0)^T) \in \mathbb{R}^{d+1}$.

$\square$

Wedges are a basic building block for results on combinatorial diameters, due to some nice properties:

1. Wedging over a facet F of a simple polyhedron P gives a simple polyhedron P'. This is because all vertices of P' are contained in $P_l \cup P_u$, and both P_l and P_u are isomorphic to the simple polyhedron P.
2. The wedge satisfies $\Delta_{\mathcal{E}}(P') \leq \Delta_{\mathcal{E}}(P) + 1$, as for any vertex $v \in P_u$ there is a neighboring vertex $u \in P_l$; in fact $v = \phi(u)$.
3. Any edge walk in P' transfers to an edge walk in P by projecting to $\mathbb{R}^d \times \{0\}$ using ϕ^{-1}. Because of this, we obtain $\Delta_{\mathcal{E}}(P) \leq \Delta_{\mathcal{E}}(P')$.

Wedges also are helpful in the analysis of circuit diameters. However, for circuit walks in wedges, the situation is more involved. Clearly all wedges P' on P over a facet F satisfy $\Delta_{\mathcal{C}}(P') \leq \Delta_{\mathcal{C}}(P) + 1$, due to the neighboring vertices $v \in P_u$ and $u \in P_l$ (see 2. above). In contrast, circuit walks in P' do not necessarily transfer to circuit walks in P. This is because it is possible to hit the interior of P_u, as depicted in Figure 2: Let a walk begin at a vertex $u \in P_l \backslash F$. Then take a step along a circuit in $\mathcal{C}(P_u)$, which will give a y in a side of the wedge. Then continue from y along a circuit in $\mathcal{C}(P_l)$ to y', which may lie in the interior of P_u. Projecting this walk down to P_l gives a circuit step in P that does not use maximal step length and is thus not a circuit walk. The point y' may also be incident to both P_u and a side of the wedge, or lead to another point where this happens, in which case P' is not $\mathcal{C}$-simple – and this is possible even if P itself is $\mathcal{C}$-simple. See Figure 3.
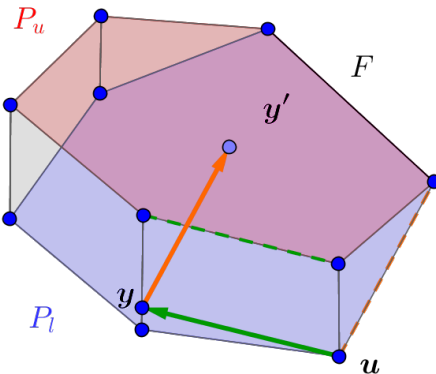

Fig. 2. A circuit walk u, y, y' in P' that does not project to a circuit walk in P_l.

Thus the corresponding circuit formulations of the above Properties 1. and 3. do not hold. In particular, the wedge operation may reduce the circuit diameter by creating 'shortcuts' between vertices of P_l by walks that take an intermediary step into the interior of P_u. In fact, it is possible to construct a polyhedron P and a wedge P' with $\Delta_{\mathcal{C}}(P) > \Delta_{\mathcal{C}}(P')$ (based on the construction for Lemma 15 in [BLF16b]).

But as we will see, we will not need this property for our arguments. On the other hand, we will require that polyhedra constructed by wedging are $\mathcal{C}$-simple polyhedra (compare 1.). The wedge operation may create a P' that is not $\mathcal{C}$-simple, even if the underlying polyhedron P is $\mathcal{C}$-simple. In the following, we explain how to deal with this problem. First, we require some new terminology that strengthens the $\mathcal{C}$-simple property.

Definition 12 (Wedge-simple). *Let P be a $\mathcal{C}$-simple polyhedron. P is a [1-]wedge-simple polyhedron with respect to a facet F if there is a wedge P' on P over F that is $\mathcal{C}$-simple. P is wedge-simple if for all facets F there is a wedge P' on P over F that is $\mathcal{C}$-simple. We can then recursively define P to be k-wedge-simple for $k \geq 2$, if for all facets F there is a wedge P' on P over F that is $(k-1)$-wedge-simple.*

With this terminology, we are ready to prove that a $\mathcal{C}$-simple polyhedron can be perturbed to obtain a wedge-simple polyhedron.

Lemma 13. *Let P be a $\mathcal{C}$-simple polyhedron. Then there is a wedge-simple polyhedron P^* in the same dimension and with the same number of facets with $\Delta_{\mathcal{C}}(P) \leq \Delta_{\mathcal{C}}(P^*)$.*

Proof. Let P' be a wedge on P over facet F, and consider a mild perturbation of the facets of P'. Perturbing a side of P' has the same effect as perturbing the corresponding facet in P before wedging; perturbing either base P_l or P_u has the same effect as perturbing the facet F of P. Thus, it is possible to guarantee some properties of P' by using a mild perturbation of P before the application of the wedge operation.
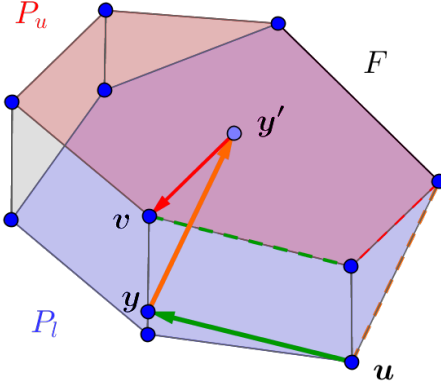
Fig. 3. Polyhedron $P = P_l$ is $\mathcal{C}$-simple but not wedge-simple: the walk u, y, y', v is a non-simple circuit walk in P' – two new facets are seen in the step from y' to v.

First, recall Lemma 2 telling us that any polyhedron can be perturbed to get a $\mathcal{C}$-simple polyhedron with at least the same circuit diameter. Thus if P' is not $\mathcal{C}$-simple, it can be perturbed mildly, which gives a polyhedron that is $\mathcal{C}$-simple and of at least the same circuit diameter. But then P' is just a wedge over a mildly perturbed P, by the above arguments. Thus it is always possible to perturb an original polyhedron such that the wedge over a given facet F is $\mathcal{C}$-simple.

So let now P allow a wedge P' over facet F that is $\mathcal{C}$-simple. Further, let $F' \neq F$ be a different facet of P and consider a wedge P'' over facet F'. If P'' is not $\mathcal{C}$-simple, again a mild perturbation of P'' will make it $\mathcal{C}$-simple. This perturbation is also a perturbation of P, so that it is necessary to consider what happens with the wedge over facet F.

Note that a sufficiently small mild perturbation will maintain $\mathcal{C}$-simplicity using similar arguments as in the proof of Lemma 2: we perturb by less than the minimum distance between points in $Y_{P'}$. But now a perturbation of P is also a perturbation of the wedge P' over F, and thus P' stays $\mathcal{C}$-simple for this perturbation.

This means that one can iteratively apply perturbations to the original polyhedron P to guarantee that the wedges over all facets F are $\mathcal{C}$-simple. With each of these perturbations, all facets for which there was a $\mathcal{C}$-simple wedge keep this property. There are only a finite number of facets of a given polyhedron. This proves the claim. $\qquad \square$

Additionally, Lemma 13 transfers to k-wedge-simplicity.

Corollary 14. *Let P be a $\mathcal{C}$-simple polyhedron and let $k \in \mathbb{N}$ be given. Then there is a k-wedge-simple polyhedron P^* in the same dimension and with the same number of facets with $\Delta_{\mathcal{C}}(P) \leq \Delta_{\mathcal{C}}(P^*)$.*

Proof. Let $P = P_0, P_1, \ldots, P_k$ be a sequence of wedges P_{i+1} on P_i for $0 \leq i < k$ and suppose w.l.o.g. P_k is not $\mathcal{C}$-simple. This can be amended by a slight perturbation of P_k (Lemma 2), which translates to a perturbation of P_{k-1}, which in turn translates to a perturbation of P_{k-2} and so on up to a perturbation of $P = P_0$. Repeated application of the proof of Lemma 13 thus gives the claim. $\qquad \square$

The above statements sum up to the equivalence of the following conjecture to the original formulation in Conjecture 1.

Conjecture 15 (Wedge-Simplicity). For any k-wedge-simple d-dimensional polyhedron with f facets and any $k \in \mathbb{N}$ the circuit diameter is bounded above by $f - d$.

In light of the many applications of wedging over polyhedra in the studies of the combinatorial diameter, Conjecture 15 may be useful in its own right. It tells us that we may restrict our studies

to $\mathcal{C}$-simple polyhedra for which an arbitrarily large number of wedging operations still gives a $\mathcal{C}$-simple polyhedron.

We conclude this section by explaining how k-wedge-simplicity transfers from a polyhedron P to its faces.

Lemma 16. *Let P be a wedge-simple d-dimensional polyhedron and F be any d'-face of P for $1 < d' < d$. Then F is also wedge-simple.*

Proof. Let F be a d'-dimensional face of the wedge-simple polyhedron P. In order to prove that F is wedge-simple, we need to show that there is a $\mathcal{C}$-simple wedge on F over each of its facets. To this end let G be a facet of F. We know that $G = F \cap \tilde{G}$, where $\tilde{G}$ is a facet of P. By wedge-simplicity of P, there is a wedge W on P over $\tilde{G}$ that is $\mathcal{C}$-simple. Recalling Definition 10, let $H^{\leq}$ be the closed halfspace intersected with $P \times L = P \times [0, \infty)$ to produce the wedge W, and let H be the defining hyperplane for $H^{\leq}$ in $\mathbb{R}^{d+1}$. In particular this means that $H \cap (P \times \{0\}) = \tilde{G} \times \{0\}$; that is, the hyperplane intersects the lower base P_l in the facet $\tilde{G}$. Hence we have:

$$
\begin{aligned}
H \cap (F \times \{0\}) &= H \cap ((P \times \{0\}) \cap (F \times \{0\})) \\
&= (H \cap (P \times \{0\})) \cap (F \times \{0\}) \\
&= (\tilde{G} \times \{0\}) \cap (F \times \{0\}) \\
&= G \times \{0\}
\end{aligned}
$$

So we can use the same halfspace $H^{\leq}$ intersected with $F \times L$ to produce a wedge W_F on F over G that is contained in W.

Further W_F is a face of W. To see this, let $H_F \subset \mathbb{R}^d$ be a supporting hyperplane of P that intersects P exactly in F: $H_F \cap P = F$. Then we have the following implications:

$$
\begin{aligned}
H_F \cap P &= F \\
\Rightarrow (H_F \times L) \cap (P \times L) &= (F \times L) \\
\Rightarrow (H_F \times L) \cap (P \times L) \cap H^{\leq} &= (F \times L) \cap H^{\leq} \\
\Rightarrow (H_F \times L) \cap W &= W_F \\
\Rightarrow (H_F \times \mathbb{R}) \cap W &= W_F \quad (\text{since } W, W_F \subseteq \mathbb{R}^d \times L)
\end{aligned}
$$

Clearly $W \subseteq P \times L$ is contained in $H^{\leq}_{\tilde{F}} \times \mathbb{R}$, since P is contained in $H^{\leq}_{\tilde{F}}$. This means that W_F is a face of W, with supporting hyperplane $H_F \times \mathbb{R}$. Circuit walks in W_F are therefore also circuit walks in W, and so if W is $\mathcal{C}$-simple, then W_F must be $\mathcal{C}$-simple as well. Since the facet G of F was selected arbitrarily at the beginning, the proof works for any facet; this proves that F is wedge-simple. $\square$

Lemma 17. *Let P be a k-wedge-simple d-dimensional polyhedron and F any d'-face of P for $1 < d' < d$. Then F is also k-wedge-simple.*

Proof. Following the proof of Lemma 16, let F be a d'-dimensional face of the k-wedge-simple polyhedron P, and let G be a facet of F. Call $\tilde{G}$ the facet of P such that $G = F \cap \tilde{G}$, W the wedge on P over $\tilde{G}$, and W' the wedge on F over G, constructed using the same closed halfspace as W. Then we saw above that W' is a $(d'+1)$-face of W.

This means that we can proceed with this proof using induction on k. The base case, $k = 2$, is handled immediately by Lemma 16: If P is 2-wedge-simple, then W is wedge-simple, and thus so is W', implying F is 2-wedge-simple by the arbitrary choice of G.

The same argument can be used to prove the inductive step: if it is true for $(k-1)$, then using the same constructions, if P is k-wedge-simple, then W is $(k-1)$-wedge-simple, and thus so is W'. Since G was chosen arbitrarily at the beginning, this implies that F itself is k-wedge-simple. $\square$

3   The Conjectures

Next, we prove the equivalence of several variants of Conjecture 1. Most of them are circuit analogues of variants of the Hirsch conjecture for the combinatorial diameter. We begin with a discussion and formal statements for these conjectures and then prove their equivalence.

3.1 Non-revisiting circuit walks

One of the most useful variants in the studies of the Hirsch conjecture for the combinatorial diameter is the *non-revisiting conjecture*. A walk is *non-revisiting* if no facet is left during the walk then entered again at a later step; the non-revisiting conjecture was that any two vertices are connected by such a walk. In particular this means that if two vertices u, v lie in the same face of a polyhedron, there would be a walk connecting the two vertices that stays in this face and adheres to the given bound.

In an edge walk in a simple polyhedron, at each step exactly one facet is left and exactly one other facet is entered. The non-revisiting conjecture requires that the entered facet is new, i.e. it has not been left before. For circuit walks the situation is a bit more complicated. Circuit steps that are not along edges will leave multiple facets simultaneously, and, in a non-$\mathcal{C}$-simple polyhedron, may enter multiple facets as well. To transfer the concept of a non-revisiting walk, we consider the facets that are entered during a walk.

Recall that the connection of the non-revisiting conjecture for edge walks and the original Hirsch conjecture, in fact, comes from the 'positive' interpretation of the above: One enters exactly one new facet in each step. As one starts at a vertex, to which d facets are incident, entering a new facet in each step immediately gives the bound $f - d$. We use this interpretation to give a viable formulation for circuit walks.

Conjecture 18 (Non-revisiting). For any polyhedron P and two vertices $u, v \in P$, there is a circuit walk from u to v that enters a new facet in each step, that is, each step produces an active facet that was inactive at all previous steps.

Each circuit step enters at least one new facet, and may leave any number of old facets. So generally, it is possible to enter 'old' facets in a step as long as one also enters a new facet. However, for $\mathcal{C}$-simple polyhedra, only exactly one new facet is entered in each step. Then the above formulation is equivalent to asking for a circuit walk from u to v that does not enter a facet it left before.

It is easy to see that Conjecture 18 implies Conjecture 1 with the same arguments as before: One begins at a vertex, to which (at least) d facets are incident, and enters a new facet in each step, which gives a bound of $f - d$. As we will see in Theorem 3, Conjecture 1 also implies Conjecture 18.

3.2 Any start

In contrast to edge walks, which only walk between vertices of a polyhedron, one may consider circuit walks that begin at a feasible point in a polyhedron that is not a vertex: indeed many steps in circuit walks do not begin at vertices. To study partial sequences of circuit walks (and for many other reasons), we here present a variant of the conjectures that deals with the necessary generalization of the starting point. We say that a set of facets is linearly independent if the corresponding outer normals are linearly independent.

Conjecture 19 (Any start). For any d-dimensional polyhedron P with f facets and any finite set M of points in P, the length of a circuit walk from any point $u \in M$ to any vertex in P is bounded above by $f - d'$, where d' is the number of linearly independent facets active at v.

Note the number of active facets for any vertex u of a d-dimensional polyhedron is at least d. The above conjecture gives rise to a generalized notion of circuit walk that may start anywhere in the polyhedron, not only at a vertex.

Let us briefly look at an example to see why the above formulation is plausible: Consider a simplex in $\mathbb{R}^d$ and recall it has $d + 1$ vertices and $d + 1$ facets, which implies $f - d = 1$. It has combinatorial diameter 1, which transfers to circuit diameter 1. However, the number of steps to a vertex can be much larger for non-vertices. For example, let a walk begin at a feasible point u in the strict interior of a facet F. Then $d' = 1$ and $f - d' = d$. Walking to the unique vertex that is not incident to F requires exactly d steps (if we assume the simplex to be $\mathcal{C}$-simple with respect to a set M of points that includes the starting point). Figure 4 depicts an example in dimension 3.
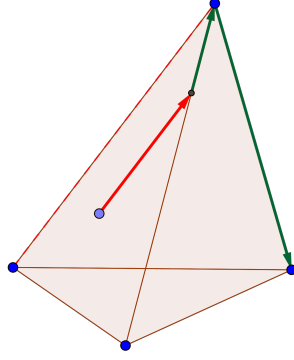
Fig. 4. A simplex in $\mathbb{R}^3$. The circuit distance of a point on the boundary to any vertex is at most three.

Consider a non-revisiting walk (as in Conjecture 18) starting at a non-vertex. Picking up a new facet in each step that did not appear before would transfer to a bound of $f - d'$, as only $d' \leq d$ facets are active in the beginning. It is clear that Conjecture 19 is at least as strong as Conjecture 1, as it encompasses the corresponding statement for M as the set of all vertices and due to $d' \geq d$ for all vertices. But Conjecture 18 also implies Conjecture 19:

Lemma 20. *Let P be a d-dimensional polyhedron with f facets, let u be a feasible point in P that is incident to d' linearly independent facets of P, and let v be a vertex of P. Suppose further that the non-revisiting conjecture (Conjecture 18) is true. Then there is a circuit walk from u to v that enters a new facet in each step.*

Proof. Our strategy for the proof is as follows: We construct a polyhedron $P' \subset P$ such that u, v are vertices of P'. Then we show that a non-revisiting walk from u to v in P' transfers to a non-revisiting circuit walk in P.

Let $F_1^v, \ldots, F_d^v$ be d linearly independent facets incident to v in P with outer normals $a_1^v, \ldots, a_d^v$. Let further $F_1^u, \ldots, F_{d'}^u$ be d' linearly independent facets incident to u in P and let $a_1^u, \ldots, a_{d'}^u$ be the corresponding outer normals. For $d' \geq d$, there is nothing to prove. For $d' < d$, there are $d - d'$ outer normals among $a_1^v, \ldots, a_d^v$, without loss of generality $a_1^v, \ldots, a_{d-d'}^v$, such that the set $\{a_1^u, \ldots, a_{d'}^u, a_1^v, \ldots, a_{d-d'}^v\}$ is linearly independent.

Let now $F_{d'+i}^{\geq} = \{x \in \mathbb{R}^d : (a_i^v)^T x \geq (a_i^v)^T u\}$ for $i \leq d - d'$. Note that by definition of the a_i, $(a_i^v)^T v \geq (a_i^v)^T u$ for all $i \leq d - d'$. Thus $P' = P \cap \bigcap_{i=1}^{d-d'} F_i^{\geq}$ contains both u and v, and both are vertices of P'. Informally P' is the intersection of P with a cone (starting at u) of facets that are parallel to facets incident to v.

By validity of Conjecture 18 there is a walk from vertex u to vertex v in P' that is non-revisiting. As P and P' only differ in facets incident to the starting point u of the walk, this implies that in each step of such a walk there is a facet from the original polyhedron P that bounds the step length. Thus the corresponding walk is also a circuit walk in P and it is non-revisiting in P. $\qquad\square$

3.3 Dantzig figures and the circuit d-step conjecture

Next, we consider the connection of Conjecture 1 to the so-called d-step conjecture and Dantzig figures. It is well known that for the maximal combinatorial diameter of d-dimensional polyhedra, it suffices to consider polyhedra with $2d$ facets. The maximal value of $f - d$ is realized in a polyhedron with $2d$ facets. This leads to the circuit equivalent of the d-step conjecture.

Conjecture 21 (d-step). For any d-dimensional polyhedron with $2d$ facets the circuit diameter is bounded above by d.

With $f = 2d$, clearly $f - d = d$. Conjecture 21 treats a special class of polyhedra, and thus is a specialization of Conjecture 1. But just as for the combinatorial diameter, we will see that both conjectures are equivalent.

In fact, one may even restrict the studies to the so-called *Dantzig figures*. First, let us define these figures. We follow [YKK84].

Definition 22 (Dantzig figure). *Let P be a d-dimensional polyhedron with $2d$ facets, of which exactly d are incident to a vertex u and the other d are incident to a vertex v. Then the tuple (P, u, v) is a* Dantzig figure.

Essentially, Dantzig figures are the intersection of two d-dimensional cones of d facets. In fact, it will suffice to consider the distances of u and v in such a Dantzig figure.

Conjecture 23 (Dantzig figure). For any d-dimensional Dantzig figure and vertices u, v not sharing a facet, the circuit distance from u to v is bounded above by d.

Note that for a $\mathcal{C}$-simple Dantzig figure, the circuit distance from u to v is at least d. If it is equal to d, then a corresponding walk is non-revisiting.

3.4 Equivalence of the Conjectures

We now prove the equivalence of Conjectures 1, 18, 21, and 23. The proof methods are inspired by the corresponding proof for the combinatorial diameter in [YKK84], but we have to pay significantly more attention to technical detail.

Theorem 3. *The following statements are equivalent:*

1. *Let u, v be two vertices of a k-wedge-simple polyhedron P for $k \geq f$. Then there is a non-revisiting circuit walk from u to v.*
2. *$\Delta_{\mathcal{C}}(f, d) \leq f - d$ for all $f \geq d$*
3. *$\Delta_{\mathcal{C}}(2d, d) \leq d$ for all d*
4. *For all d-dimensional Dantzig figures (P, u, v), the circuit distance of u and v is at most d.*

$1. \Rightarrow 2.$ First, recall that $\Delta_{\mathcal{C}}(f, d)$ is realized by a k-wedge-simple polyhedron. In a non-revisiting circuit walk from vertex u to v, a new facet is entered in each step. As P is k-wedge-simple, it in particular is $\mathcal{C}$-simple, so this is exactly one new facet per step. u is incident to exactly d facets. Thus there are at most $f - d$ steps. $\qquad\square$

$2. \Rightarrow 3.$ $\Delta_{\mathcal{C}}(2d, d) \leq d$ just states the special case of $\Delta_{\mathcal{C}}(f, d) \leq f - d$ for $f = 2d$. $\qquad\square$

$3. \Rightarrow 4.$ The d-dimensional Dantzig figure (P, u, v) in particular has $2d$ facets. $\qquad\square$

$4. \Rightarrow 1.$ Let P be a k-wedge-simple d-dimensional polyhedron with $d + m$ facets and let u, v be two of its vertices. Use $u = u_0$ and consider a face F_0 of smallest dimension that contains both u and v. Let $\dim F_0 = d_0$, the number of its facets $f_{d_0 - 1}(F_0) = d_0 + m_0$, $d_0 \leq d$ and $m_0 \leq m$.

By choice of F_0, there are no $(d_0 - 1)$-dimensional faces of F_0 containing both u and v. Thus we have $m_0 = d_0 + k$, where k is the number of $(d_0 - 1)$-dimensional faces that are not incident to either u or v. If $k > 0$, let G be such a facet of F_0.

Now construct a wedge F_1 over F_0 with respect to G such that it is $(k - 1)$-wedge simple, which is possible as P is k-wedge-simple. The $(d_0 + 1)$-dimensional polyhedron F_1 has $k - 1$ facets which are not incident to x or to $v_1 = \phi(v)$, where ϕ is the projection of $v \times \{0\}$ to the upper base.

By repeating this process of replacing a polyhedron by a wedge with smaller number of faces not incident to the vertices u and v_i, after at most k steps one obtains a Dantzig figure (P, u, v_k) of dimension $d_0 + k$. By validity of 4., a circuit walk from u to v_k has length at most $d_0 + k$. As we began with a k-wedge-simple polyhedron with $k \geq f$, we know (P, u, v_k) is $\mathcal{C}$-simple. Because of this, such a walk from u to v_k is non-revisiting.

Recall that (P, u, v_k) also is a wedge, so all circuit walks but those that hit the interior of the upper base transfer to circuit walks in the previous polyhedron. But circuit walks that hit the interior of the upper base first must leave one of the facets incident to v_k, and thus they are

revisiting. The transfer to the original circuit walks is possible via the canonical projection along the $(d_0 + k)$-th coordinate to the lower base; by doing so, all circuit steps from the upper base are replaced by the corresponding steps in the lower base, and steps along $(0, \ldots, 0, \pm 1)^T$ are ignored. Thus there is a non-revisiting walk from u to v in the original polyhedron. $\square$

3.5 A connection of unbounded and bounded circuit diameters

In the proof of Lemma 20, we added extra facets incident to a boundary point u of P to obtain a polyhedron $P' \subset P$ in which u is a vertex. The added facets were parallel to existing facets, such that $\mathcal{C}(P) = \mathcal{C}(P')$. By performing a similar construction it is possible to transform an unbounded polyhedron P to a bounded polytope P' with $\mathcal{C}(P) = \mathcal{C}(P')$ and without cutting off any vertices. We will do so by adding facets through u that are parallel to facets through a particular vertex v. In doing so, we obtain an intimate connection of the circuit diameters of bounded polytopes and unbounded polyhedra.

In the following, for a facet with outer normal a_i, we call a facet with outer normal $-a_i$ an *opposite facet*. We begin by examining how many opposite facets have to be added to a single vertex of an unbounded polyhedron P to obtain a bounded polytope P'. Here we say a facet *blocks* an edge direction incident to vertex v if the half-line starting at v in edge direction intersects the facet. Essentially, an edge direction only is unbounded if there is no blocking facet.

Lemma 24. *Let P be a d-dimensional unbounded polyhedron with f facets, and let u, v be two vertices of P that do not share a facet. Then there is a bounded polyhedron $P' \subset P$, where P' is constructed by adding at most $d-1$ facets incident to u that are opposite facets for facets incident to v. By this construction, $u, v \in P'$ are vertices and $\mathcal{C}(P) = \mathcal{C}(P')$.*

Proof. By construction, P' still contains both u and v, as all facets that are added are opposite facets of facets that are incident to v. In particular, the facets are parallel to existing ones, so $\mathcal{C}(P) = \mathcal{C}(P')$. Thus it suffices to prove that at most $d-1$ facets are necessary to create a bounded polytope.

Consider the cone of unbounded directions in P. In particular, it is contained in the inner cone of v. Thus it suffices to block the edge directions of the inner cone in P' by the addition of extra facets. A simple way to do so would be to add opposite facets incident to u to *all* facets incident to v. In fact, this would give a bounded box that contains P'.

However, not all of these facets are necessary. Let S be a simple cone coming from the selection of exactly d linearly independent facets of the $d^* \geq d$ facets incident to v. Then S contains the inner cone and it suffices to block the corresponding d edge directions of S.

The graph of P is connected and thus there is an edge incident to v that leads to a neighboring vertex. It is possible to choose the facets $F_1, \ldots, F_d$ for S as a superset of those $d-1$ facets that define such an edge. This implies that we only have to block $d-1$ edge directions to validate the claim.

Let now e be one of the edge directions of S incident to v, where e is defined by the intersection of $d-1$ facets. There is exactly one facet F_i with outer normal a_i such that $a_i^T e < 0$. By adding the opposite facet incident to u of F_i, the corresponding edge direction is blocked, i.e. it is not unbounded anymore. By doing this for all $d-1$ edge directions that have to be blocked, one obtains a bounded polytope. $\square$

Note that the number of facets that are necessary for the construction may be lower than $d-1$ if the unbounded cone is of lower dimension than d or if multiple edges lead from v to neighbors in the graph of P. Lemma 24 is our key ingredient to connect the maximal circuit diameter $\Delta_{\mathcal{C}}^u(f, d)$ of an unbounded d-dimensional polyhedron with f facets and the maximal circuit diameter $\Delta_{\mathcal{C}}^b(f, d)$ of a bounded d-dimensional polytope with f facets.

Theorem 4. *If all $\mathcal{C}$-simple bounded (f', d')-polytopes with $f' \leq f + d - 1$ and $d' \leq d$ satisfy the non-revisiting conjecture (Conjecture 18), then $\Delta_{\mathcal{C}}^u(f, d) \leq f - 1$.*

Proof. Let P be an unbounded d-dimensional polyhedron with f facets and let u, v be two of its vertices. We assume u and v do not share a facet; otherwise consider the minimal face that contains u and v in place of P. Further, let P' be constructed as in the proof of Lemma 24,

where new facets are added to u. Then P' has at most $f + (d-1)$ facets. Now perturb polytope P' to obtain the $\mathcal{C}$-simple polytope P''. Let u' be a vertex of P'' that corresponds to u in the non-perturbed P', and to which all (at most $d-1$) extra facets are incident. (The existence of such a u' in P'' can be guaranteed by first fixing a set of d facets that contains all extra facets and relaxing all other facets slightly. Any subsequent perturbation then keeps the single point of intersection u' of these facets feasible, i.e. u' is a vertex of P''.) Let further v' be a vertex of P'' that corresponds to v in P'.

Now consider a non-revisiting circuit walk from u' to v' in P''. As it is non-revisiting, none of the extra facets is the only blocking facet in any step – in other words the step length is always bounded by one of the original facets. This means that the circuit walk transfers to a circuit walk from u to v in P, with the same number of steps. The circuit walk thus has length at most $f - 1$, as none of the initial d facets is revisited.

<div align="right">□</div>

Note that the constructed walk for the unbounded polyhedron P may be revisiting. An interesting special case arises for Dantzig figures. For this case, Theorem 4 can be refined.

Corollary 25. *If all d-dimensional bounded spindles $P(u, v)$ with $2d - 1$ facets incident to u and d facets incident to v have a non-revisiting circuit walk from u to v of length at most d, then the d-step conjecture holds in dimension d, even for unbounded Dantzig figures.*

Note the non-revisiting condition in the above corollary is needed to transfer a circuit walk in the bounded polytope to be a circuit walk in the unbounded polyhedron. In dimension 4, all spindles have length at most 4 [SST12]. Showing that such a walk can be realized in a non-revisiting manner gives the circuit 4-step conjecture for bounded and unbounded polyhedra. In the following section, we give two proofs of the circuit 4-step conjecture: first by a careful analysis of the Klee-Walkup polyhedron U_4 and second via Corollary 25 by showing that 4-spindle walks can in fact be made non-revisiting.

4   The Circuit 4-step Conjecture

4.1   Proof via the Klee-Walkup polyhedron

The first unbounded counterexample to the Hirsch conjecture was given by Klee and Walkup in [KW67], where they constructed a 4-dimensional polyhedron with 8 facets and combinatorial diameter 5. In [SY15b], Stephen and Yusun prove that this polyhedron satisfies the Hirsch bound in the circuit diameter setting. We detail the proof here, and then consider the more general 4-step conjecture afterwards.

Denote by U_4 the polyhedron defined by the system of linear inequalities $\{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} \geq \mathbf{b}\}$,

$$
\text{where } A = \begin{pmatrix} -6 & -3 & 0 & 1 \\ -3 & -6 & 1 & 0 \\ -35 & -45 & 6 & 3 \\ -45 & -35 & 3 & 6 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ -8 \\ -8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

Its graph is shown in Figure 5: here the vertices are indexed by the four facets containing each one, while the points labelled with R's represent extreme rays. It is clear from the graph that vertices V5678 and V1234 are at graph distance five apart.

The result we need is the following:

Theorem 26. *The circuit diameter of the Klee-Walkup polyhedron U_4 is at most 4, independent of realization.*
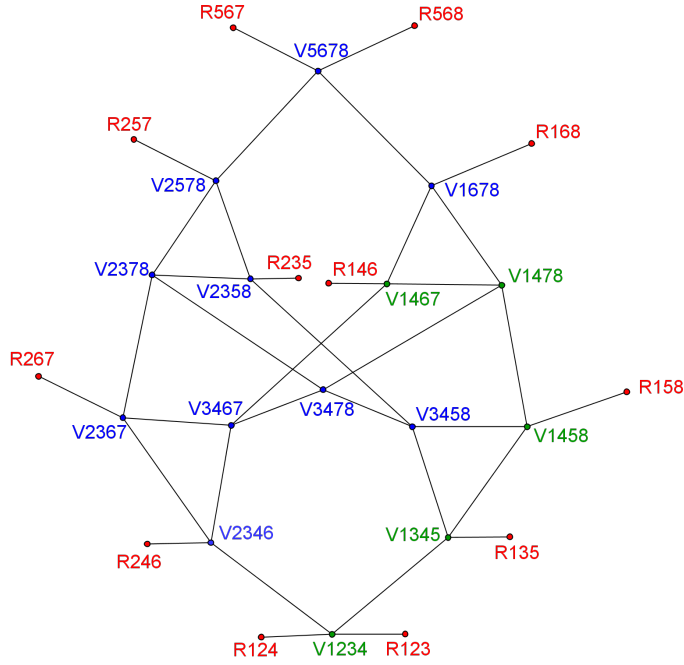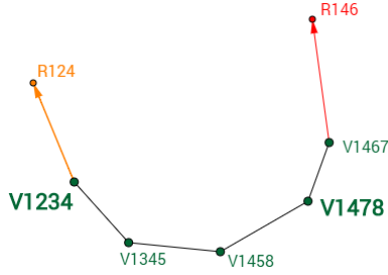
Fig. 5. The graph of U_4.

Fig. 6. The 2-face determined by facets `1` and `4`.

Proof. First we demonstrate the existence of a circuit walk of length 4 from `V5678` to `V1234`. Observe that we can take two edge steps as follows: `V5678` → `V1678` → `V1478`. Vertices `V1478` and `V1234` are both contained in the 2-face determined by facets `1` and `4`, so we can complete the walk on this face. Note that this 2-face is an unbounded polyhedron on six facets. Figure 6 is a topological illustration of this face, showing the order of the vertices and rays.

Now consider a vector **g** corresponding to the edge direction from `V1458` to `V1345` – this is the blue vector in Figure 7. Note that this is always a circuit direction in any realization of U_4 since it corresponds to an actual edge of the polyhedron.

To see that **g** is a feasible direction at `V1478`, consider vector **h** in the edge direction from `V1478` to `V1458`, and vector **r** in the direction of ray `R124`. Observe that **g** and −**h** are the two incident edge directions at `V1458`, and so **r** must be a strict conic combination of **g** and −**h**, i.e. $\mathbf{r} = \alpha_1(\mathbf{g}) + \alpha_2(-\mathbf{h})$ for $\alpha_1, \alpha_2 > 0$. By rearranging terms we see that **g** is a strict conic combination of **h** and **r**: $\mathbf{g} = (\alpha_2/\alpha_1)\mathbf{h} + (1/\alpha_1)\mathbf{r}$, with $\alpha_2/\alpha_1, 1/\alpha_1 > 0$. Feasibility of **r** and **h** at `V1478` implies that **g** is a feasible direction at `V1478`.

Now starting at `V1478` traverse **g** as far as feasibility allows. This direction is bounded since we exit the polyhedron when taking **g** from `V1458`. We will eventually exit the 2-face at a point along the boundary, and at one of the following positions:
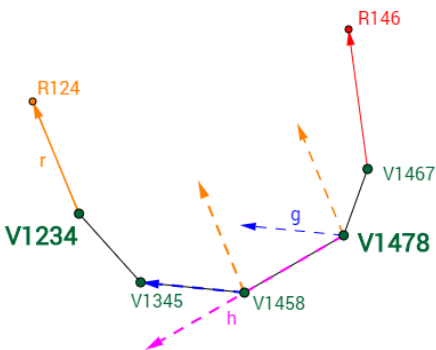
  − exactly at `V1234`,

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Fig. 7. Feasibility of the circuit direction **g**.

– on the edge connecting `V1234` and `V1345`, or
– on the ray `R124` emanating from `V1234`.

Hitting exactly `V1234` gives a circuit walk of length 3 from `V5678`, while the other two cases give circuit walks of length 4 since we only need one step to `V1234`. These two situations are illustrated in Figure 8.
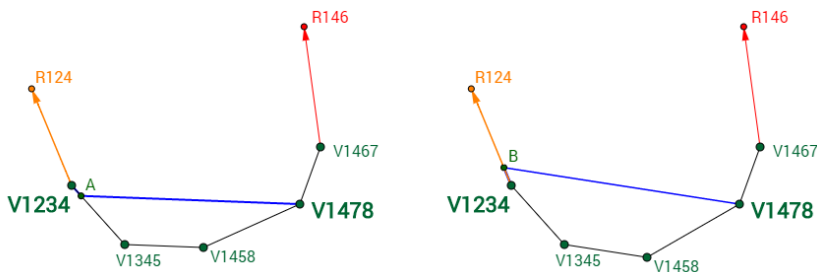

Fig. 8. Getting from `V1478` to `V1234` in at most 2 steps.

The argument is the same for the reverse direction (`V1234` to `V5678`). We can construct a similar walk by first traversing edges `V1234` → `V2346` → `V3467`, and then taking a maximal step in the circuit direction arising from the edge connecting `V1467` and `V1678`. Here we stay in the 2-face determined by facets 6 and 7. We can then arrive at `V5678` in at most two steps from `V3467`. □

We refer to a polyhedron with f facets in dimension d as an (f, d)-polyhedron. One consequence of Theorem 26 is the general circuit 4-step conjecture, however we will need the following fact to prove it:

Lemma 27. *Up to isomorphism, U_4 is the only non-Hirsch simple (8,4)-polyhedron.*

Proof. The simple bounded (8,4)-polytopes are enumerated in Grünbaum and Sreedharan [GS67]. A simple unbounded (8,4)-polyhedron can be truncated with an additional facet that cuts off the vertex at infinity, this produces a simplicial (9,4)-polytope. Observe that the diameter of this polytope will remain at least 5, as any route between `V1234` and `V5678` through the new facet will have to add the new ninth facet along with facets 5, 6, 7 and 8. Klee and Kleinschmidt in [KK87] mention that there is a unique simple polytope with $d = 4$, $f = 9$ and diameter 5, following directly from the complete enumeration of all polytopal simplicial 3-spheres, completed by Altshuler et al. [ABS80]. Thus this must be exactly that polytope, which we denote by Q_4. The result follows, as any non-Hirsch 4-polyhedron with 8 facets comes from Q_4 by projecting to infinity the ninth facet that does not contain either of the two vertices at distance 5. □

Polytopes such as Q_4 that satisfy $\Delta_{\mathcal{E}}(P) = f - d$ are known as *Hirsch-sharp* polytopes.

Theorem 5 (Circuit 4-step). $\Delta_{\mathcal{C}}(8,4) = 4$.

Proof. By Lemma 2, it suffices to consider $\mathcal{C}$-simple polyhedra – let P be a 4-dimensional polyhedron with 8 facets. If P is bounded then it has combinatorial diameter 4 ([SST12]), so suppose P is unbounded. By Lemma 27, P is combinatorially equivalent to U_4, and by Theorem 26 it has circuit diameter 4. $\square$

4.2 Proof via facial paths in 4-prismatoids

Here we present a second proof of Theorem 5. Recall that a *spindle* is a polytope with two distinguished vertices $\mathbf{x}$ and $\mathbf{y}$ such that each facet is incident to exactly one of $\mathbf{x}$ and $\mathbf{y}$. Polar to this, a *prismatoid* is a polytope with two distinguished facets Q^+ and Q^- (called its *bases*) that together contain all the vertices of the polytope. The *length* of a spindle is the graph distance between the two special vertices, while the *width* of a prismatoid is the dual graph distance between the two bases. These constructions were essential in finding counterexamples to the combinatorial Hirsch conjecture ([San12]). In [SST12] Santos et al. prove that 4-dimensional prismatoids have width at most 4. We strengthen this result here to get a non-revisiting path and use it to show $\Delta_{\mathcal{C}}(8,4) = 4$.

Lemma 28. *In a 4-prismatoid with parallel faces Q^+ and Q^-, there exists a facial path from Q^+ to Q^- such that at each step at least one new vertex of Q^- is encountered.*

Proof. Suppose a 4-prismatoid Q is given, with bases Q^+ and Q^-. If Q has width 2 then there is a facet of Q that is adjacent to both bases. The claim then follows as the number of vertices of Q^- incident to each facet in the facial path is strictly increasing.

If Q has width 3, suppose the facial path of length 3 is $Q^+ \to F \to G \to Q^-$. Then F must have at most 2 vertices from Q^- – any more and it would itself be adjacent to Q^-, and there would be a shorter path between the bases. Also, G must have at least 3 vertices in common with Q^- to be adjacent to it. Hence the number of vertices of Q^- incident to each facet in this facial path is also strictly increasing.

Suppose now that Q has width 4. Consider the pair (G^+, G^-) of geodesic maps constructed from the normal fans of the two bases.[2] Letting H be the common refinement of G^+ and G^-, label its vertices as *positive* if it is a vertex of G^+, *negative* if it is a vertex of G^-, and a *crossing vertex* otherwise. The main result of [SST12] was proven by showing that there is always a crossing vertex incident to both a positive and a negative vertex. Now this corresponds to a minimal facial path in Q of the form $Q^+ \to F_1 \to F_2 \to F_3 \to Q^-$.

To see why this proves the claim, first observe that under the current assumptions, F_2 can only be a tetrahedron with two vertices on Q^+ and two vertices on Q^-. Otherwise, if F_2 were incident to more than two vertices of say, Q^+, then it would be adjacent to Q^+ and we would have a shorter facial path between the bases, contradicting the assumption that Q has width 4. Denote by $\mathbf{x}$ and $\mathbf{y}$ ($\mathbf{z}$ and $\mathbf{w}$) the two vertices of Q^+ (Q^-) on F_2.

Let us now consider each step of the path. The first step $Q^+ \to F_1$ and the last step $F_3 \to Q^-$ clearly satisfy the condition we require. Moreover, going from F_2 to F_3, the number of vertices on Q^- increases from 2 to at least 3 – a strict increase as well.

As for the step from F_1 to F_2, the crucial observation is that the triangle of F_2 that is incident to F_1 contains two vertices of Q^+ ($\mathbf{x}$ and $\mathbf{y}$) and one of Q^- (assume without loss of generality that it is $\mathbf{z}$). This means that F_1 cannot contain $\mathbf{w}$ as well, or else F_1 would contain F_2 entirely. Therefore $\mathbf{w}$ is the new vertex of Q^- seen when moving from F_1 to F_2. $\square$

Note that although similar, the notion of seeing a new vertex of Q^- in each step is not the same as the interpretation of non-revisiting paths discussed in Section 3.1. Here we do not require that the new vertex at each step be new with respect to the entire path, just that it is a new vertex of Q^- at that step. Now we use the polar result for spindles, which in turn implies the next result:

---

[2] We refer the reader to [SST12] for a more detailed discussion of the techniques used.

Corollary 29. *Let $P(u,v) = P(u) \cap P(v) \subset \mathbb{R}^4$ be a bounded spindle coming from the intersection of two cones $P(u), P(v)$ at u, respectively v. Then there is an edge walk from u to v such that in each step, a new facet of $P(v)$ becomes active.*

Lemma 30. *Let $P(u,v) = P(u) \cap P(v) \subset \mathbb{R}^4$ be an unbounded spindle coming from the intersection of two cones $P(u), P(v)$ at u, respectively v. Let further $P(v)$ be simple. Then there is a circuit walk of length at most 4 from u to v.*

Proof. Let $a_1, \ldots, a_4$ be the outer normals of facets $F_1, \ldots, F_4$ incident to v. Let further $Q_i = \{x \in \mathbb{R}^d : (-a_i)^T x \leq (-a_i)^T u\}$, informally an opposite halfspace of the one created by F_i, but now moved to be incident to u_i. Set $P'(u) = P(u) \cap Q$ with $Q = \bigcap_{i=1}^{4} Q_i$ and $P'(u,v) = P'(u) \cap P(v)$. Clearly, $P'(u,v) \subset P(u,v)$ is a bounded spindle with simple cone $P(v)$ so that we may apply Corollary 29.

Thus there is a circuit walk from u to v in $P'(u,v)$ of length at most 4 such that in each step at least one of the facets of $P(v)$ becomes active. This means that the 'extra' facets introduced as Q never are the only facets to bound the step length. Combining this fact with $P'(u,v) \subset P(u,v)$, we see that the circuit walk from u to v in $P'(u,v)$ of length at most f is a circuit walk in $P(u,v)$, as well. This proves the claim. $\qquad\square$

Validity of Theorem 5 follows by combining Lemma 30 with Theorem 3.

5 Discussion

The (combinatorial) polynomial Hirsch conjecture remains a fundamental question. We consider a diameter question where we make the natural relaxation of edge walks to circuit walks. In this setting, the Hirsch bound of $f - d$ is again a possibility. We recover the Klee-Walkup equivalences for the Hirsch bound holding in this setting, and show that, in particular, the circuit version holds even for the unbounded 4-step conjecture, which fails for the combinatorial version.

Presently we do not see a clear path to fully resolve the general conjecture, though the circuit 5-step conjecture may be approachable via either of the approaches that worked for 4-step. An enumerative approach would argue that any simple (10,5)-polyhedra are obtained fairly directly from U_4, allowing us to leverage the construction in this paper. While this seems likely, see for example the discussion in Kim and Santos [KS10], it may require substantial enumeration to prove. The discussion in Firsching's thesis [Fir15] outlines the state-of-the-art for enumeration. Alternatively, it could be resolved by proving that 5-dimensional spindles with 9 facets on one side and 5 on the other side satisfy the non-revisiting conjecture.

It is interesting to consider *circuit Hirsch-tight* polyhedra, i.e. polyhedra that meet the $f - d$ bound. The combinatorial Hirsch-tight polytopes were intensively studied [FH99,HK98b,HK98a] before the demise of the bounded Hirsch conjecture. These include *trivial* Hirsch-sharp polytopes (where $f \leq 2d$) like the d-cube and the d-simplex, and *non-trivial* Hirsch-sharp polytopes (where $f > 2d$), which include the polytope Q_4, and others obtained by performing operations on Q_4 (see [KS10]). In the circuit setting, the d-simplex remains Hirsch-tight independent of realization since $f - d = 1$. A regular d-cube is also Hirsch-tight, but it is not obvious whether this remains true for non-regular realizations. Similarly, it is open whether there is a collapsing realization of U_4, i.e. one with a diameter of less than 4.

While almost all realizations of U_4 are Hirsch-tight, it is difficult to determine if there is a Hirsch-tight realization of Q_4 [SY15a]. Surprisingly, if there is a Hirsch-tight realization of Q_4, it will be tight in only one direction: if u and v are the vertices at combinatorial distance 5, it is always possible to find a circuit walk of length 4 from *either* u to v or from v to u. However, this does not transfer to the opposite direction – note that reversing a (maximal) circuit walk does not give a circuit walk.

Finally, we mention again the connections of bounded and unbounded diameters for both combinatorial and circuit walks. For the combinatorial diameter, we have the relation $\Delta_{\mathcal{E}}^b(f,d) \leq \Delta_{\mathcal{E}}^u(f,d)$, but we are not aware of any upper bound on $\Delta_{\mathcal{E}}^u(f,d)$ based on $\Delta_{\mathcal{E}}^b(f,d)$. This means that, for example, it could be possible that bounded polytopes satisfy a linear bound in $f - d$, while there could exist a class of unbounded polyhedra of quadratic or even worse diameter.

For circuit walks, the situation may be different. We showed that the circuit non-revisiting conjecture implies $\Delta_{\mathcal{C}}^b(f,d) + d - 1 \geq \Delta_{\mathcal{C}}^u(f,d)$; recall the proof of Theorem 4 required this for the ability to transfer circuit steps in a bounded spindle to be circuit steps in the original polyhedron. It would be interesting to see whether this dependence can be dropped.

Acknowledgments

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