Expected Crossing Numbers

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Abstract

The expected value for the weighted crossing number of a randomly weighted graph is studied. A variation of the Crossing Lemma for expectations is proved. We focus on the case where the edge-weights are independent random variables that are uniformly distributed on $[0, 1]$.

Keywords: graph, crossing number, weighted crossing number, crossing lemma.

1 Introduction

The *crossing number* of a graph is the minimum number of internal intersections of edges in a drawing of the graph on the plane. Computing the crossing number, even for complete graphs, is a surprisingly challenging problem and an active area of research [RS09,Vrt10].

The notion of the weighted crossing number, when the edges have weights and each crossing counts as the product of the corresponding weights, has been

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used in various situations, since it mimics the possibility of having many edges in parallel. In this note we study the expected value of the weighted crossing number of the complete graph K_n on n vertices, where the weights of edges are independent random variables. We consider the situation where the weights are i.i.d. variables with the uniform distribution on $[0, 1]$. The first non-trivial case is K_5; here we outline a strategy to compute the expected value. Then we show that the expected crossing number of K_n retains the $\Theta(n^4)$ asymptotics of the usual crossing number $\mathrm{cr}(K_n)$ of complete graphs. This is proved by using a similar recurrence as used for the usual crossing number of complete graphs and, alternatively, by proving and applying a variation of the Crossing Lemma for expectations.

2 Preliminaries

Given a graph $G = (V, E)$, we denote its crossing number by $\mathrm{cr}(G)$. This is the minimum over all drawings of G in the Euclidean plane $\mathbb{R}^2$ of the number of crossings of edges in the drawing. All drawings are assumed to have simple polygonal arcs representing the edges of the graph, and it is assumed that each pair of edges involves at most one intersection of their representing arcs. Here and in the remainder of the paper, we consider only internal intersections of edges. Formally, a *crossing* in a drawing $\mathcal{D}$ is an unordered pair $\{e, f\}$ of edges whose arcs in $\mathcal{D}$ intersect each other internally. We let $\mathbb{X}(\mathcal{D})$ denote the set of all crossings and set $\mathrm{cr}(\mathcal{D}) = |\mathbb{X}(\mathcal{D})|$.

Given non-negative weights $w : E \to \mathbb{R}_+$ on the edges of G, we define the *crossing weight* of a drawing $\mathcal{D}$ of G as:

$$\mathrm{cr}(\mathcal{D}, w) = \sum_{\{e,f\}\in\mathbb{X}(\mathcal{D})} w(e)w(f).$$

We define the *weighted crossing number* of a weighted graph G as:

$$\mathrm{cr}(G, w) = \min_{\mathcal{D}} \mathrm{cr}(\mathcal{D}, w). \tag{1}$$

For a fixed graph, the function $\mathrm{cr}(G, \cdot)$ is also called the *crossing function* for G. We take the domain of $\mathrm{cr}(G, \cdot)$ to be $\mathbb{R}_+^E$. We remark that $\mathrm{cr}(G, 0) = 0$, $\mathrm{cr}(G, w) \geq 0$ and $\mathrm{cr}(G, \cdot) \equiv \mathbf{0}$ if and only if $\mathrm{cr}(G) = 0$. The function $\mathrm{cr}(G, \cdot)$ is piecewise quadratic in w, and the chambers defined by these pieces correspond to (groups of) optimal drawings for the contained weightings; the forms in the chambers are neither convex nor concave. If $\mathbf{1} \in \mathbb{R}_+^E$ is the constant all-1 function, then $\mathrm{cr}(G) = \mathrm{cr}(G, \mathbf{1})$.

The crossing function of any n-vertex graph is just a specialization of the crossing function $\mathrm{cr}(K_n, w)$ of the complete graph K_n, where we put weight 0 for the non-edges in the graph. In this sense the crossing functions of complete graphs contain information about crossing numbers of all graphs. This universality property was the main goal to introduce this notion in [Moh08,Moh10] and to propose its study.

Note that we allow the edges to be represented by any (polygonal) line, they need not be straight lines. The related question of rectilinear crossing numbers is also interesting and well-studied. While rectilinear crossing number is in some cases larger than the usual crossing number, they do not differ in the computations performed in this paper. As in the unweighted case, minimal drawings can be obtained without using double crossings (pairs of edges that cross more than once).

3 Computation of the expected crossing number

We begin by considering the expected crossing number of the complete graph K_n for some small values of n. We take the weights on the edges to be independently identically distributed random variables, with uniform distributions on the interval $[0, 1]$. Let us denote the expected value of $\mathrm{cr}(\mathrm{K}_n, w)$ under this distribution as $\mathrm{Eu}(n)$.

For $n \leq 4$, the graph can be drawn without crossings, so $\mathrm{Eu}(n) = 0 = \mathrm{cr}(\mathrm{K}_n)$. For $n \geq 5$, we have $0 < \mathrm{Eu}(n) < \mathrm{cr}(\mathrm{K}_n)$. In this section we outline how to compute $\mathrm{Eu}(5)$ directly from the definition of expectation. Our somewhat cumbersome case analysis can also be viewed as determination of the piecewise quadratic chambers for the crossing function of K_5.

We will denote the random weight assigned to the ith edge by $\mathbf{X}_i$, $i = 1, \ldots, 10$. We note that $\mathrm{cr}(\mathrm{K}_5) = 1$ and by symmetry, for any two non-adjacent edges, K_5 can be drawn so that those two edges are the single pair of crossing edges. Hence:

$$\mathrm{Eu}(5) = \mathbb{E}[\min_{\text{Edges } i,j \text{ do not share a vertex}} (\mathbf{X}_i \mathbf{X}_j)] \tag{2}$$

We abbreviate the quantity inside the expectation (which has 15 terms) as $m(\mathbf{X})$. This is a problem in order statistics. The direct way to obtain $\mathrm{Eu}(5)$ is to evaluate:

$$\int_0^1 \int_0^1 \ldots \int_0^1 m(\mathbf{x}) \mathbf{d}x_1 \ldots \mathbf{d}x_9 \mathbf{d}x_{10} \tag{3}$$

where $m(\mathbf{x})$ is the function of $\mathbf{x} \in \mathbb{R}^{10}$ corresponding to the random variables of $m(\mathbf{X})$. To do this we break (3) into 10! terms based on the increasing order of the variables, i.e. we compute (3) via the sum:

$$\sum_{\sigma \in S_{10}} \int_0^1 \int_0^{x_{\sigma(10)}} \int_0^{x_{\sigma(9)}} \cdots \int_0^{x_{\sigma(2)}} m(\mathbf{x}) \mathbf{d}x_{\sigma(1)} \ldots \mathbf{d}x_{\sigma(9)} \mathbf{d}x_{\sigma(10)} \quad (4)$$

Here the permutations $\sigma \in S_{10}$ index the possible orderings of the random variables $\mathbf{X}$. This sum has 10! terms, but they can be grouped into a manageable number of cases. The proof proceeds by dividing cases based on the relative orderings of some of the remaining variables. The computed expectation is $\frac{35921}{1108800}$. The details of the calculation are available in [MS10].

As noted in Section 4.1, the computed value of $\mathrm{Eu}(5)$ is used in a lower bound for $\mathrm{Eu}(n)$ for $n \geq 5$.

4 Asymptotics

Some standard arguments used for crossing number estimates work also for the expectations. In this section we show that simple adaptations of these arguments show that $\mathrm{Eu}(n)$ is $\Theta(n^4)$. Since $\mathrm{cr}(\mathrm{K}_n)$ is $O(n^4)$ and an upper bound for $\mathrm{Eu}(n)$, we need only show the lower bound. We remark that the asymptotic upper bound $\mathrm{cr}(\mathrm{K}_n)$ can be obtained trivially from the fact that there are only $O(n^4)$ pairs of edges in K_n, but that much better constructive upper bounds exist and are an ongoing research challenge, see for instance [AAK06,PR07].

4.1 Asymptotics via a recurrence

We recall that we denote the crossing weight of a given drawing $\mathcal{D}$ of a graph weighted by w as $\mathrm{cr}(\mathcal{D}, w)$, and the weighted crossing number of G weighted by w (i.e. the minimum over all drawings) by $\mathrm{cr}(G, w)$.

Given a drawing $\mathcal{D}$ of K_n with weights w, we can consider the induced drawings of copies of $\mathrm{K}_n - v \approx \mathrm{K}_{n-1}$ obtained by removing each vertex $v \in V = V(K_n)$ from K_n in turn. Then

$$\sum_{v \in V} \mathrm{cr}(\mathcal{D}|_{\mathrm{K}_n - v}, w|_{\mathrm{K}_n - v}) = (n-4)\,\mathrm{cr}(\mathcal{D}, w) \quad (5)$$

since each pair of disjoint edges $ij, i'j'$ of K_n appear in all but four of the terms on the left side of (5).

Now consider K_n for $n > 4$ with a fixed weighting w. There is some optimal drawing $\mathcal{D}^*$ of K_n such that $\mathrm{cr}(\mathrm{K}_n, w) = \mathrm{cr}(\mathcal{D}^*, w)$. Now:

$$\begin{aligned}\mathrm{cr}(\mathrm{K}_n, w) = \mathrm{cr}(\mathcal{D}^*, w) &= \frac{1}{n-4}\sum_{v\in V} \mathrm{cr}(\mathcal{D}^*|_{\mathrm{K}_n - v}, w|_{\mathrm{K}_n - v}) \\ &\geq \frac{1}{n-4}\sum_{v\in V} \min_{\mathcal{D}} \mathrm{cr}(\mathcal{D}|_{\mathrm{K}_n - v}, w|_{\mathrm{K}_n - v}) = \frac{1}{n-4}\sum_{v\in V} \mathrm{cr}(\mathrm{K}_n - v, w|_{\mathrm{K}_n - v}).\end{aligned}$$

If the weights in w are i.i.d. random variables, we can take expectations on both sides to get $\mathrm{Eu}(n) \geq \frac{n}{n-4}\,\mathrm{Eu}(n-1)$. Applying this inequality recursively, we find for $n \geq 6$ that $\mathrm{Eu}(n) \geq \frac{1}{5}\binom{n}{4}\,\mathrm{Eu}(5)$.

4.2 Asymptotics via the Crossing Lemma

The version given below of the Crossing Lemma (with the specific constant $1024/31827 > 0.032$) is due to Pach et al. [PRTT06]; this result improves the constant as compared to previous versions.

Theorem 4.1 *Let G be a graph of order n with $m \geq \frac{103}{16}n$ edges. Then*

$$\mathrm{cr}(G) \geq \frac{1024}{31827}\,\frac{m^3}{n^2}.$$

Let π be a probability distribution with expectation $\mathbb{E}(\pi) = \mu$. We define the *complementary probability distribution* π^* by setting $\pi^*(\mu + x) = \pi(\mu - x)$. For the purpose of the following argument, let us assume that our probability distribution is symmetric, i.e., $\pi = \pi^*$. Then, given a random weight function w, the *complementary weight function* w^*, defined as $w^*(e) = 2\mu - w(e)$, has the same distribution as w. Let us define w' to be either w or w^*, so that $w'(e) \geq \mu$ holds for at least half of the edges $e \in E(G)$. Finally, let w_1 be defined as $w_1(e) = 0$ if $w'(e) < \mu$, and $w_1(e) = 1$ if $w'(e) \geq \mu$. Since $\mathrm{cr}(G, w) + \mathrm{cr}(G, w^*) \geq \mathrm{cr}(G, w') \geq \mu^2\,\mathrm{cr}(G, w_1)$, the following holds:

$$\begin{aligned}\mathbb{E}(\mathrm{cr}(G, w)) &= \tfrac{1}{2}\,\mathbb{E}(\mathrm{cr}(G, w) + \mathrm{cr}(G, w^*)) \geq \tfrac{1}{2}\,\mathbb{E}(\mathrm{cr}(G, w')) \\ &\geq \frac{\mu^2}{2}\,\mathbb{E}(\mathrm{cr}(G, w_1)) \geq \frac{\mu^2}{2}\cdot\frac{1024}{31827}\,\frac{(m/2)^3}{n^2} = \frac{64\mu^2}{31827}\,\frac{m^3}{n^2}.\end{aligned}$$

This gives a version of the Crossing Lemma for expectations. With a little more care we can improve the above bound and also get rid of the symmetry condition. In order to do this, we replace the mean by the *median*, i.e. the largest number ν such that $Prob[w(e) \geq \nu] \geq \frac{1}{2}$.

Theorem 4.2 *Let G be a graph of order n with $m \geq \frac{103}{16(1-4^{-1/3})} n$ edges. Suppose that each edge $e \in E(G)$ gets a random weight $w(e)$, where the weights of distinct edges are independent non-negative random variables (not necessarily i.d.) whose median is at least $\nu > 0$. Then*

$$\mathbb{E}(\mathrm{cr}(G, w)) \geq \frac{128\nu^2}{31827} \cdot \frac{m^3}{n^2}.$$

The proof is included in [MS10].

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