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A MAJORIZATION BOUND FOR THE EIGENVALUES OF SOME GRAPH LAPLACIANS*

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Abstract. Grone and Merris conjectured that the Laplacian spectrum of a graph is majorized by its conjugate vertex degree sequence. In this paper, we prove that this conjecture holds for a class of graphs, including trees. We also show that this conjecture and its generalization to graphs with Dirichlet boundary conditions are equivalent.

Key words. graph Laplacian, majorization, graph spectrum, degree sequence, Dirichlet Laplacian

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1. Introduction.

One way to extract information about the structure of a graph is to encode the graph in a matrix and study the invariants of that matrix, such as the spectrum. In this paper, we study the spectrum of the combinatorial Laplacian matrix of a graph.

The *combinatorial Laplacian* of a simple graph $G = (V, E)$ on the set of n vertices $V = \{v_1, \ldots, v_n\}$ is the $n \times n$ matrix $L(G)$ defined by

$$L(G)_{ij} = \begin{cases} \deg(v_i) & \text{if } \ i = j, \\ -1 & \text{if } \ \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\deg(v)$ is the *degree* of v, that is, the number of edges on v. The matrix $L(G)$ is positive semidefinite, and so its eigenvalues are real and nonnegative. We list them in nonincreasing order and with multiplicity:

$$\lambda_1(L(G)) \geq \lambda_2(L(G)) \geq \cdots \geq \lambda_{n-1}(L(G)) \geq \lambda_n(L(G)) = 0.$$

When the context is clear, we can write $\lambda_i(G)$, or simply λ_i. We abbreviate the sequence of n eigenvalues as $\lambda(L(G))$.

We are interested in the Grone–Merris (GM) conjecture that the spectrum $\lambda(L(G))$ is majorized by the conjugate partition of the nonincreasing sequence of vertex degrees of G [5]. This question is currently being studied, see, for example, [4], but has yet to be resolved. We extend the class of graphs for which the conjecture is known to hold to include trees, among other graphs. We also show that if GM holds for graph Laplacians, it also holds for more general Dirichlet Laplacians (cf. [2]) as conjectured by Duval [3].

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2. Background and definitions.

2.1. Graphs. Given a graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, there are several ways to represent G as a matrix. There is the *edge-incidence matrix*, an $n \times m$ matrix that records in each column the two vertices incident on a given edge. For directed graphs, we can consider a signed edge-incidence matrix:

$$\partial(G)_{ve} = \begin{cases} 1 & \text{if } v \text{ is the head of edge } e, \\ -1 & \text{if } v \text{ is the tail of edge } e, \\ 0 & \text{otherwise.} \end{cases}$$

There is also an $n \times n$ matrix $A(G)$ called the *adjacency matrix*, which is defined by

$$A(G)_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal of $A(G)$ is zero.

We can encode the (vertex) degree sequence of G in nonincreasing order as a vector $d(G)$ of length n and in an $n \times n$ matrix $D(G)$ whose diagonal is $d(G)$ and whose off-diagonal elements are 0. Then the combinatorial Laplacian $L(G)$ that we study is simply $D(G) - A(G)$. It is easy to check that if we (arbitrarily) orient G and consider the matrix $\partial(G)$ above, we also have $L(G) = \partial(G)\partial(G)^t$.

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G}$ on V whose edges are exactly those not included in G.

Remark 2.1. The Laplacian is sometimes defined with entries *normalized* by dividing by the square roots of the degrees. However, we do not do that here.

2.2. Majorization. We recall that a *partition* $p = p(i)$ is a nonincreasing sequence of natural numbers, and its *conjugate* is the sequence $p^T(j) := |\{i : p(i) \leq j\}|$. Then p^T has exactly $p(1)$ nonzero elements. When convenient, we can add or drop trailing zeros in a partition. For nonincreasing real sequences s and t of length n, we say that s is *majorized* by t (denoted $s \trianglelefteq t$) if, for all $k \leq n$,

$$\sum_{i=1}^{k} s_i \leq \sum_{i=1}^{k} t_i \tag{2.1}$$

and

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i. \tag{2.2}$$

The concept of majorization extends to vectors by comparing the nonincreasing vectors produced by sorting the elements of the vector into nonincreasing order. Given a vector v, call the sorted vector v' which contains the elements of v sorted in nonincreasing order (with multiplicity) sort(v).

In the context of majorization of unsorted vectors, we will often want to refer to the *concatenation* of two vectors x and y (i.e., the vector which contains the elements of x followed the elements of y). This is denoted x, y, as, for example, in Lemma 2.3.

There is a rich theory of majorization inequalities which occur throughout mathematics; see, for example, [9]. Matrices are an important source of such inequalities. Notably, the relationship between the diagonal and spectrum of a Hermitian matrix is characterized by majorization; see, for example, [7].

We will use the following lemmas about majorization, which can be found in [9].

LEMMA 2.2. *If x and y are vectors and P is a doubly stochastic matrix and $x = Py$, then $x \trianglelefteq y$.*

This yields two simple corollaries.

LEMMA 2.3. *For any vectors $x \trianglelefteq y$ and any vector z, we have $x, z \trianglelefteq y, z$.*

LEMMA 2.4. *If x and y are nonincreasing sequences, and $x = y$ except that at indices $i < j$ we have $x_i = y_i - a$ and $x_j = y_j + a$, where $a \geq 0$, then $x \trianglelefteq y$.*

Lemma 2.4 says that for nonincreasing sequences, transferring units from lower to higher indices reduces the vector in the majorization partial order. In particular, if x, x', y, y' are all *nonincreasing* sequences, $x' \trianglelefteq x$ and $y' \trianglelefteq y$, then

$$x' + y' \trianglelefteq x' + y \trianglelefteq x + y. \tag{2.3}$$

LEMMA 2.5. *Let A and B be positive semidefinite (more generally, Hermitian) matrices. Then*

$$\lambda(A), \lambda(B) \trianglelefteq \lambda(A + B).$$

This is Theorem G.1.b in Chapter 9 of [9], although the majorization is reversed in the printing available to the author.

LEMMA 2.6. *For positive semidefinite (more generally, Hermitian) matrices A and B,*

$$\lambda(A + B) \trianglelefteq \lambda(A) + \lambda(B).$$

This is a theorem of Fan (Theorem G.1 in Chapter 9 of [9]).

LEMMA 2.7. *Let A be an $m \times n$ $0 - 1$ (or incidence) matrix with row sums $r_1, \ldots, r_m$ and columns sums $c_1, \ldots, c_n$ both indexed in nonincreasing order. Let r^T be the conjugate of the partition $(r_1, \ldots, r_m)$ and c be the partition $(c_1, \ldots, c_n)$. Then*

$$c \trianglelefteq r^T. \tag{2.4}$$

This is known as the Gale–Ryser theorem (Theorem C.1 in Chapter 9 of [9]).

2.3. The GM conjecture. In the notation of this section, the GM conjecture is

$$\lambda(G) \trianglelefteq d^T(G). \tag{2.5}$$

Note that

$$\sum_{i=1}^{n} d_i^T = \sum_{i=1}^{n} d_i = \mathrm{trace}(L(G)) = \sum_{i=1}^{n} \lambda_i.$$

If we ignore isolated vertices (which contribute only zero entries to λ and d), we will have $d_1^T = n$. Using this fact, it is possible to show that

$$\lambda_1 \leq d_1^T. \tag{2.6}$$

Three short proofs of this are given in [4]. The authors then continue to prove the second majorization inequality:

$$\lambda_1 + \lambda_2 \leq d_1^T + d_2^T. \tag{2.7}$$

However, their proof would be difficult to extend.

There are several other facts which fit well with the GM conjecture. One is that if the GM conjecture holds, then the instances where (2.5) holds with equality are well understood; these would be the threshold graphs of section 3.1. Also, since d and λ are, respectively, the diagonal and spectrum of $L(G)$, we have $d \trianglelefteq \lambda$. Combining this with GM gives $d \trianglelefteq d^T$, a fact that has been proved combinatorially. We refer the reader to [4] for further discussion.

3. GM on classes of graphs. In this section, we give further evidence for the GM conjecture by remarking that it holds for several classes of graphs, including threshold graphs, regular graphs, and trees.

3.1. Threshold graphs. The GM conjecture was originally formulated in the context of *threshold* graphs, which are a class of graphs with several extremal properties. An introduction to threshold graphs is [8]. Threshold graphs are the graphs that can be constructed recursively by adding isolated vertices and taking graph complements. It turns out that they are also characterized by degree sequences: the convex hull of possible (unordered) degree sequences of an n vertex graph defines a polytope. The extreme points of this polytope are the degree sequences that have a unique labelled realization, and these are exactly the threshold graphs.

Threshold graphs are interesting from the point of view of spectra. Both Hammer and Kelmans [6] and Grone and Merris [5] investigated the question of which graphs have integer spectra. They found that threshold graphs are one class of graphs that have integer spectra and showed for these graphs that $\lambda(G) = d^T(G)$. We could interpret the GM conjecture as saying that threshold graphs are extreme in terms of spectra and that these extreme spectra can be understood as conjugate degree sequences.

3.2. Complements. Threshold graphs are built using graph complements of existing graphs, and so it is not surprising that the GM conjecture behaves well under taking complements. Indeed, the relationship between $\lambda(G)$ and $\lambda(\overline{G})$ is the same as between $d_n^T(G)$ and $d_n^T(\overline{G})$. For a graph G with n vertices, the ith largest vertex of G is the $(n-i)$th largest vertex of $\overline{G}$, and we have $d_i(G) = n - 1 - d_{n-i}(\overline{G})$. Translating this to the conjugate partition d^T yields $d_i^T(G) = n - d_{n-1-i}^T(\overline{G})$ with $d_n^T(G) = d_n^T(\overline{G}) = 0$.

Now notice that $L(G) + L(\overline{G}) = nI_n - J_n$, where J_n is the $n \times n$ matrix of ones. The matrix $nI_n - J_n$ sends the special eigenvector e_n (n ones) to 0 and acts as the scalar n on $e_n^\perp$. Both $L(G)$ and $L(\overline{G})$ also send e_n to 0, giving us $\lambda_n(G) = \lambda_n(\overline{G}) = 0$. Since $L(G)$ and $L(\overline{G})$ sum to nI_n on $e_n^\perp$, they have the same set of eigenvectors on $e_n^\perp$, and for each eigenvector, the corresponding eigenvalues for $L(G)$ and $L(\overline{G})$ sum to n. Thus $\lambda_i(G) = n - \lambda_{n-1-i}(\overline{G})$. As a consequence, GM holds for G if and only if GM holds for $\overline{G}$.

3.3. Regular and nearly regular graphs. For some small classes of graphs, it can be easily shown that the GM conjecture holds. Consider a k-regular graph G on n vertices (in a *k-regular* graph, all vertices have degree k). Then the degree sequence $d(G)$ is k repeated n times, and its conjugate $d^T(G)$ is n repeated k times followed by $n-k$ zeros. Thus d^T majorizes every nonnegative sequence of sum kn whose largest term is at most n, and, in particular, $\lambda \trianglelefteq d^T$. Indeed, this proof shows that GM holds for what we might call *nearly regular* graphs, that is, graphs whose vertices have degree either k or $(k-1)$.

3.4. Graphs with low maximum degree. Using facts about the initial GM inequalities, we can prove that GM must hold for graphs with low maximal degree. For example, if a graph has maximum vertex degree 2, then $d_3^T = d_4^T = \cdots = d_n^T = 0$, and so, for $k = 2, 3, \ldots, n$,

$$\sum_{i=1}^{k} \lambda_i \le \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i^T = \sum_{i=1}^{k} d_i^T.$$

More generally, the GM inequalities for $k \ge \text{max\_deg}(G)$ hold trivially. Thus GM holds for graphs of maximum degree 2 by (2.6). Using Duval and Reiner's result (2.7), we get that GM holds for graphs of maximum degree 3.

3.5. Trees and more. It is tempting to try to prove GM inductively by breaking graphs into simpler components on which GM clearly holds. In this section, we show that if G is "almost" the union of two smaller graphs on which GM holds, then GM holds for G as well. We apply this construction to show that GM holds for trees.

Take two graphs $A = (V_A, E_A)$ and $B = (V_B, E_B)$ on disjoint vertex sets V_A and V_B. Define their *disjoint sum* to be $A + B = (V_A \cup V_B, E_A \cup E_B)$. Assuming V_A and V_B are not empty, this is a disconnected graph. Now take two graphs $G = (V, E_G)$ and $H = (V, E_H)$ on the same vertex set V. Define their *union* as $G \cup H = (V, E_G \cup E_H)$.

Given the spectra and conjugate degree sequences of A and B, the spectrum of $A + B$ is (up to ordering) $\lambda(A + B) = (\lambda(A), \lambda(B))$, while the conjugate degree sequence of $A + B$ is $d^T(A + B) = d^T(A) + d^T(B)$ (taking each vector to have length n). Then if $\lambda(A) \trianglelefteq d^T(A)$ and $\lambda(B) \trianglelefteq d^T(B)$, we see that

$$\lambda(A + B) \trianglelefteq \lambda(A) + \lambda(B) \trianglelefteq d^T(A) + d^T(B) = d^T(A + B),$$

where the first majorization follows from Lemma 2.6 and the second from (2.3).

In a typical situation, where neither A or B is very small, we would expect the majorization $\lambda(A + B) \trianglelefteq d^T(A + B)$ to hold with considerable slack. We can use this slack to show that if we add a few more edges to $A + B$, the majorization will still hold.

THEOREM 3.1. *Take graphs A or B on disjoint vertex sets V_A and V_B. Let $G = A + B$, and on $V = V_A \cup V_B$ let C be a graph of "new edges" between V_A and V_B. Assume that GM holds on A, B, and C, i.e., that $\lambda(A) \trianglelefteq d^T(A)$, $\lambda(B) \trianglelefteq d^T(B)$, and $\lambda(C) \trianglelefteq d^T(C)$. Additionally, assume that $d_i^T(C) \le d_i^T(A), d_i^T(B)$ for all i and that $d_1^T(B) \le d_m^T(A)$, where m is the largest nonzero index of $d^T(C)$ (equivalently, m is the maximum vertex degree in C). Let $H = C \cup G$. Then*

$$\lambda(H) \trianglelefteq d^T(H). \tag{3.1}$$

Proof. Our strategy is to understand $d^T(H)$ in terms of the conjugate degree sequences of its constituent graphs. In particular, we show that

$$\text{sort}(d^T(A), d^T(B)) + d^T(C) \trianglelefteq d^T(H). \tag{3.2}$$

Then we can apply the majorizations of λ by d^T for A, B, C to the above terms and apply (2.3) to get

$$\begin{aligned} &\mathrm{sort}(\lambda(A), \lambda(B)) + \lambda(C) \trianglelefteq \mathrm{sort}(d^T(A), \lambda(B)) + \lambda(C) \\ \trianglelefteq\ &\mathrm{sort}(d^T(A), d^T(B)) + \lambda(C) \trianglelefteq \mathrm{sort}(d^T(A), d^T(B)) + d^T(C) \trianglelefteq d^T(H). \end{aligned}$$

The two terms on the left-hand side of this equation are spectra of $L(G)$ and $L(C)$, respectively. Hence by Lemma 2.6 their sum majorizes the spectrum of $L(H) = L(G) + L(C)$:

$$\lambda(H) \trianglelefteq \lambda(G) + \lambda(C) \trianglelefteq d^T(H).$$

It remains to prove (3.2), which is a statement entirely about conjugate degree sequences. For convenience, we will use the terminology of Ferrer's diagrams to describe these nonincreasing nonnegative integer sequences. That is, if s is such a sequence, we will describe s as a diagram of rows and columns with row i (reading from top to bottom) of length $s(i)$ and hence column j (reading from left to right) of length $s^T(j)$.

We begin with the following.

Claim 3.2. *Let k be the larger of* $\mathrm{max\_deg}(A)$ *and* $\mathrm{max\_deg}(B)$. *We have*

$$(d_1^T(G), d_2^T(G), \ldots, d_k^T(G), d_1^T(C), \ldots, d_m^T(C)) \trianglelefteq d^T(H).$$

Proof of claim. The term on the right-hand side is the concatenation of two partitions, $d^T(G)$ and $d^T(C)$. The columns of $d^T(G)$ index the vertices of G, and the length of a column gives the degree of the corresponding vertex. Since this claim is purely about degree sequences, we introduce a series of intermediate "partial graphs" where edges are allowed to have only one end. Degree sequences and their conjugates are still well defined for such objects.

Consider two copies of V, calling them V^1 and V^2. Take $G_0 = G$ on V^1 and $C_0 = C$ on V^2. Let $l = 2\,\mathrm{max\_deg}(C)$. For $i = 1, 2, \ldots, l$, define graphs G_i and C_i by moving one end of one edge from each nonisolated vertex of C_{i-1} on V^1 to V^2. That is, let G_i be G_{i-1} plus these additional ends of edges, and let C_i be C_{i-1} with these ends of edges removed. Then we will have $G_l = H$, and C_l will be the empty graph 0_{V^2} on V^2.

We can now prove the claim via a chain of l majorizations:

$$d^T(G), d^T(C) = d^T(G_0), d^T(C_0) \trianglelefteq \cdots \trianglelefteq d^T(G_l), d^T(C_l) = d^T(H), d^T(0_{V^2}) = d^T(H)$$

if we can show that, for each $i = 1, 2, \ldots, l$,

$$d^T(G_{i-1}), d^T(C_{i-1}) \trianglelefteq d^T(G_i), d^T(C_i). \tag{3.3}$$

Compare the partitions in (3.3): The first row of $d^T(C_{i-1})$ on the left-hand side is removed, and each element from that row is inserted into a separate column of $d^T(G_{i-1})$ (representing a distinct vertex) to get $d^T(G_i)$. Where there are columns of equal length in $d^T(G_{i-1})$, they should be ordered so that those acquiring new elements come first. To see that this operation increases (or leaves unchanged) the partition in the majorization partial order, observe that after ignoring the (unchanged) contents of $d^T(C_i)$ it is equivalent to sorting the new row into the partition $d^T(G_{i-1})$, using Lemma 2.4 to move its final (rightmost) element to the proper column and repeating as necessary.

This completes the proof of the Claim 3.2. We note that

$$\begin{aligned} d^T(G) &= d^T(A) + d^T(B) \\ &= (d_1^T(A) + d_1^T(B), d_2^T(A) + d_2^T(B), \ldots, d_k^T(A) + d_k^T(B), 0, \ldots, 0), \end{aligned}$$

and hence

$$(d_1^T(A) + d_1^T(B), d_2^T(A) + d_2^T(B), \ldots, d_k^T(A) + d_k^T(B), d_1^T(C), \ldots, d_m^T(C)) \trianglelefteq d^T(H).$$

If we sort the vector on the left into nonincreasing order, the first m terms will remain fixed by the assumptions that $d_m^T(A) \geq d_1^T(B) \geq d_1^T(C)$. Since we have assumed that $d_i^T(C) \leq d_i^T(B)$ for all i, we can apply Lemma 2.4 to the reordered sequence to get

$$\begin{aligned} &(d_1^T(A) + d_1^T(C), d_2^T(A) + d_2^T(C), \ldots, d_m^T(A) + d_m^T(C), \\ &\qquad d_{m+1}^T(A), \ldots, d_k^T(A), d_1^T(B), \ldots, d_k^T(B)) \trianglelefteq d^T(H). \end{aligned}$$

The right-hand term decomposes as

$$(d_1^T(A), \ldots, d_k^T(A), d_1^T(B), \ldots, d_k^T(B)) + (d_1^T(C), \ldots, d_m^T(C), 0, \ldots, 0).$$

Since we assume $d_m^T(A) \geq d_1^T(B)$, the first m entries of $(d^T(A), d^T(B))$ will remain unchanged if the vector is sorted. This gives (3.2) and completes the proof of Theorem 3.1. ∎

More generally, we could replace the conditions in the statement of Theorem 3.1 with the condition (3.2), which can be checked combinatorially. The conditions in the theorem statement and (3.2) are most likely to be satisfied if C is small relative to A and B.

A useful case is when C consists of k disjoint edges. Then $m = 1$ and $d_1^T(C) = 2k$. Without loss of generality, we can take $d_1(A) \geq d_1(B)$, and the only condition that we will need to check is that $d_1(A), d_1(B) \geq d_1(C)$; i.e., both A and B must have at least $2k$ nonisolated vertices.

The strategy for applying Theorem 3.1 to show that a given graph H satisfies GM is to find a "cut" C for it that contains few edges and divides H into relatively large components. For example, we have the following result.

Corollary 3.3. *The GM conjecture holds for trees.*

Proof. Proceed by induction on the diameter of the graph. If T has diameter 1 or 2, then there is a vertex v which is the neighbor of all the remaining vertices, and T is a threshold graph. So GM holds with equality for T.

Otherwise, we can find some edge e that does not have a leaf vertex. Since T is a tree, e is a cut edge and divides T into two nontrivial connected components, A and B. We apply induction to A and B and apply Theorem 3.1 to $H = (A + B) \cup C$, where C is the graph on the vertex set of T containing the single edge e. ∎

Remark 3.4 (small graphs). The facts in this section allow us to check that GM holds for some small graphs without directly computing eigenvalues. For example, since the GM condition is closed under complement (see section 3.2) for graphs on up to 5 vertices, it is enough to observe that either G or $\overline{G}$ has maximum degree ≤ 3. Out of 156 graphs on 6 vertices, 146 can be decomposed into smaller graphs $(A + B) \cup C$ using Theorem 3.1. Calculating the eigenvalues of the remaining 10 does not yield a counterexample.

For any particular larger graph G, we could attempt to verify that GM holds for G by breaking G (or $\overline{G}$) into smaller graphs across cuts that have relatively few edges and applying Theorem 3.1.

4. Simplices and pairs. The most recent work relating to the GM conjecture has been to study the spectra of more general structures than graphs, such as simplicial complexes and simplicial family pairs. In this section, we show that the generalization of GM to graphs with Dirichlet boundary conditions is equivalent to the original conjecture and may be useful in approaching GM.

4.1. Simplicial complexes. In [4], the authors look at *simplicial complexes*, which are higher-dimensional analogues of simple graphs (see, for example, [10]). A set of faces of a given dimension i is called an *i-family.* Given a simplicial complex Δ, we can denote the i-family of all faces in Δ of dimension i as $\Delta^{(i)}$. For example, a graph is a one-dimensional complex, and its edge set is the 1-family $\Delta^{(1)}$. Define the degree sequence d of an i-family to be the list of the numbers of i-faces from the family incident on each vertex and sorted into nonincreasing order. We can then define $d(\Delta, i)$ as the degree sequence of $\Delta^{(i)}$, which we can abbreviate to $d(\Delta)$ or d when the context is clear.

We define the *chain group* $C_i(\Delta)$ of formal linear combinations of elements of $\Delta^{(i)}$ and generalize the signed incidence matrix ∂ of section 2.1 to a signed boundary map $\partial_i : C_i(\Delta) \to C_{i-1}(\Delta)$. This allows us to define a *Laplacian* on $C_i(\Delta)$, namely $L_i(\Delta) = \partial_i \partial_i^T$, and study its corresponding spectrum $s(\Delta, i)$, sometimes abbreviated $s(\Delta)$ or s.

Duval and Reiner [4] looked at *shifted* simplicial complexes, which are a generalization of threshold graphs to complexes. They showed that for a shifted complex Δ and any i, we have $s(\Delta, i) = d^T(\Delta, i)$. They then conjectured that GM also holds for complexes, i.e., that for any complex and any i we have

$$s(\Delta, i) \trianglelefteq d^T(\Delta, i). \tag{4.1}$$

They also show that some related facts, such as (2.6), generalize to complexes.

4.2. Simplicial pairs. In [3], Duval continues by studying *relative (family) pairs* (K, K'), where the set $K = \Delta^{(i)}$ for some i is taken modulo a family of $(i-1)$-faces $K' \subseteq \Delta^{(i-1)}$. When $K' = \emptyset$, this reduces to the situation of the previous section.

Remark 4.1. In the case $i = 1$, this is the edge set of a graph (K) with a set of *deleted* boundary vertices K'. An edge attached to a deleted vertex will not be removed—it remains as part of the pair, but we now think of the edge as having a hole on one (or both) ends.

This type of graph with a boundary appears in conformal invariant theory. In this language, the relative Laplacian of an (edge, vertex) pair is sometimes referred to as a *Dirichlet Laplacian* and its eigenvalues as *Dirichlet eigenvalues*; see, for example, [2]. Recently [1] used the spectrum of the Dirichlet Laplacian in the analysis of "chip-firing games," which are processes on graphs that have an absorbing (Dirichlet) boundary at some vertices.

We can form chain groups $C_i(K)$ and $C_{i-1}(K, K')$ and use these to define a (signed) boundary operator on the pair $\partial(K, K') : C_i(K) \to C_{i-1}(K, K')$. Hence we get a Laplacian for family pairs $L(K, K') = \partial(K, K')\partial(K, K')^T$. Considered as a matrix, $L(K, K')$ will be the principal submatrix of $L(K)$ whose rows are indexed by the i-faces in $\Delta^{(i-1)} - K'$. Finally, we get a spectrum $s(K, K')$ for family pairs from the eigenvalues of $L(K, K')$.

Duval defines the degree $d_v(K, K')$ of vertex v (in the case of a graph, v is allowed to be in K') relative to the pair (K, K') as the number of faces in K that contain v such that $K - \{v\}$ is in $\Delta^{(i-1)} - K'$. This allows him to define the degree sequence

$d(K,K')$ for pairs and to conjecture that GM holds for relative pairs:

$$s(K,K') \trianglelefteq d^T(K,K'). \tag{4.2}$$

4.3. The GM conjecture for relative pairs. It turns out that at least in the case of (edge, vertex) pairs that (4.2) follows from the original GM conjecture for graphs.

THEOREM 4.2. *GM for graphs ⇒ GM for (edge, vertex) pairs.*

Proof. Let $G=(V,E)$ be a graph with $D \subseteq V$ a set of "deleted" vertices. Let $U = V - D$ be the remaining "undeleted" vertices. We will assume that GM holds only on the undeleted part of the graph, i.e., $G|_U$. So we have $s(G|_U) \trianglelefteq d^T(G|_U)$. We can ignore the edges in $G|_D$ completely, since they have no effect on either $s(E,D)$ or $d(E,D)$. The remaining edges connect vertices in D to vertices in U. Define G' to be the graph on V whose edge are exactly the edges of G between D and U. Let a be the degree sequence of the deleted vertices in G' and b be the degree sequence of the undeleted vertices in G'.

We can compute $d^T(E,D)$ in terms of the degree sequences and spectra of $G|_U$, G', and $G|_D$, since $d_i^T(E,D)$ is the number of vertices (deleted or not) attached to at least i nondeleted vertices. The number of such vertices in U will be $d_i^T(G|_U)$, and the number in D will be $d_i^T(G') = a^T$. Hence $d^T(E,D) = d_i^T(G|_U) + a^T$.

Now consider the Laplacian $L(E,D)$. This is the submatrix of $L(G)$ indexed by U. An edge (i,j) in $G|_U$ contributes to entries ii, ij, ji, jj in both $L(E,D)$ and $L(G)$. An edge in G', say from $i \in U$ to $j \in D$, contributes only to entry ii, and an edge in $G|_D$, does not affect $L(E,D)$. So we have $L(E,D) = L(G|_U) + \mathrm{Diag}(b)$, and by Lemma 2.6 we have

$$s(E,D) \trianglelefteq s(G|_U) + b. \tag{4.3}$$

We complete our equivalence by appealing to the Gale–Ryser theorem, (2.4), to claim that $b \trianglelefteq a^T$. This follows from the fact that a and b are row and column sums (in nonincreasing order) of the $|D| \times |U|$ bipartite incidence matrix for G'. Combining with the assumption that $s(G|_U) \trianglelefteq d^T(G|_U)$ and (4.3), we get

$$s(E,D) \trianglelefteq s(G|_U) + b \trianglelefteq d^T(G|_U) + a^T = d^T(E,D). \qquad \square$$

This proof relies on the bipartite structure of G', and so it is not immediately obvious how to extend it to higher-dimensional complexes. It would be interesting to do this.

Remark 4.3. Because the induction used to prove Theorem 4.2 requires only that the "undeleted" part of the graph satisfy GM, it is tempting to attack the original GM conjecture by showing that if GM holds for a pair $(G,\{v\})$, then GM holds for G.

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