Flop Sums

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In my last article, I mentioned a problem brought to me by a friend at Casino Regina. The problem arose out of a discussion he was having with another player regarding the sum of the three cards comprising the flop in hold'em. In determining the sum, we use the usual blackjack convention in which an ace counts 1 or 11. They noticed, or thought they noticed, that the cards frequently sum to 21. They made a small wager based on intuitive guesses about the probability of the sum being 21.

Just to make certain you understand the question, I'll give two examples. If the flop is 4-J-A, the sum is either 15 or 25. If the flop is 2-9-K, the sum is 21. The question then is: What is the probability the sum of the flop in hold'em is 21? By the way, we are going to work out the value under the assumption that there is an observor who knows none of the cards held by any of the players. Thus, the question we consider is the number of three-card hands from a standard 52-card deck that sum to 21. Since we are going to the trouble of counting how many three-card hands sum to 21, let's count the number of three-card hands attaining any of the possible sums for three cards.

The computations here are straightforward and based on seven building blocks. The building blocks arise because there are 16 cards that have value 10 and 4 cards of each of the ranks A, 2, ..., 9. The notation we use to describe the seven patterns for three-card hands is T for cards of value 10 and x, y, z for other ranks.

For a hand of the form T-T-T, we are choosing three cards from 16 so that there are $C(16, 3) = 560$ of them. There are $4C(16, 2) = 480$ hands of the form T-T-x because we are choosing two cards from 16, and one card from four.

There are $6 \cdot 16 = 96$ hands of the form T-x-x because we are choosing one card from 16, and two cards from four cards of rank x. There are $4 \cdot 4 \cdot 16 = 256$ hands of the form T-x-y because of choosing one card from four for each of the ranks x and y, and one card from 16 for the 10-valued card. There are four hands of the form x-x-x since we are choosing three cards from four cards of rank x. There are 24 hands of the form x-x-y since we are choosing one card from four cards of rank y, and choosing one of six pairs of rank x. Finally, there are 64 hands of the form x-y-z because we have four choices for each rank.

The above building blocks are used in all the calculations. We must be aware of the fact that aces can count as either 1 or 11 because this enhances the possibility for errors. We shall show how to check the results.

Let's illustrate the process by going through the details for one specific value. Motivated by the source of the question, let's consider how many three-card hands sum to 21. One combination is A-A-9 of which there are 24 since this has the form x-x-y. The combinations with a single ace are A-8-2, A-7-3, A-6-4, A-5-5 and A-T-T. Three have the form x-y-z, one has the form x-x-y, and the other has the form T-T-x. Thus, there are 696 of them. There are no additional combinations with T-T. The combinations with a single T are T-9-2, T-8-3, T-7-4 and T-6-5. There are 256 for each of them producing 1,024 hands with a single T. The combinations with neither A nor T are 9-9-3, 8-8-5, 9-6-6, 7-7-7, 9-8-4, 9-7-5 and 8-7-6. There are 24 for each of the first three, four for the fourth, and 64 for the last three. This gives 268 of them.

Adding all of the numbers gives 2,012 three-card hands whose sum is 21. The maximum sum is 33, the minimum sum is 3 and all numbers between occur. For each of the possible sums we go through what was done in the preceding illustration. The table below contains all the information in the column headed **Number**.

Sum	**Number**	**Rough Probability**
33	4	5,525
32	96	230
31	504	44
30	840	26
29	784	28
28	920	24
27	1,108	20
26	1,264	17.5
25	1,472	15
24	1,652	13.4
23	1,860	11.9
22	1,896	11.7
21	2,012	11
20	1,688	13.1
19	1,640	13.5
18	1,540	14.4
17	1,448	15.3
16	1,304	17
15	1,172	18.9
14	984	22.5
13	828	26.7
12	620	36
11	440	50
10	352	63
9	268	82
8	200	111
7	136	163
6	92	240
5	48	460
4	24	921
3	4	5,525

To obtain the probability of a certain sum occurring, we simply divide the number of hands with that sum by 22,100. Instead of displaying the exact probability in the column headed **Rough Probability**, we round off the probability to the form 1 divided by the value displayed in the column. For example, the entry in the column corresponding to the sum 21 is 11. This means the probability of having a sum of 21 is about 1/11 (the exact value is 503/5525). Similarly, the probability of having a sum of 19 is about 1/13.5 which is 2/27. This way of expressing the probability is convenient for setting the odds against it happening. Referring to the same two examples above, the odds against a sum of 21 is about 10-to-1, and the odds against a sum of 19 is about 12.5-to-1.

As far as checking our results is concerned, summing all of the numbers in the column headed **Numbers** yields 27,200. There are only $C(52, 3) = 22,100$ 3-card hands leading us to ask, "What is going on here?" Now we have to account for the effect of aces counting as 1 or 11. The hand A-A-A, of which there are 4, sums to either 33, 23, 13 or 3. Thus, these 4 hands actually contribute 16 to the table. So these hands are over-counted 12 times. Any hand with two aces has three different sums so that they are overcounted twice. There are $6 \cdot 48 = 288$ 3-card hands with two aces. Thus, we have a further 576 over-counted hands.

Finally, any hand with a single ace has two sums. There are $4C(48, 2) = 4,512$ three-card hands with one ace. So this contributes 4,512 to the over-count. The total over-count is $12 + 576 + 4,512 = 5,100$. Subtracting 5,100 from 27,200 yields 22,100 as the actual number of distinct hands contributing to the table. Thus, everything checks. Poker players love proposition bets. They are not usually as complex as this one, however.