# Counting Starting Poker Hands

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If you ask poker players how many starting hold'em hands there are, you will find that a majority of the players say there are 169. Not only is this the correct number, but you would discover that most of them can tell you how to derive the number. The typical argument goes something like this. Any two pairs of the same rank are really the same hand as far as analyzing their values, and there are 13 possible ranks. This gives us 13 starting hands that are pairs. There are 78 ways to choose two distinct ranks from 13. Two cards of different ranks come in two flavours: suited and offsuit. Other than this distinction, they really are the same hand. Thus, there are $78 + 78 + 13 = 169$ different starting hold'em hands.

The preceding verification is perfectly correct and easily understood, but it has one weakness. It is what is known as an ad hoc argument. That is, if you were to ask the same poker players how many starting Omaha hands there are, they would quickly realize the limitations in trying to extend the argument for starting hold'em hands to starting Omaha hands. They would soon find themselves bogged down in messy cases and subcases.

This suggests an obvious question. Is there a universal method for counting the number of starting hands for different poker games? The answer is a resounding "yes".

There are several aspects of this general method that appeal to me a great deal. First, the method works for all situations. Second, the method is efficient and fast. Third, the method arises in a beautiful subject of mathematics called *group theory*, and this is the only application of group theory to poker that I have seen.

I now am going to describe the method followed by applications to hold'em (the method better produce 169 as its answer), pineapple, seven-card stud, and Omaha. The method depends on a careful analysis of what it means for two hands to be equivalent.

Let's look at the hold'em situation. There actually are 1,326 ways of forming two-card hands. (There are 52 choices for the first card, 51 choices for the second card, and then we divide by two because any given hand can come in either of two orders.) However, we agree that many of those hands are equivalent to each other because we ignore suits. For example, the seven of clubs and eight of diamonds behave similarly to the seven of hearts and eight of clubs. To be specific, we agree that two hands are equivalent if you can tranform one hand into the other by performing some permutation of the suits.

In total, there are 24 possible permutations of the four suits. The complete collection of all 24 possible permutations is called a *permutation group.* (For those who wish to show off, it is called the symmetric group of degree 4.)

We are going to use a nifty way to describe all the permutations. I believe an example will make it clear. Look at the notation $(C\ H\ S)(D)$. This is the permutation that changes clubs to hearts, hearts to spades, spades back to clubs, and leaves diamonds alone. So for an expression inside parentheses, we change the first to the second, the second to the third, and so on until reaching the last element which gets changed into the first. If there is just one symbol in the parentheses, it is left unchanged. This description is called the cyclic structure of the permutation.

Even though there are 24 permutations, many of them have similar cyclic structure and that is all we need to do the counting. Here are the distinct types of cyclic structures. The permutation leaving all suits unchanged is called the identity permutation. There are six permutations with cyclic structure $(x\ y)(z)(w)$ and we call them Type 1. There are three permutations with cyclic structure $(x\ y)(z\ w)$ and we call them Type 2. There are eight permutations with cyclic structure $(x\ y\ z)(w)$ and we call them Type 3. Finally, there are six permutations with cyclic structure $(x\ y\ z\ w)$ and we call them Type 4.

We illustrate the remaining ingredients we need by considering hold'em. As mentioned above, there are 1,326 two-card hands. Each of the 24 permutations of the four suits induces a permutation of the 1,326 two-card hands. For example, the permutation $(C\ H)(D\ S)$ changes the hand $3\clubsuit - 6\diamondsuit$ to $3\heartsuit - 6\spadesuit$. The same permutation changes the hand $4\clubsuit - 4\heartsuit$ to itself. The latter is called a *fixed point*. There is then a famous counting theorem for permutation groups that tells us that the number of different starting hands is obtained by counting all the fixed points over the 24 permutations and then dividing by 24.

The reason the computation is fast is that two permutations with the same cyclic structure have the same number of fixed points, and counting the number of fixed points is easy. We look at two examples to show you how easy it is to count fixed points.

The permutation $(C\ D)(H)(S)$ is a typical Type 1 permutation. How many two-card hands does it fix? If both cards are chosen from hearts and spades, then the permutation fixes it because those two suits are left unchanged. That gives us $C(26, 2) = 325$ fixed hands. If the hand has a club of rank x and it is fixed, then it must have a diamond of rank x, and vice versa. So there are 13 hands like this. The permutation then fixes 338 hands. There are six permutations of Type 1 so that altogether they fix 2,028 two-card hands (see the entry in the table below).

Consider a Type 2 permutation with a three-card hand. You should see that the permutation can fix no three-card hands.

The table below contains the information for the four games (so that you can see if you get the same answers for each case). Note that the identity permutation fixes all hands because it changes no suits. An entry in the table gives the number of hands fixed by all the permutations of a given cyclic structure for the game indicated at the head of the column. The last line is obtained by dividing the preceding line by 24.

| Cyclic Types | Hold'Em | Pineapple | Seven-Card Stud | Omaha |
|---|---|---|---|---|
| Identity | 1,326 | 22,100 | 66,300 | 270,725 |
| Type 1 | 2,028 | 17,628 | 48,828 | 115,518 |
| Type 2 | 78 | 0 | 0 | 975 |
| Type 3 | 624 | 2,392 | 6,864 | 7,072 |
| Type 4 | 0 | 0 | 0 | 78 |
| Number Fixed | 4,056 | 42,120 | 121,992 | 394,368 |
| Starting Hands | 169 | 1,755 | 5,083 | 16,432 |