

Is There a Flush Out There? Part I

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Last month we took a look at a variety of probabilities for players making flushes, given certain preflop hands, in both hold'em and Omaha. This month we are going to turn the situation around and examine the chances that someone has a flush given that the board allows a flush. In Part I we will introduce a powerful counting technique, and in Part II we will provide results for many situations. In two months, we will complete the current discussion of flushes by considering how to quantify the strength of a particular flush.

Many of the probability questions arising in poker are amenable to what mathematicians call "elementary counting methods". Elementary means that we solve a problem by breaking it into simpler problems that have solutions involving either combinations or permutations, and then combine the answers to the simpler problems, using multiplication and addition, to finally arrive at a solution of the original problem. Solving counting problems successfully via elementary methods involves common sense, a few basic facts, and good bookkeeping.

Now consider the following question: Given three hearts on board, what is the probability that at least one player will have a flush if we deal 10 random two-card hands from the remaining 47 cards? The exact answer to this question is not conveniently found using elementary methods. However, there is a method, called inclusion-exclusion, that gives an exact answer. The inclusion-exclusion technique is one level of sophistication above elementary methods so that there is some skill involved in learning how to apply the method. On the other hand, once a person learns how to apply inclusion-exclusion, the range of counting problems she may now solve easily is vastly extended.

I am going to expend considerable energy on explaining inclusion-exclusion in this article so that in the future I need mention only briefly that the method is being employed. The really clever aspect of inclusion-exclusion is that we perform some relatively easy computations to obtain some numbers. All of these numbers contain a certain amount of "white noise". We then combine the numbers cleverly so that all the white noise cancels out leaving us with the exact value we are looking for.

I am going to explain inclusion-exclusion by example. We now carefully work through the problem mentioned above. Since there are three hearts and two non-hearts on board, there are 10 hearts and 37 non-hearts comprising the remaining 47 cards from which we deal 10 random two-card hands. Label the 10 players 1, 2, ..., 10. What is the probability that a particular player, say i , is dealt two hearts? The probability that her first card is a heart is $10/47$, and the probability that her second card is a heart is $9/46$. Multiplying gives a probability of $45/1,081$ that she is dealt two hearts.

Let's step aside a moment here and reflect on the value $45/1,081$ we just obtained. That means the probability that player i is *not* dealt two hearts is $1,036/1,081$. Suppose you have a vague recollection from that statistics course you took years ago that says in order to determine the probability none of the 10 players is dealt two hearts, all you have to do is raise $1,036/1,081$ to the tenth power. Subtracting the result from 1 would then give the probability that at least one player has two hearts. Doing this produces a value of .3464. The problem with this simple approach is that your vague recollection omitted the fact that the preceding is valid only if the 10 events are independent. They are not independent because what each player is dealt affects the probabilities of what other players are dealt. On the other hand, the value .3464 is an estimate of the correct answer. We shall soon see just how close to the exact value this estimate is.

We have just seen that the probability of any individual player being dealt two hearts is $45/1,081$. Let's extend this to two players i and j . What is the probability that both i and j are dealt two hearts. Given that i has been dealt two hearts, the probability that j is dealt two hearts then is the product of $8/45$ and $7/44$ because those are the numbers of hearts and cards left for j . We then multiply to obtain the probability $14/11,891$ that both i and j receive two hearts.

If we now add a third player, say k , to the cast and ask for the probability that all of i, j, k receive two hearts, the probabilities for i and j remain as they are above, but now we are choosing two hearts for k from six that are left, and two cards from 43. This means the probability that k also receives two hearts is the product of $6/43$ and $5/42$. Multiplication of all three probabilities yields a probability of $10/511,313$ that all three of them get two hearts.

For a fourth player to receive two hearts, we have $4/41$ times $3/40$ as the probability this happens given that three other players already have received two hearts. Multiplying all the probabilities gives a probability of $3/20,963,833$ that these four specific players all receive hearts. Since there are only 10 hearts left in the deck, at most five players can be dealt two hearts. When we are down to the fifth player, the probability he gets the last two hearts is $2/39$ times $1/38$ giving us a value of $1/5,178,066,751$ that some collection of five specified players all are dealt two hearts.

Let's summarize the preceding probabilities in the following list and introduce simplifying notation as well:

$$p1 = \frac{45}{1,081}, p2 = \frac{14}{11,891}, p3 = \frac{10}{511,313}, p4 = \frac{3}{20,963,833}, p5 = \frac{1}{5,178,066,751}$$

The first thing to note about the probabilities $p1, p2, p3, p4, p5$ is that they are easy to calculate. One just multiplies fractions. My earlier comment about white noise can now be explained. The value $p2$, for example, is the probability that two specified players are dealt hearts, but it places no restriction on what the other eight players are dealt. One or more of the remaining players also may be dealt two hearts. In other words, $p2$ is *not* the probability that exactly two specified players are dealt two hearts.

The magic of inclusion-exclusion is contained in the next equation. I claim that this equation is the exact probability that no one is dealt two hearts.

$$1 - 10p_1 + 45p_2 - 120p_3 + 210p_4 - 252p_5 = .63438$$

Understanding the validity of the preceding equation is the key to inclusion-exclusion. Again, we look at an example. Suppose we have a random deal D of 10 hands in which players i and j are the only two players dealt two hearts. The term 1 on the left includes every deal once because a probability of 1 includes all possibilities. Thus, D contributes once to the 1 term. The term $10p_1$ is counting, for each of the 10 players, the probability that the player has been dealt two hearts. Thus, D contributes twice, with a minus sign, to $10p_1$. There are 45 ways to choose two players from 10 so that the term $45p_2$ is counting the probability two or more players are dealt two hearts for each pair of players. This means D contributes once to this term with a plus sign. Thus, the contribution of D is cancelled out. The same thing happens for all deals with one or more players receiving two hearts. All that is left is the contribution from deals in which no one receives two hearts as claimed.

Therefore, we see that the exact value for the probability that one or more random hands is dealt two hearts is .36562, so that the earlier estimate is not too bad.