Flop Sums

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Abstract

We examine the numbers of flops having a fixed sum.

We are going to count the number of flops with a fixed sum and take the viewpoint of an external person observing the flops. In other words, we are not paying any attention to the cards in a particular player's hand. This means, in fact, we are counting the sums of all possible 3-card hands from a standard 52-card deck.

The computations here are straightforward and based on 7 building blocks. The building blocks arise because there are 16 cards that have value 10 and 4 cards of each of the ranks A, 2, ..., 9. The notation we use to describe the 7 patterns for 3-card hands is T for cards of value 10 and x, y, z for other ranks.

For a hand of the form T-T-T, we are choosing 3 cards from 16 so that there are $\binom{16}{3} = 560$ of them. There are $4\binom{16}{2} = 480$ hands of the form T-T-x because we are choosing 2 cards from 16, and 1 card from 4.

There are $6 \cdot 16 = 96$ hands of the form T-x-x because we are choosing 1 card from 16, and 2 cards from 4 cards of rank x. There are $4 \cdot 4 \cdot 16 = 256$ hands of the form T-x-y because of choosing 1 card from 4 for each of the ranks x and y, and 1 card from 16 for the 10-valued card. There are 4 hands of the form x-x-x since we are choosing 3 cards from 4 cards of rank x. There are 24 hands of the form x-x-y since we are choosing 1 card from 4 cards of rank y, and choosing 1 of 6 pairs of rank x. Finally, there are 64 hands of the form x-y-z because we have 4 choices for each rank.

The above building blocks are used in all the calculations. There is one further point of interest, namely, the fact that aces can count as either 1 or 11 does enhance the possibility for errors. We shall show how to check the results.

Let's illustrate the process by going through the details for one specific value. First consider how many 3-card hands sum to 22. One combination is A-A-T of which there are 96 since this has the form T-x-x. The combinations with a single ace are A-9-2, A-8-3, A-7-4 and A-6-5. Each has the form x-y-z so that there are 256 of them. The only combination of the form T-T-x is T-T-2 giving 480 of them. The combinations with a single T are T-9-3, T-8-4, T-7-5 and T-6-6. There are 256 for each of the first three and 96 of the last. This gives 864 of them with a single T. The combinations with neither A nor T are 9-9-4, 8-8-6, 7-7-8, 9-8-5 and 9-7-6. There are 24 for each of the first three and 64 for the last two. This gives 200 of them. Adding all of the numbers gives 1,896 3-card hands whose sum is 22.

The maximum sum is 33, the minimum sum is 3 and all numbers between occur. For each of the possible sums we go through what was done in the preceding illustration. The table below contains all the information in the column headed **Number**.

Sum	**Number**	**Rough Probability**
33	4	5,525
32	96	230
31	504	44
30	840	26
29	784	28
28	920	24
27	1,108	20
26	1,264	17.5
25	1,472	15
24	1,652	13.4
23	1,860	11.9
22	1,896	11.7
21	2,012	11
20	1,688	13.1
19	1,640	13.5
18	1,540	14.4
17	1,448	15.3
16	1,304	17
15	1,172	18.9
14	984	22.5
13	828	26.7
12	620	36
11	440	50
10	352	63
9	268	82
8	200	111
7	136	163
6	92	240
5	48	460
4	24	921
3	4	5,525

The last column needs a little explanation. To obtain the exact probability of a certain sum occurring, we simply divide the number of hands with that sum by 22,100. Instead of displaying the exact probability in the column headed **Rough Probability**, we round off the probability to the form 1 divided by the value displayed in the column. For example, the entry in the column corresponding to the sum 30 is 26. This means the probability of having a sum of 30 is about 1/26 (the exact value is 42/1105). Similarly, the probability of having a sum of 19 is about 1/13.5 which is 2/27.

This way of expressing the probability is convenient for setting the odds against it happening. Referring to the same two examples in the preceding paragraph, the odds against a sum of 30 is about 25-to-1, and the odds against a sum of 19 is about 12.5-to-1.

As far as checking our results is concerned, summing all of the numbers in the column heade **Numbers** yields 27,200. There are only $\binom{52}{3} = 22,100$ 3-card hands leading us to ask, "What is going on here?" Now we have to account for the effect of aces counting as 1 or 11. The hand A-A-A, of which there are 4, sums to either 33, 23, 13 or 3. Thus, these 4 hands actually contribute 16 to the table. So these hands are overcounted 12 times.

Any hand with two aces has three different sums so that they are overcounted twice. There are $6 \cdot 48 = 288$ 3-card hands with two aces. Thus, we have a further 576 overcounted hands.

Finally, any hand with a single ace has two sums. There are $4\binom{48}{2} = 4,512$ 3-card hands with one ace. So this contributes 4,512 to the overcount. The total overcount is $12 + 576 + 4,512 = 5,100$. Subtracting 5,100 from 27,200 yields 22,100 as the actual number of distinct hands contributing ot the table. Thus, everything checks.