Enumerating Starting Poker Hands

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Abstract

We examine the numbers of beginning hands for various poker games.

There are four games for which we are going to count the number of beginning hands: Hold'em, pineapple, 7-card stud and Omaha. The assumption we are making is that two hands are equivalent if one looks exactly the same as another upon permuting suits. Thus, for example, any two hands with 7-8 offsuit in hold'em are considered the same because they differ only with respect to the suits involved. Thus, we have a permutation group acting on the hands. The permutation group is the group of all permutations of the 4 suits. This is called the *symmetric group of degree* 4 and is denoted S_4.

The group S_4 has order 24. For a particular game, S_4 induces a permutation group acting on the set of all possible hands. Two hands are equivalent exactly when they are in the same orbit of the induced group. Therefore, the number of beginning hands is the number of orbits of the induced permutation group. Fortunately, there is a well-known theorem for permutation groups counting the number of orbits of a permutation group G. It states that the number of orbits is given by

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{Fix}(g),$$

where $|G|$ is the order of G and $\mathrm{Fix}(g)$ is the number of elements left fixed by g.

What makes the quoted theorem so useful is that $\mathrm{Fix}(g)$ is easy to calculate because it depends only on the disjoint cycle structure of the corresponding permutation in S_4. Moreover, if two permutations have the same disjoint cycle structure, they fix the same number of elements. So we now consider the disjoint cycle structure of the permutations in S_4.

The identity permutation is the unique permutation all of whose disjoint cycles have length 1. It also fixes every element. We mention that now so that we can use it below.

We now classify the remaining 23 permutatioins according to cycle structure.

Type 1 There are 6 permutations with cycle structure of the form $(a\ b)(c)(d)$.

Type 2 There are 3 permutations of the form $(a\ b)(c\ d)$.

Type 3 There are 8 permutations of the form $(a\ b\ c)(d)$.

Type 4 Finally, there are 6 permutations of the form $(a\ b\ c\ d)$.

Hold'Em

Altogether there are $\binom{52}{2} = 1,326$ 2-card hands. We know that the identity permutation fixes all 1,326 hands. We now use the symbols C, D, H and S to denote clubs, diamonds, hearts and spades, respectively. The permutation $(C\ D)(H)(S)$ that interchanges clubs and diamonds and leaves hearts and spades alone is Type 1. When will a 2-card hand be fixed by this permutation? First, any 2-card hand with no clubs or diamonds will be fixed. There are $\binom{26}{2} = 325$ such hands. If the hand contains a club of rank x, then the only way it can be fixed is if it also contains the diamond of rank x. In other words, a pair made up of a club and a diamond also is fixed. There are 13 such pairs. So the given permutation fixes 338 2-card hands. There are 6 permutations of Type 1 implying that they fix 2,028 hands altogether.

The permutation $(C\ D)(H\ S)$ is of Type 2. Using the same analysis as the preceding case, we see that this permutation fixes a hand only if it is a pair, where the suits of the pair are either clubs and diamonds, or hearts and spades. This means it fixes 26 2-card hands. Since there are 3 permutations of Type 2, they fix 78 2-card hands altogether.

The permutation $(C\ D\ H)(S)$ fixes only those 2-card hands that are both spades. There are $\binom{13}{2} = 78$ such hands. There are 8 permutations of Type 3 giving us 624 fixed hands for permutations of Type 3.

It should be clear that any permutation of Type 4 can fix no 2-card hand. They do not contribute to the sum. Adding all of the fixed hands gives 4,056. Dividing by 24 yields 169. Therefore, there are 169 starting hands in hold'em.

Pineapple

The total number of 3-card hands is $\binom{52}{3} = 22,100$. We need to analyze the number of fixed hands for the various permutations. The identity permutation fixes all 22,100 3-card hands.

The permutation $(C\ D)(H)(S)$ is a typical Type 1 permutation. If all 3 cards are chosen from hearts and spades, then the hand is fixed by the permutation. There are $\binom{26}{3} = 2,600$ such hands. If the hand has a club of rank x and it is going to be fixed, then it also must have a diamond of rank x. The other card must be either a heart or a spade. Thus, the hand consists of 13 possible pairs and any of 26 hearts or spades. This gives another 338 hands fixed by the permutation, or 2,938 hands fixed by the permutation. There are 6 permutations of Type 1 giving us 17,628 3-card hands fixed by permutations of Type 1.

The permutation $(C\ D)(H\ S)$ is a Type 2 permutation. It is not difficult to see that such a permutation cannot fix any 3-card hand.

The permutation $(C\ D\ H)(S)$ is a Type 3 permutation. If a 3-card hand is fixed and it contains any card that is not a spade, then it must contain trips of some rank with a club, diamond and heart of the given rank. There are 13 such hands. The only other 3-card hand fixed by this permutation is a hand of all spades. There are $\binom{13}{3} = 286$ such hands. Thus, a permutation of Type 3 fixes 299 3-card hands. Since there are 8 such permutations, they contribute 2,392 fixed hands to the sum.

A permutation of Type 4 cannot fix a 3-card hand. Adding all the fixed hands yields 1,755 beginning pineapple hands.

7-Card Stud

Like pineapple, we are dealing with 3 cards, but 1 of them is dealt up and the other 2 are dealt down. We shall think of such a hand as an ordered pair (x,y-z) of 3 cards. The number of these ordered pairs is then $52\binom{51}{2} = 66,300$ because there are 52 choices for the upcard, and we are choosing 2 cards from 51 for the 2 downcards.

When the permutations act on the ordered pairs, they map upcards to upcards and downcards to downcards. As usual, the identity permutation fixes all 66,300 hands.

A typical permutation of Type 1 is $(C\ D)(H)(S)$. In order for this permutation to fix a hand, the upcard must be a heart or spade, and the 2 downcards are either both chosen from hearts and spades or they form a pair of rank x with 1 club and 1 diamond. Thus, we have 26 choices for the upcard and $13 + \binom{25}{2} = 313$ choices for the downcards. Thus, the permutation fixes 8,138 hands. The 6 permutations of Type 1 then fix 48,828 7-card stud beginning hands.

A permutation of Type 2 cannot fix any 7-card stud hand because the upcard is changed. A typical permutation of Type 3 is $(C\ D\ H)(S)$. In order for this permutation to fix a 7-card stud hand, the upcard must be a spade, and the 2 downcards also must be spades. Therefore, the number of fixed hands is $13\binom{12}{2} = 858$. The 8 permutations of this type then fix 6,864 hands.

A permutation of Type 4 cannot fix any hand. Summing the above numbers yields 121,992. Dividing by 24 gives us 5,083 different 7-card stud beginning hands.

Omaha

There are $\binom{52}{4} = 270,725$ 4-card hands. The identity permutation fixes all of them.

The permutation $(C\ D)(H)(S)$ is a typical Type 1 permutation. Any 4-card hand with only hearts and spades is fixed by this permutation. There are $\binom{26}{4} = 14,950$ such hands. If a club of rank x is in the hand and the hand is fixed, then the x of diamonds also must be in the hand. Thus, any hand with a pair chosen from clubs and diamonds, and 2 cards chosen from hearts and spades is fixed by the permutation. There are $13\binom{26}{2} = 4,225$ 4-card hands of this form. Finally, if a hand has 2 pairs chosen from clubs and diamonds, then it also is fixed. There are 78 hands of this form. Altogether, the permutation fixes 19,253 4-card hands. Since there are 6 Type 1 permutations, they fix 115,518 4-card hands.

The permutation $(C\ D)(H\ S)$ is a typical Type 2 permutation. If a 4-card hand is fixed by this permutation, then a club of rank x in the hand forces a diamond of rank x and vice versa. Similarly, a heart of rank y forces a spade of rank y and vice versa. There are 26 such pair combinations and we must choose 2 of them. This gives us $\binom{26}{2} = 325$ 4-card hands fixed by the permutation. There are 3 Type 2 permutations so that there are 975 4-card hands fixed by Type 2 permutations.

The permutation $(C\ D\ H)(S)$ is a typical Type 3 permutation. If a hand is fixed by this permutation and the hand has any club, diamond or heart of some rank x, then it must have the other 2 suits of rank x. So this permutation fixes any 4-card hand consisting of 4 spades, or consisting of a club, diamond, heart of rank x together with any spade. Hence, it fixes $\binom{13}{4} + 13^2 = 884$ 4-card hands. There are 8 permutations of this type giving us 7,072 fixed hands from Type 3 permutations.

The permutation $(C\ D\ H\ S)$ fixes only a hand made up of four cards all of the same rank. There are 13 such hands and 6 permutations like this. This gives 78 4-card hands fixed by Type 4 permutations.

Adding all the fixed hands produces 394,368 of them for Omaha. Dividing by 24 gives us 16,432 beginning Omaha hands.

Draw Poker

There are $\binom{52}{5} = 2,598,960$ 5-card hands. The identity permutation fixes all of them.

The permutation $(C\ D)(H)(S)$ is a typical Type 1 permutation. Any 5-card hand with only hearts and spades is fixed by this permutation. There are $\binom{26}{5} = 65,780$ such hands. If a club of rank x is in the hand and the hand is fixed, then the x of diamonds also must be in the hand. Thus, any hand with a pair chosen from clubs and diamonds, and 3 cards chosen from hearts and spades is fixed by the permutation. There are $13\binom{26}{3} = 33,800$ 5-card hands of this form. Finally, if a hand has 2 pairs chosen from clubs and diamonds and any heart or spade, then it also is fixed. There are $78 \cdot 26 = 2,028$ hands of this form. Altogether, the permutation fixes 101,608 5-card hands. Since there are 6 Type 1 permutations, they fix 609,648 5-card hands.

The permutation $(C\ D)(H\ S)$ is a typical Type 2 permutation. It cannot fix any hand with an odd number of cards. Thus, the Type 2 permutations fix zero 5-card hands.

The permutation $(C\ D\ H)(S)$ is a typical Type 3 permutation. If a hand is fixed by this permutation and the hand has any club, diamond or heart of some rank x, then it must have the other 2 suits of rank x. So this permutation fixes any 5-card hand consisting of 5 spades, or consisting of a club, diamond, heart of rank x together with any 2 spades. Hence, it fixes $\binom{13}{5} + 13\binom{13}{2} = 2,301$ 5-card hands. There are 8 permutations of this type giving us 18,408 fixed 5-card hands from Type 3 permutations.

The permutation $(C\ D\ H\ S)$ fixes no 5-card hand which is not hard to see.

Adding all the fixed hands produces 3,227,016 of them for draw poker. Dividing by 24 gives us 134,459 beginning draw poker hands.