Online Bad-Beat Jackpot Computations

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Abstract

We provide details of the computations for the probability of a bad-beat jackpot occurring using the rules at PartPoker.

This file provides the details for the probability in the article "Online Bad-Beat Jackpots: I" that appeared in *Canadian Poker Player*, Vol. 1, No. 12, December, 2004.

We first compute the total number of possible semideals for 10-handed hold'em. The total number of possible boards is $\binom{52}{5} = 2,598,960$. There are $\binom{47}{20} = 9,762,479,679,106$ ways of choosing 20 cards to form the 10 hands. Finally, there are $19!! = 654,729,075$ to partition 20 chosen cards into 10 hands of two cards each. Multiplying then produces

$$\binom{52}{5}\binom{47}{20}19!! \quad = \quad 16,611,978,703,557,549,675,134,772,000 \tag{1}$$

different semideals for 10-handed hold'em.

All we need do now is determine the number of semideals that qualify for a bad-beat jackpot and divide by the preceding number to get the probability that a semideal qualifies for a bad-beat jackpot. First, we need the rules under which a hand qualifies for such a jackpot. We are going to use the rules as of 29 October 2004 at the PartPoker website.

The minimum qualifying hand is quad 8s. A player must use the best hand she has, and both hole cards must be used in forming her best hand. Thus, a player holding 4,5 of hearts with 6,7,8,9 of hearts on board does NOT have a qualifying hand because her best hand is a 9-high straight flush, and this hand uses only one card from her hand. On the other hand, a player holding 10-K with a board of 10-10-10-J-K does have a qualifying hand because her kicker in her hand ties the kicker on board.

Here is a list of board types that will allow two players to have qualifying hands, thereby, allowing a bad-beat jackpot to occur.

- A full house on board
- Three-of-a-kind on board with two other ranks allowing a straight
- Two-pair on board

- One pair on board with three other ranks allowing a straight
- Five distinct ranks on board allowing two distinct straights in the same suit

A full house on board

Write the full house in the form x-x-x-y-y. Both ranks x and y must be at least 8 because of the qualifying rules. Further, for the player holding a single card of rank x, her other card must be of rank y or better in order for the hand to qualify. For example, a board of the form x-x-x-A-A cannot produce two qualifying hands because a player holding the case x has a second card of rank smaller than A when another player holds A-A.

So we see that x can be any of 7 ranks, and the number of ranks possible for y always is 5. This follows because when x is any of 8 through K, y is any rank from 8 through K different from x, and when x is ace, then y is any of 8 through Q. This implies that there are 35 possible (x,y) pairs.

The table below gives the number of possible kickers for a player holding a single x given that another player holds y-y. The rows of the table correspond to the rank x and the columns correspond to the rank y.

	8	9	10	J	Q	K
8	-	20	16	12	8	4
9	20	-	16	12	8	4
10	20	16	-	12	8	4
J	20	16	12	-	8	4
Q	20	16	12	8	-	4
K	20	16	12	8	4	-
A	20	16	12	8	4	-

We can now use the information in the table to determine the number of qualifying semideals when the board is a full house. First, for each legal pair (x,y), there are 24 possible full houses on board because there are 4 ways of choosing x-x-x and 6 ways of choosing y-y. Once a particular full house is chosen for the board, there is no choice for the y-y hand. There is no choice for the card of rank x, but there is the choice of the number of kickers shown in the table for the other card in the latter hand. Hence, if we sum the entries in the table and multiply by 24, we obtain the number of ways of choosing a full house on board and two accompanying hands that qualify for a bad-beat jackpot. Doing so yields $420 \cdot 24 = 10,080$ choices.

For each of the preceding choices that produces a qualifying bad-beat jackpot now must be completed to a semideal. There are 43 cards left in the deck. We choose 16 of them to form the remaining 8 hands. This produces

$$\binom{43}{16} 15!! = 537,530,846,018,616,450$$

ways to form 8 hold'em hands from 43 cards. Multiplying by the 10,080 from above, leads to

$$10,080\binom{43}{16}15!! \quad = \quad 5,418,310,927,867,653,816,000 \tag{2}$$

bad-beat qualifying semideals with a full house on board.

Three-of-a-kind on board

If the board has the form x-x-x-y-z, then the only possible way we can get two qualifying hands is for one player to have an x, where x is a rank of 8 or higher, and a kicker that is at least as big as both y and z, and for the other player to have a straight flush. Suppose that x is 8. Then for each of the ranks 4, 5, 6, and 7, there are 3 choices for another rank so that together with 8 the three ranks allow a straight. For example, with 4 we can choose one of 5, 6 or 7. Finally, from the 4 ranks 9, 10, J and Q, we can choose any 2 that allow a straight with 8. In other words, there are 18 pairs (y,z) so that 8-y-z allows a straight.

The same count works when x is 9 or 10. When x is J, there are only 3 choices of two larger ranks so that we get 15 pairs for (y,z) in this case. When x is Q, we have only 12 pairs (y,z) that allow a straight with Q. When x is K, there are only 9 choices for the pairs (y,z). When x is A, there are only 6 pairs (y,z) allowing a straight.

For a legal set of ranks x-y-z, where the trips on board have rank x, there are 4 choices for the trips of rank x and there are 3 choices for the suit that allows the straight flush. Hence, each legal set of ranks gives us 12 boards that allow a qualifying bad beat. For each such board, we need to know how many qualifying semideals arise. One complicating factor is that a few of them allow three qualifying hands, that is, one player with quads and two players with straight flushes. We handle them separately.

The next table contains information on the number of ways two qualifying hands may be formed given a board of the type being discussed in this section. The columns headed "(x,y,z)" correspond to the ranks x-x-x-y-z comprising the board. The columns headed "quads" give the number of ways of forming a hand with one card of rank x and a kicker that is at least as high in rank as z. The columns headed "SF" give the number of straight flush hands that are possible. Finally, the columns headed "number" give the total number of ways of choosing the two qualifying hands. It simply is the product of the numbers in the two columns to the immediate left.

We now discuss how we obtain the entries for a typical portion of the table, and do so with two examples. First consider the entries for (8,4,6). This means that the board has 8-8-8-4-6 with a 4-6-8 suited. There is only one way that a player can make a straight flush, namely, the player must have the 5-7 of that suit. The player with quads has the missing card of rank 8 plus a kicker of rank 6 or higher. One 7 has been used in the straight flush so that the player has a choice of 3 sixes, 3 sevens and any card of rank 9 up through A. This produces 30 choices as indicated in the table.

(x,y,z)	quads	SF	number	(x,y,z)	quads	SF	number
(8,4,5)	33	1	33	(8,4,6)	30	1	30
(8,4,7)	27	1	27	(8,5,6)	29/30	2	59
(8,5,7)	26/27	2	53	(8,5,9)	23	1	23
(8,6,9)	22/23	2	45	(8,6,10)	19	1	19
(8,7,10)	18/19	2	37	(8,7,J)	15	1	15
(8,9,J)	14/15	2	29	(8,9,Q)	11	1	11
(8,10,J)	14/15	2	29	(8,10,Q)	11	1	11
(8,J,Q)	11	1	11	(9,5,6)	29	1	29
(9,5,7)	26	1	26	(9,5,8)	23	1	23
(9,6,7)	25/26	2	51	(9,6,8)	22/23	2	45
(9,6,10)	19	1	19	(9,7,10)	18/19	2	37
(9,7,J)	15	1	15	(9,8,J)	14/15	2	29
(9,8,Q)	11	1	11	(9,10,Q)	10/11	2	21
(9,10,K)	7	1	7	(9,J,Q)	10/11	2	21
(9,J,K)	7	1	7	(9,Q,K)	7	1	7
(10,6,7)	25	1	25	(10,6,8)	22	1	22
(10,6,9)	19	1	19	(10,7,8)	21/22	2	43
(10,7,9)	18/19	2	37	(10,7,J)	15	1	15
(10,8,J)	14/15	2	29	(10,8,Q)	11	1	11
(10,9,Q)	10/11	2	21	(10,9,K)	7	1	7
(10,J,K)	6/7	2	13	(10,J,A)	3	1	3
(10,Q,K)	6/7	2	13	(10,Q,A)	3	1	3
(10,K,A)	3	1	3	(J,7,8)	21	1	21
(J,7,9)	18	1	18	(J,7,10)	15	1	15
(J,8,9)	17/18	2	35	(J,8,10)	14/15	2	29
(J,8,Q)	11	1	11	(J,9,Q)	10/11	2	21
(J,9,K)	7	1	7	(J,10/Q,K)	6/7	2	26
(J,10,A)	3	1	3	(J,Q,A)	3	1	3
(J,K,A)	3	1	3	(Q,8,9)	17	1	17
(Q,8,10)	14	1	14	(Q,8,J)	11	1	11
(Q,9,10)	13/14	2	27	(Q,9,J)	10/11	2	21
(Q,9,K)	7	1	7	(Q,10/J,K)	6/7	2	26
(Q,10,A)	3	1	3	(Q,J/K,A)	3	1	6
(K,9,10)	13	1	13	(K,9,J)	10	1	10
(K,9,Q)	7	1	7	(K,10,J)	9/10	2	19
(K,10,Q)	6/7	2	13	(K,10,A)	3	1	3
(K,J,Q)	6/7	2	13	(K,J,A)	3	1	3
(K,Q,A)	3	1	3	(A,10,J)	9	1	9
(A,10/J,Q)	6	1	12	(A,10/J/Q,K)	3	1	9

Consider the example for (8,5,6). The board has 8-8-8-5-6 with a suited 8-5-6. A player may have either 4-7 of the suit or 7-9 of the suit giving 2 hands producing a qualifying straight flush. In the case the player with the straight flush has 4-7, then the player with the missing 8 may have any of 30 kickers that give a qualifying hand. On the other hand, if the player with the straight flush has 7-9, then there are only 29 cards available as the kicker for the player with the missing 8. This is indicated by the entry of 29/30 in the table. The total number of two hand combinations is then 59 as indicated in the table.

The remaining entries are calculated in a similar manner. Each of the triples (x,y,z) corresponds to 12 boards as we saw earlier. The value in the number position gives us the number of ways of choosing two qualifying hands. Thus, if we sum the entries in the number positions, we obtain 1,482. We then complete the choices to semideals by multiplying by $12\binom{43}{16}15!!$ to obtain

$$1,495\binom{43}{16}15!! \quad = \quad 9,559,448,565,595,074,946,800 \tag{3}$$

semideals with trips on board such that one player makes qualifying quads and one other player makes a straight flush.

The following triples were not included in the above table because it is possible that three players may make qualifying hands: (8,6,7), (8,7,9), (8,9,10), (9,7,8), (9,8,10), (9,10,J), (10,8,9), (10,9,J), (10,J,Q), (J,9,10), (J,10,Q), and (Q,10,J). We use inclusion-exclusion to determine the precise number of qualifying semideals for these triples. Let's look at (8,6,7) in detail. There is just one way to choose two hands making straight flushes, there are 27 ways to make quad 8s with a 4-5 straight flush since there are 27 possible kickers that work, there are 26 ways to make quad 8s with a 5-9 straight flush, and there are 25 ways to make quad 8s with a 9-10 straight flush since there are 25 possible kickers in this case. Hence, there are 79 ways for two or more hands to have qualifying hands taking the three possibilities two at a time.

Following the same line of reasoning, for (8,7,9) there are 67 ways, for (8,9,10) there are 55 ways, for (9,7,8) there are 67 ways, for (9,8,10) there are 55 ways, for (9,10,J) there are 43 ways, for (10,8,9) there are 55 ways, for (10,9,J) there are 43 ways, for (10,J,Q) there are 31 ways, for (J,9,10) there are 43 ways, for (J,10,Q) there are 31 ways, and for (Q,10,J) there are 31 ways.

Summing all of the ways gives us 600 ways that the given types of boards can produce two or more qualifying hands taking the properties two at a time. We then multiply by 12 to give us 7,200 combinations of boards and two or more qualifying hands taking the properties two at a time. Multiplying by $\binom{43}{16}15!!$ gives a count on the semideals allowing two or more qualifying hands. Carrying out the calculation produces

$$7,200\binom{43}{16}15!! = 3,870,222,091,334,038,440,000$$

semideals.

Any semideal that has three qualifying hands is counted three times in the preceding count. Hence, we need to count these and subtract twice the value from the sum just obtained to give us the exact number of qualifying semideals for the 12 missing triples (x,y,z). In order to have three qualifying hands, there must be two straight flushes and there is no freedom in how they are chosen. The quads then has the number of choices according to the available number of kickers. Looking at the numbers above, when (x,y,z) had 53, there are 25 here. Continuing, 45 gives 21, 37 gives 17, 29 gives 13, and 21 gives 9. Thus, we have 184 choices altogether. We multiply by 12 and then complete the three hands to semideals by multiplying by $\binom{41}{14}13!!$.

Carrying out the calculation produces

$$2,208\binom{41}{14}13!! = 10,514,889,107,500,377,600$$

semideals. As stated above, we subtract twice the latter from the number two above to obtain

$$3,838,677,424,011,537,307,200 \tag{4}$$

qualifying semideals corresponding to the 12 triples (x,y,z) that were not included in the big table above because they would allow three qualifying hands.

The last board with trips that must be considered are those of the form x-x-x-y-z, where x is of rank 2 through 7 (thereby, not allowing quads that qualify), that allow two straight flushes. The only possible triples are (3,4,5), (4,5,6), (5,6,7), (6,7,8), and (7,8,9), where we do not use 8 or 9 as the rank of the trips on board for the last two triples. For the first three triples, we have 3 choices for the rank of the trips, for (6,7,8) we have two choices for the rank of the trips, and for (7,8,9) we must use 7 as the rank for the trips. Thus, we have 12 choices for the multiset x-x-x-y-z. As before we have 4 choices for the cards comprising the trips and 3 choices for the suit of the 3 suited cards. Hence, we have 144 boards of this form that allow two straight flushes.

Once the board is chosen, there is no choice for the hands making up the two straight flushes. We then complete the 144 choices to semideal by multiplying by $\binom{43}{16}15!!$ to obtain

$$144\binom{43}{16}15!! \quad = \quad 77,404,441,826,680,768,800 \tag{5}$$

semideals qualifying for the bad-beat jackpot in this case.

We now obtain the total number of semideals qualifying for a bad-beat jackpot with three-of-a-kind on board by taking the sum of (3), (4), and (5). Doing so gives us

$$13,475,530,431,433,293,022,800 \tag{6}$$

semideals.

Two-pair on board

Two-pair on board has much in common with the precedng section because this means there are exactly three ranks on board. There is an additional complication from the fact that it is possible (though highly unlikely) to have four qualifying hands from a single board.

The board has the form x-x-y-y-z. The first subcase we consider is that both x and y are rank 8 or bigger. There are 21 ways to choose x and y from the 7 possible ranks. There are then 11 ways to choose the rank z. This gives us 231 ways of choosing the ranks for the board, that is, the rank set {x,y,z} can be chosen in 231 ways.

Consider x = 8 and y = 10. Then z has 6 choices so that the 3 ranks x,y,z do not allow a straight, z has 4 choices so that suited x,y,z allows only one straight flush hand—although it may happen in one of two different ways—at a time, and z has one choice so that suited x,y,z allows two simultaneous straight flushes. Checking this over all possible combinations of x and y, we find that there are 155 x,y,z sets not allowing a straight, 66 of them allowing just one straight flush at a time, and 10 of them allowing simultaneous straight flushes. The values sum to 231 as they should.

For the 155 rank multisets x-x-y-y-z not allowing a straight, the only way we can have a qualifying semideal is for two players to have quads. There are 6 ways to choose each of the pairs, and 4 ways to choose the card of rank z. Hence, there are 144 boards for each rank multiset. To obtain the qualifying semideals in this case, we have

$$144 \cdot 155 \binom{43}{16} \quad = \quad 11,997,688,483,135,519,164,000 \tag{7}$$

semideals.

We move to the rank multisets that allow a straight so that with some of the corresponding boards, straight flushes become possible. Of the 66 that allow just one straight flush at a time, 24 allow them in two different ways and 42 of them allow a straight flush in only one way. Of course, some of the corresponding boards don't allow a flush at all. We take care of this first.

If the board is going to allow a flush, there are 4 choices for the common suit. This leaves each of the other cards forming a pair to be chosen in any of 3 ways. Hence, 36 of the boards corresponding to x-x-y-y-z allow a flush and 108 do not allow a flush. Thus, for the 66 rank multisets, there are

$$66 \cdot 108 \binom{43}{16} 15!! \quad = \quad 3,831,519,870,420,698,055,600 \tag{8}$$

semideals qualifying for a bad-beat jackpot with two players making quads.

For these 66 rank multisets, there may be 3 qualifying hands. For 42 of them, there is only one way to choose each of the 3 hands. This gives $42 \cdot 36$

choices. For 24 of them, the straight flush may be chosen in one of two ways. This gives $24 \cdot 36 \cdot 2$ choices. Completing these choices to semideals is obtained by multiplying the number of choices by $\binom{41}{14}13!!$. This produces

$$36 \cdot 90 \binom{41}{14} 13!! = 15,429,456,842,527,728,000$$

semideals with 3 qualifying hands for these rank multisets.

Now we take the qualifying hands two at a time so that we may employ inclusion-exclusion to determine the exact number of qualifying semideals.

For the 42 rank multisets, every combination of two qualifying hands has a unique choice for the hands. This means there are $3 \cdot 42 \cdot 36$ choices. For the other 24 rank multisets, one combination has two unique choices, while the other two combinations have 2 choices because the straight flush can happen with either of 2 hands. This gives us $5 \cdot 24 \cdot 36$ choices. All of them are completed to semideals in $\binom{43}{16}15!!$ ways. This gives us

$$36 \cdot 246 \binom{43}{16} 15!! = 4,760,373,172,340,867,281,200$$

semideals for which we are counting two qualifying hands at a time.

Any semideal with 3 qualifying hands is counted 3 times in the value just calculated. Thus, we subtract twice the value computed prior to that. Doing so gives us

$$4,729,514,258,655,811,825,200 \tag{9}$$

qualifying semideals corresponding to the 66 rank multisets that allow a straight flush when there are 3 suited cards on board.

This now takes us to the 10 rank multisets that allow simultaneous straight flushes. This means that 4 simultaneous qualifying hands could occur. First, let's dispose of the 108 boards without a flush being possible. As above, we obtain

$$10 \cdot 108 \binom{43}{16} 15!! \quad = \quad 580,533,313,700,105,766,000 \tag{10}$$

semideals with 2 players holding qualify quads.

We start the inclusion-exclusion with semideals having 4 qualifying hands. Since all 4 hands are completely determined, the number of semideals for this case is

$$10 \cdot 36 \binom{39}{12} 11!! = 14,634,986,164,999,200.$$

We now move to semideals having at least 3 qualifying hands. Those with two of the hands being quads have 3 choices for the straight flush hand (the

2 cards can bracket the 3 successive ranks on board explaining the 3 choices). We then have $3 \cdot 10 \cdot 36 = 1,080$ choices. Those with 2 straight flushes have 2 choices for the quads. This gives us $20 \cdot 36 = 720$ choices. Hence, the number of semideals with at least 3 qualifying hands is

$$1,800\binom{41}{14}13!! \quad = \quad 8,571,920,468,070,960,000. \tag{11}$$

Those with at least two qualifying hands arise in several ways. Those with two quad hands have unique choices for both qualifying hands. This gives $10 \cdot 36$ choices. Those with one quads and one straight flush have 6 choices for the hands. This gives 2,160 choices. Those with 2 straight flushes have unique choices for both qualifying hands. This also gives 360 choices. Altogether we have have 2,880 choices. Thus, there are

$$1,548,088,836,533,615,376,000$$

semideals of this type.

We use inclusion-exclusion to remove the duplicate counting in the preceding three numbers. We obtain

$$1,530,988,900,555,968,453,600 \tag{12}$$

semideals qualifying the bad-beat jackpot for the 10 rank multisets that allow up to 4 hands qualifying hands.

The above computations take care of the subcase that both pairs have rank 8 or more. We now consider x-x-y-y-z, where x is of rank 8 or more and y is of rank 7 or less. The only way we can get two qualifying hands now is for straight flushes to provide at least one of the qualifying hands. This is going to restrict the sets considerably.

It is not hard to enumerate the 52 rank multisets that allow straights under the restrictions on the ranks of the pairs. All but 4 of them allow only one straight flush at a time. Let's look at the 4 exceptional rank multisets first. They are 7-7-8-8-9, 6-7-7-8-8, 6-6-7-8-8, and 7-7-8-9-9.

For each of them, if there are 3 qualifying hands, then the hands are uniquely determined. This means that the number of semideals with 3 qualifying hands is

$$144\binom{41}{14}13!! = 685,753,637,445,676,800.$$

We now count using two qualifying hands. If one of the qualifying hands is quads, then there are 3 ways of choosing a straight flush hand. This gives $4 \cdot 3 \cdot 36$ choices. If the two qualifying hands are straight flushes, then the 2 hands are determined. This gives $4 \cdot 36$ choices. Altogether we have

$$16 \cdot 36\binom{43}{16}15!! = 309,617,767,306,723,075,200$$

semideals taking the qualifying hands two at a time.

We now use inclusion-exclusion to eliminate duplicate counting. This produces

$$308,246,260,031,831,721,600 \tag{13}$$

semideals that qualify for the bad-beat jackpot for the 4 rank multisets allowing simultaneous straight flushes.

From the 48 rank multisets not yet considered, 36 allow a straight in a unique way, and 12 allow a single straight flush, but in 2 different ways. Thus, the former 36 sets have 2 uniquely determined hands that qualify. The latter 12 have 2 choices for the straight flush hand. Thus, there are $60 \cdot 36$ choices altogether for these rank multisets. We then obtain

$$2,160\binom{43}{16}15!! \quad = \quad 1,161,066,627,400,211,532,000 \tag{14}$$

semideals producing bad-beat jackpots for these 48 rank multisets.

We now move to rank multisets x-x-y-y-z for which both x and y are smaller than 8. The only way to get a qualifying semideal in this case is for 2 straight flushes to arise. This limits the rank multisets to 6-6-7-7-8, 5-6-6-7-7, 5-5-6-7-7 and the additional 7 sets we get by subtracting 1 from each rank successively until 3 is the smallest rank in the multiset. Hence, we have 10 multisets. There are 36 choices for each that lead to 3 suited cards. The two straight flush hands are uniquely determined so that the number of qualifying semideals arising from this situation is

$$360\binom{43}{16}15!! \quad = \quad 193,511,104,566,701,922,000. \tag{15}$$

One pair on board

With only one pair on board, it is impossible to have a qualifying semideal without at least one player holding a straight flush. This tends to simplify matters, but now there are 4 ranks on board which tends to complicate matters. Hopefully, these two opposing forces neutralize each other.

There are several ways we could partition the appropriate boards and we shall do so by considering the number of suited cards on board. It is clear that there are either 3 or 4 suited cards on board. We consider the case of 3 suited cards on board first.

It is possible for 3 qualifying hands to arise. We consider this subcase first. If the board has suited 3-4-5, then there must be a pair of rank 8 or more, but A-A is excluded because one of the straight flush hands is A-2, thereby, eliminating the possibility of quad aces. Thus, there are 4 choices for the suit, 6 choices for the rank of the big pair, and 3 choices for the pair of that rank.

This gives 72 boards of this form. In a similar way, we discover that there are 72 boards for suited 4-5-6, and 60 boards for suited 5-6-7.

The remaining suited triples of the form x,x+1,x+2 also allow the pair to be formed with any of the suited cards of large enough rank. For example, 6-7-8 can be completed either with another 8 or a pair of rank chosen from {J,Q,K,A}.

We now list the board types and the number of such boards in parentheses: 6-7-8-x-x (48); 6-7-8-8-x (360); 7-8-9-x-x (36); 7-8-9-9-x (360); 7-8-8-9-x (360); 8-9-10-x-x (24); 8-9-10-10-x, 8-9-9-10-x and 8-8-9-10-x (360); 9-10-J-x-x (24); 9-10-J-J-x, 9-10-10-J-x, and 9-9-10-J-x (360); 10-J-Q-x-x (24); and the 3 built on 10-J-Q with one of them paired, 360 each.

Summing the number of preceding boards gives us 4,680. In order to have 3 simultaneous qualifying hands, 3 hands are completely determined. This, gives us

$$4,680\binom{41}{14}13!! = 22,286,993,216,984,496,000$$

semideals with 3 qualifying hands, where there is one pair and 3 suited cards on board.

Now let's count semideals with at least 2 qualifying hands and use inclusion-exclusion to get the exact number of qualifying semideals with 3 suited cards on board. First, consider the case that we have 2 qualifying straight flushes. The 3 suited cards must be of consecutive ranks in order to allow 2 straight flushes. The smallest is 3-4-5 and the largest is 10-J-Q giving us 8 choices for the rank set. There is a choice of 4 suits for the 3 cards. Now if the pair is of a different rank, there are 10 choices for rank and 3 choices for the pair. If the pair is one of the successive ranks, there are 3 choices for the rank, 3 choices for the card forming the pair, 10 choices for the rank of the last card, and 3 choices for a card of that rank. Hence, the number of boards is $8 \cdot 4(30 + 270) = 9,600$.

Two hands are uniquely determined and this gives us

$$9,600\binom{43}{16}15!! = 5,160,296,121,778,717,920,000$$

semideals with 2 qualifying straight flushes, where there is one pair and 3 suited cards on board.

Finally, we count semideals with at least 2 qualifying hands, where one of the hands is a straight flush and the other is quads. One way to have a straight flush is to have 3 consecutive suited cards. We may use any of suited A-2-3 through suited Q-K-A. There are 12 choices. For these 12 choices, we first determine the number of ways of choosing pairs of a different rank that allow quads of rank 8 through A. For 2-3-4 through 5-6-7, we have a choice of 7 ranks. For A-2-3 and 6-7-8, we have a choice of 6 ranks, for 7-8-9 a choice of 5 ranks, and for the remaining a choice of 4 ranks for each set, except J-Q-K with 3 choices and Q-K-A with 2 choices. Thus, the number of boards in this situation is $4 \cdot 62 \cdot 3 = 744$.

Some of the boards, such as A-2-3-Q-Q, uniquely determine the choices for the two qualifying hands. Other boards, such as 2-3-4-J-J, have 2 ways the two qualifying hands may be chosen since either a suited A-5 or 5-6 gives a straight flush. Finally, the remaining boards allow 3 ways to choose qualifying hands. Going through them individually, there are 216 allowing a unique choice, 192 allowing 2 choices for the hands, and 336 allowing 3 choices. The number of situations for which we complete a board and 2 hands to a semideal is then $216 + (2 \cdot 192) + (3 \cdot 336) = 1,608$.

The preceding number is for the subcase that the pair is a rank distinct from one of the ranks of the 3 suited cards. Consider the alternate situation, namely, the pair is one of the ranks of the 3 suited cards. The rank set 6-7-8 allows only the 8 as the pair. The rank set 7-8-9 allows either 8 or 9 as the pair rank. The other 5 sets of 3 consecutive ranks allow any of the 3 ranks to be paired. For each of the 4-card situations arising, the fifth card may be any of the unused 10 ranks, and cannot be in the flush suit. Thus, it has 30 choices. Altogether, the number of boards is $18 \cdot 4 \cdot 3 \cdot 30 = 6,480$.

Any board involving a suited Q-K-A has only 1 choice for the straight flush, any board involving J-Q-K has only 2 choices for a straight flush, whereas, all other boards allow 3 ways of choosing the straight flush hand. Thus, we have 16,200 situations for which we complete a board and 2 hands to a semideal.

The preceding takes care of the situation when the 3 suited cards have consecutive ranks. First, let's describe all the sets on 3 non-consecutive ranks that allow a straight (we take them to be suited when we return to straight flushes). There are 10 ranks—A through 10—that may serve as the smallest rank in a straight. So fix any of these 10 ranks, which may be done in 10 ways, and then choose any 2 of the next 4 ranks, which may be done in 6 ways. This gives us 60 sets of 3 ranks that allow a straight by adding 2 additional ranks. However, this includes 10 sets of 3 consecutive ranks. We also have omitted J-Q-A and J-K-A. Hence, there are 52 sets of 3 non-consecutive ranks that allow a straight by adding two ranks.

As before, we have to insert a pair into the board so that quads may arise. If the 3 ranks have no rank bigger than 7, then the pair must be formed by a pair of large rank. If the 3 ranks have large ranks, then the pair may arise in 2 ways. So we again divide all of this up according to a pair of distinct rank, or a pair of rank agreeing with one of the suited ranks.

Let's look at pairs of distinct ranks first. The 10 sets of 3 ranks with 2 or 3 as the smallest element, allow any of 7 ranks for the rank of the pairs. Of the 5 sets with 4 as the smallest rank, 2 allow 7 ranks, but 3 allow only 6 ranks because these 3 sets contain an 8. In going through the sets, it is not difficult to verify that 12 allow 7 ranks, 10 allow 6 ranks, 5 allow 5 ranks, 5 allow 4 ranks, 9 allow 3 ranks, and 11 allow 2 ranks. This yields 238 rank multisets for the current subcase.

What we now need to check is how many choices of 2 hands each multiset produces. A set of 3 non-consecutive ranks may allow a straight in 1 or 2 ways, but never in 3 ways, by adding 2 ranks. Thus, some of the 238 rank multisets have only 1 way of choosing 2 hands that qualify, while the remaining rank multisets allow 2 ways of choosing 2 qualifying hands. Going through them leads to 70 rank multisets allowing 2 different choices for the qualifying hands and 168 rank multisets allowing only 1 choice for the 2 qualifying hands.

Since each rank multiset has 4 choices for suit and 3 choices for the pair, we have $12(168+140) = 3,696$ situations for which we need completions from a board and 2 hands to semideals.

Now we consider the subcase that the pair is of the same rank as one of the cards among the 3 non-consecutive suited cards. Going through the possible rank multisets, we find there are 55 with a unique way of choosing the 2 qualifying hands, and there are 27 with 2 ways of choosing the qualifying hands. Each rank multiset has 4 choices for the suit, 3 choices for the suit of the pair, and 30 choices for the last card. This gives us $12 \cdot 30(55+54) = 39,240$ situations for which we need completions from a board and 2 hands to semideals.

We now organize the preceding material about semideals with at least 2 qualifying semideals, where one of the qualifying hands is a straight flush and the other is quads. We determined that there are 1,608 situations of a board and two hands to complete to a semideal when the board has 3 suited cards of consecutive ranks and a pair of a distinct rank. When the pair is one of the ranks of the 3 suited cards, there are 16,200 situations to complete. Moving to 3 suited cards of non-consecutive ranks, we found 3,696 situations to complete to semideals when the pair was of a distinct rank, and 39,240 situations when the pair was one of the ranks of the 3 suited cards.

Summing the preceding numbers gives us 60,744 situations of a board and 2 hands to be completed to a semideal. Doing so gives us

$$60,744\binom{43}{16}15!! = 32,651,773,710,554,837,638,800$$

semideals when the board has 3 suited cards and a pair, where there is a qualifying hand of quads and a straight flush. Earlier we saw that there are

$$5,160,296,121,778,717,920,000$$

semideals with two qualifying straight flushes. If we sum the preceding two values and subtract twice the number of semideals with 3 qualifying hands—earlier we obtained 22,286,993,216,984,496,000 for that number—we obtain the exact number of semideals with at least 2 qualifying hands, where there are 3 suited cards and a pair on board. Performing this operation yields

$$37,767,495,845,899,586,566,800 \tag{16}$$

semideals for the subcase of 3 suited cards on board.

We now consider 4 suited cards (there cannot be more than 4 with a pair on board) on board. The first subcase is when the board allows 3 qualifying hands. This means there are 2 straight flushes and quads. Look at the subcase that the pair has a rank not involved in the straight flushes. Suited 3-4-5 allows 8 through K for the pair ranks. Suited 4-5-6 allows 8 through A for the pair ranks. This continues through suited 9-10-J which allows K-A for the pair ranks. Note that 10-J-Q allows no ranks for pairs. We obtain 33 rank multisets for which 3 qualifying hands are uniquely determined. For each rank multiset, there are 4 choices for the suit and 3 choices for the second card in the pair. This gives 396 boards and 3 uniquely determined hands. They can be completed to semideals in

$$396\binom{41}{14}13!! = 1,885,822,502,975,611,200$$

ways.

Now we do the same thing for the pairs being one of the cards among 3 suited cards of consecutive rank. For example, 7-8-9 allows either the 8 or the 9 to be paired. The fourth suited card can have any rank from {A,2,3,4,J,Q,K}. By looking at all of the appropriate x,x+1,x+2 consecutive rank sets, we obtain 81 rank multisets. There are 4 choices for the suit and 3 choices for the other card of the pair. This gives us $12 \cdot 81 = 972$ boards and 3 uniquely determined hands. The number of completions then is

$$972\binom{41}{14}13!! = 4,628,837,052,758,318,400$$

completions to semideals.

We now move to 4 suited cards without 3 consecutive ranks allowing 2 simultaneous straight flushes, and a pair of rank larger than 7. There are 4 patterns of suited ranks that work: x,x+2,x+3,x+5; x,x+2,x+3,x+6; x,x+3,x+4,x+6; and x,x+3,x+4,x+7. By taking the various appropriate values of x, we find that there are 60 rank multisets. There are 4 choices for suit and 3 choices for the suit of the pairing card. This gives another 720 boards for which there are 3 uniquely determined hands that qualify for the bad-beat jackpot. Completing these to semideals then may be done in

$$720\binom{41}{14}13!! = 34,287,68,187,228,384,000$$

ways.

The three previous numbers obtained must be summed to obtain the total number of semideals having 3 qualifying hands when there are 4 suited cards and a pair on board. Taking the sum then gives us

$$9,943,427,742,962,313,600$$

such semideals.

We are nearing the end of the case for a pair on board. We now need to count qualifying hands two at a time and use inclusion-exclusion to get the exact number. First we count the boards that allow simultaneous straight flushes. We cannot have 4 suited cards with 4 consecutive ranks because then a straight flush formed by a single card of rank 1 smaller does not qualify as the player's best hand uses 4 cards on board. By looking at the 4 forms for the sets of 4 ranks allowing 2 simultaneous straight flushes, where there are not 3 consecutive ranks, we see that there are 30 such rank sets. For 3 consecutive ranks x,x+1,x+2, the fourth rank can be anything other than x-1 or x+3, except for several special cases. This gives an additional 62 rank sets for 92 altogether. The pair can be any of the ranks from the 4 suited cards, that is, there are 4 choices. Since there are 4 choices of suit, 4 choices of rank, and 3 choices of suit for the pairing card, we have $48 \cdot 62 = 2,976$ boards with unique choices of 2 hands for completion to semideals, where the 2 hands give two straight flushes.

Now we count boards that allow a straight flush and quads of rank 8 or higher. We have to be careful here because of the additional sets of 4 suited cards allowing a single qualifying straight flush. For example, 4 suited cards of consecutive ranks with an appropriate pair can produce a qualifying straight flush and qualifying quads. There are 15 rank multisets x,x+1,x+2,x+3 including a count of the rank that has a pair. Each of these rank multisets has 4 choices for suit of the flush and 3 choices for the pairing card. This gives 180 boards with 2 unique qualifying hands to be completed to a semideal.

We now examine 4 suited cards with 3 of consecutive rank. We give the information in the table shown here. In order to use the information, we must describe the entries of the table with some care. The column headed "sequence" gives the 3 consecutive suited ranks. The column headed "y" gives the range of the fourth suited rank to form the four suited cards. The column headed "pair" tells us how many choices there are for ranks to be paired over the entire range of suited cards being described. The column headed "SF" tells us how many ways EACH of the rank sets can produce straight flushes.

sequence	y	pair	SF
A-2-3	5–7	3	1
A-2-3	8–K	12	1
2-3-4	8–K	6	2
3-4-5	A	1	2
3-4-5	8–K	6	3
4-5-6	8–A	1	3
5-6-7	9–A	6	3
6-7-8	2–4	3	3
6-7-8	10–A	10	3
7-8-9	2–5	8	3
7-8-9	J–A	12	3
8-9-10	2–6	15	3
8-9-10	Q–A	12	3
9-10-J	2–7	18	3
9-10-J	K–A	8	3
10-J-Q	2–8	22	3
10-J-Q	A	4	2
J-Q-K	2–9	26	2
Q-K-A	2–10	30	1

Let's look at one row as an example. The second row corresponds to rank sets having A-2-3 with a fourth rank chosen from rank 8 through K. This consists of 6 distinct rank sets. For each such rank set, we may choose either A or y to be paired, thereby, producing 12 qualifying pairs. Finally, each of the rank sets has only 1 way of producing a straight flush when it is suited.

We now want to calculate the number of boards and choice of 2 hands that need to be completed to semideals for a qualifying bad-beat jackpot. If we multiply the entry in the "pair" column and the "SF" column for a given row, we have the number of rank multisets for that particular sequence of 3 consecutive ranks corresponding to the possible choices for a single hand making a straight flush. Doing this and taking the sum yields 482. There are 4 choices for the suit of the 4 cards, and 3 choices for the suit of the other card in the pair. Altogether, we get $12 \cdot 482 = 5,784$ boards and choices for two hands to complete to semideals.

We now look at sets of 4 ranks not containing 3 consecutive ranks. The rank of A is special in that it can be involved in small straights, big straights, and be part of a qualifying quads hand. So we take care of all the rank multisets involving an A. We display them in the next table whose entries are read in the

sequence	y	pair	SF
A-2-4	5–7	3	1
A-2-4	8–K	12	1
A-2-5	6–7	2	1
A-2-5	8–K	12	1
A-3-4	6–7	2	2
A-3-4	8–K	12	1
A-3-5	6–7	2	2
A-3-5	8–K	12	1
A-4-5	7	1	3
A-4-5	8	2	2
A-4-5	9–K	10	1
10-J-A	2–6	15	1
10-J-A	7	3	2
10-J-A	8	4	3
10-Q-A	8–9	8	2
10-Q-A	2–7	18	1
J-Q-A	8–9	8	2
J-Q-A	2–7	18	1
10-K-A	8–9	8	1
10-K-A	2–7	18	1
J-K-A	8–10	12	1
J-K-A	2–7	18	1

same way as the previous table.

The situations described in the preceding table take care of all the sets of 4 suited cards for which an A is part of a straight flush. An A may be in the set in the remaining cases, but it will not be part of a straight flush. For a set of 4 suited cards to give rise to a straight flush without 3 consecutive ranks, there must be x,x+1,x+3 or x,x+1,x+4 or x,x+2,x+3 or x,x+2,x+4 or x,x+3,x+4 in the set. We again display the information in a table as shown next. The entries

x	x,x+1,x+3	x,x+1,x+4	x,x+2,x+3	x,x+2,x+4	x,x+3,x+4
2	8–K(6,2)	8–K(6,1)	8(1,3)	8(1,2)	8(1,3)
2	-	-	9–K(5,2)	9–K(5,1)	9(1,2)
2	-	-	-	-	10–K(4,1)
3	8(1,3)	8–K(6,1)	8–9(2,3)	8–9(2,2)	9(2,3)
3	9–K(5,2)	-	10–K(4,2)	10–K(4,1)	10(2,2)
3	-	-	-	-	J–A(4,1)
4	9(1,2)	9–A(12,1)	9–10(2,3)	9–10(4,2)	10(4,3)
4	10–A(5,1)	-	J–A(4,2)	J–A(8,1)	J(4,2)
4	-	-	-	-	Q–A(6,1)
5	9(2,1)	10–A(10,1)	10–J(4,3)	10–J(4,2)	J(3,3)
5	10–A(10,2)	-	Q–A(6,2)	Q–A(6,1)	Q(3,2)
5	-	-	2(1,2)	2(1,1)	2–3(4,1)
5	-	-	-	-	K–A(6,1)
6	10(2,1)	J–A(8,1)	J–Q(6,3)	J–Q(6,2)	Q(3,3)
6	J–A(8,2)	2(1,1)	K–A(6,2)	K–A(6,1)	K(3,2)
6	2(1,2)	-	2–3(4,2)	2–3(4,1)	A–4(9,1)
7	J(3,1)	Q–A(9,1)	Q–K(6,3)	Q–K(6,2)	K(3,3)
7	Q–A(9,2)	2–3(4,1)	A–4(9,2)	A–4(9,1)	2–5(8,1)
7	2–3(4,2)	-	-	-	-
8	Q(4,1)	K–A(8,1)	K(4,3)	K(4,2)	-
8	K–A(8,2)	2–4(9,1)	2–5(12,2)	2–5(12,1)	2–6(15,1)
8	2–4(9,2)	-	-	-	-
9	K(4,1)	2–5(12,1)	2–6(15,2)	2–6(15,1)	2–7(18,1)
9	2–5(12,2)	-	-	-	-
10	2–6(15,2)	2–6(15,1)	2–7(18,2)	2–7(18,1)	2–8(22,1)
J	2–7(18,1)	-	2–8(22,1)	-	-

are different this time so an explanation is necessary. The column headed "x" simply gives the various values of x. The other columns correspond to sets of 3 ranks producing straights. The entries in the array are given as a range of ranks for the fourth rank with some numbers in parentheses following the range of the ranks. The first entry in parentheses gives the total occurrences of ranks that can be paired to meet the qualifying standard of a pair of rank 8 or bigger. The second entry in parentheses gives the number of 2-card hands that can make a straight flush for each of the rank sets.

An example may help clear this up. Look, for example, at the top row corresponding to x = 6. Go to the middle column. This column corresponds to the ranks 6,8,9. We find the entry J–Q(6,3). This means that if the fourth rank is either J or Q, there are 3 ways of choosing a 2-card hand to make a straight flush (5-7 or 7-10 or one of 10-Q and 10-J) for either of the rank sets 6,8,9,J or 6,8,9,Q. We also have 6 choices of ranks altogether for the qualifying pair.

The entries in the parentheses provide the basis for counting what we need. The first entry gives the number of rank multisets for the 4 ranks determined by the row and the column. Note that the choice of a qualifying hand for quads is completely determined once the suits are known. The second entry is the number of choices for hands that provide a straight flush. There are 12 choices for the suit of the straight flush and the suit of the other card in the pair. So multiplying the first and second entry of the numbers in a given parenthese, and then multiplying by 12 gives us the number of hand-board combinations for the particular set of 4 ranks in the table. Hence, we multiply the entries and sum. This yields 10,728 situations to be completed to semideals.

In addition, we must do the same for the entries of the table for the straight flushes involving A. We multiply the entries in the last two columns and sum giving us 235. Multiplying by 12 yields 2,820 situations to be completed.

Going back over the material on 4 suited cards on board, we found that there are

$$9,943,427,742,962,313,600$$

semideals with 3 qualifying hands when there are 4 suited cards and a pair on board. We now need to calculate the semideals with 2 or more qualifying hands and subtract twice the preceding number to get the exact number of qualifying semideals with 4 suited cards and a pair on board. To do the latter, we found there are 2,976 boards to complete with 2 straight flushes uniquely determined, 5,784 completions to a straight flush and quads with 3 suited cards of consecutive rank, 10,728 completions for a straight flush and quads without 3 consecutive ranks and A is not involved, and 2,820 completions when A is involved.

Adding all of these gives 22,308 situations to complete to semideals with 2 hands determined. This yields

$$11,991,238,112,983,295,766,600$$

semideals. Subtracting twice the number above gives

$$11,971,351,257,497,371,139,400 \tag{17}$$

qualifying semideals when there are 4 suited cards on board and a pair. This complete the section of a pair on board.

High card on board

When there is a board with 5 cards of distinct ranks, then there can be a qualifying semideal only if there are 2 straight flushes. Since there cannot be more than 2 straight flushes for any given board, the situation is a little simpler for this case.

The only way we can have 2 straight flushes with 3 suited cards on board is for the 3 suited cards to have successive ranks. The smallest is 3-4-5 and the largest is 10-J-Q. This gives 8 choices here. We have 4 choices for suit giving us 32 choices for the 3 suited cards. The other 2 cards must have distinct ranks, and each must be in another suit. This gives us $3^2\binom{10}{2} = 405$ choices here. Thus, there are 12,960 boards.

Each board has a unique choice for the 2 qualifying hands. Thus, the number of qualifying semideals is

$$12,960\binom{43}{16}15!! \quad = \quad 6,966,399,764,401,269,192,000 \tag{18}$$

for 3 suited cards on board out of 5 cards of distinct ranks.

We now consider 4 suited cards on board. As noted before, we cannot have 4 consecutive suited cards of the form x,x+1,x+2,x+3 because a player with a straight flush using x-1 does not have a qualifying hand. Look at the subcase of 3 consecutive ranks first. The sets 3-4-5 and 10-J-Q both allow any of 7 ranks for the other suited card. All other sets allow any of 8 ranks for the other suited card. This gives us 62 sets of 4 ranks that allow simultaneous straight flushes, where 3 of the cards have consecutive ranks. The card not in the suit can have any of 9 ranks. There are 4 choices for the suit of the flush, and there are 3 choices for the suit of the other card. Altogether this gives us $62 \cdot 9 \cdot 12 = 6,696$ boards with 4 suited cards of which 3 have consecutive ranks.

Suppose there are not suited cards of 3 consecutive ranks. If there are 2 straight flushes possible, then both hands must share exactly 2 of the suited cards. This implies these 2 cards have consecutive ranks. Thus, we have 2 cards of ranks x,x+1 belonging to both straight flushes. Since there are not 3 consecutive ranks, there must be 1 card chosen from ranks x-3 and x-2, and there must be 1 card chosen from x+3 and x+4. The range of x is from 4 through 10. Thus, there are 28 choices for the 4 ranks. The card not in the suit can be any of 9 ranks. As usual, there are 12 choices for the suits. Altogether we have $28 \cdot 9 \cdot 12 = 3,024$ boards for this case.

Adding the two subcases, we have 9,720 boards of 4 suited cards, with no pairs, allowing 2 simultaneous straight flushes. This then gives

$$5,224,799,823,300,951,894,000 \tag{19}$$

semideals for the subcase just described.

We now are left with the last subcase which is 5 suited cards on board. For the same reasons mentioned above, we cannot have 4 or 5 consecutive ranks. They do not allow 2 qualifying hands.

We consider sets with 3 consecutive ranks. The question boils down to how many ways we can choose the remaining 2 ranks in these cases. For any x,x+1,x+2, we cannot choose either x-1 or x-2 as this leads to 1 possible straight flush not qualifying. We also cannot choose x+3 as this gives us 4 consecutive ranks. Hence, we are choosing any 2 ranks from 7 which can be done in 21 ways. This does not work for 10-J-Q for which we can choose any 2 from 6 which can be done in 15 ways. Thus, we have 162 sets of 4 ranks. There are 4 choices for the suit. This yields 648 boards with 5 suited cards, of which 3 have consecutive ranks, that allow simultaneous straight flushes.

Now suppose there are not 3 consecutive ranks. First, suppose we have 2 ranks that both straight flushes use. They must have the form x,x+1. The smallest x can be is 4 and the largest it can be is 10. We must pick any one of x-2 or x-3, but not both, and any one of x+3 or x+4, but not both. The remaining element can be any of the other 5 ranks. This gives 20 choices for each of the x,x+1. That gives us 140 rank sets in total, when 2 consecutive ranks are used by both straight flushes.

This leaves us with the case that both straight flushes use only one rank in common. When this rank is A, there must be 2 ranks chosen from each of 2,3,4,5 and 10,J,Q,K without creating 3 consecutive ranks. This can be done in 5 ways for each of the 2 collections. This gives 25 rank sets altogether. The other ranks that can be used solo by 2 straight flushes are 5 through 10. We have to be a little careful that we don't create 3 consecutive ranks. Let x be the common rank. Of the 5 choices for choosing 2 ranks smaller than x, 2 of them use x-1. In these cases, we cannot use either of the choices for ranks larger than x that use x+1. Thus, for the 2 choices of smaller rank, there are 3 choices of larger rank giving us 6. For the other 3 choices of smaller rank, we can use all 5 choices for larger rank. Hence, there are 21 choices altogether. This gives us 151 rank sets sharing exactly 1 rank for 2 simultaneous straight flushes.

Summing the numbers obtained above, we obtain 939 rank sets with 5 suited cards allowing simultaneous straight flushes. We multiply by 4 to get the number of boards as 3,756. They now complete to qualifying semideals in

$$3,756\binom{43}{16}15!! \quad = \quad 2,018,965,857,645,923,386,200 \tag{20}$$

ways.

We are now ready to determine the probability of a qualifying bad-beat semideal under the rules stipulated at the beginning of this file. The number of qualifying semideals is given by the sum of (2), (6), (7), (8), (9), (10), (12), (13), (14), (15), (16), (17), (18), (19), and (20). This sum is

$$107,175,922,726,512,897,457,200. \tag{21}$$

We divide (21) by (1) and obtain

$$0.000006451725266$$

for this probability. This is approximately 1 chance in 155,000. Thus, a bad-beat qualifying semideal is unlikely under these rules.