Distribution of Aces Among Dealt Hands

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Abstract

We provide details of the computations for the distribution of aces among nine and ten hold'em hands.

There are 4 aces and 48 non-aces in a standard card deck. The total numbers of choices for cards to be dealt to 9 and 10 players in hold'em is $\binom{52}{20}$ for 10 players and $\binom{52}{18}$ for 9 players. Similarly, there are $\binom{48}{20}$ ways to choose cards not containing any aces to be dealt to 10 players, and $\binom{48}{18}$ ways to choose cards not containing any aces to be dealt to 9 players. Thus, the probability that no aces are dealt to 10 players in hold'em is

$$\frac{\binom{48}{20}}{\binom{52}{20}} = \frac{7,192}{54,145}.$$

Continuing in the same way, the probability that no aces are dealt to 9 players in hold'em is

$$\frac{\binom{48}{18}}{\binom{52}{18}} = \frac{2,728}{15,925}.$$

The number of ways of choosing one ace to be included in 20 cards is $4\binom{48}{19}$, and the number of ways of choosing one ace to be included in 18 cards is $4\binom{48}{17}$. Thus, the probability exactly one ace is among the hands in 10-handed hold'em is

$$\frac{4\binom{48}{19}}{\binom{52}{20}} = \frac{3,968}{10,829}.$$

Similarly, the probability there is exactly one ace among the hands in 9-handed hold'em is

$$\frac{4\binom{48}{17}}{\binom{52}{18}} = \frac{6,336}{15,925}.$$

There are $6\binom{48}{18}$ ways to choose exactly 2 aces among 20 cards and $6\binom{48}{16}$ ways to choose exactly 2 aces among 18 cards. We have a slight complication in that the 2 selected aces may be dealt in one hand or may be spread over 2 hands. The number of ways of partitioning m cards, where m is even, into $m/2$ hands of 2 cards each is $(m-1)!!$, where $(m-1)!! = (m-1)(m-3)\cdots 5\cdot 3\cdot 1$. This implies that there are 19!! ways of partitioning 20 given cards into hold'em hands. If the 2 aces are in the same hand, then there are 17!! ways of partitioning the remaining 18 cards into hold'em hands. If the 2 aces are in different hands, then this can be done in $18 \cdot 17 \cdot 15!!$ ways. Note that $17!! + (18 \cdot 17 \cdot 15!!) = 19!!$. Thus, we have shown that for a given collection of 20 cards with exactly 2 aces, 1/19 of the possible ways of partitioning the 20 cards into hold'em hands puts the 2 aces in the same hand, and 18/19 of the possible ways puts them in different hands.

Performing the same analysis with 18 cards yields that 1/17 of the possible partitions into hold'em hands puts the 2 aces in the same hand. Since there are 6 ways to choose 2 aces, we obtain the following probabilities.

The probability that out of 10 hold'em hands there is a pair of aces and no other aces present is

$$\frac{6\binom{48}{18}}{19\binom{52}{20}} = \frac{992}{54,145}.$$

The probability that out of 10 hold'em hands precisely 2 players have a single ace is

$$\frac{6 \cdot 18\binom{48}{18}}{19\binom{52}{20}} = \frac{17,856}{54,145}.$$

The probability that out of 9 hold'em hands there is a pair of aces and no other aces present is

$$\frac{6\binom{48}{16}}{17\binom{52}{18}} = \frac{297}{15,925}.$$

The probability that out of 9 hold'em hands precisely 2 players have a single ace is

$$\frac{6 \cdot 16\binom{48}{16}}{17\binom{52}{18}} = \frac{4,752}{15,925}.$$

There are $4\binom{48}{17}$ ways to choose 20 cards so that exactly 3 of them are aces. There are $4\binom{48}{15}$ ways to choose 18 cards so that exactly 3 of them are aces. Either we end up with one hand having a pair of aces and another hand having a single ace, or we end up with all 3 aces in sepaprate hands. We again use proportions to determine how many of each type there are.

Suppose we have a given set of 20 cards containing precisely 3 aces. There are $3 \cdot 17!!$ partitions with a pair of aces because there are 3 ways of choosing 2 of the aces and 17!! ways of partitioning the remaining 18 cards. Thus, 3/19 of the time we have a pair of aces, and 16/19 of the time we have the 3 aces spread over 3 hands. This is for the 10-handed case.

In a similar fashion, if we have a given set of 18 cards containing precisely 3 aces, there are $3 \cdot 15!!$ ways of partitioning the cards into 2-card hands so that there is a pair of aces. Hence, 3/17 of the time we have a pair of aces, and 14/17 of the time we have the 3 aces spread over 3 hands. This is for the 9-handed case.

We now have some more probabilities we may derive. The probability that out of 10 hold'em hands there is a pair of aces and a single ace in another hand is

$$\frac{3 \cdot 4\binom{48}{17}}{19\binom{52}{20}} = \frac{1,152}{54,145}.$$

The probability that out of 10 hold'em hands there are 3 aces in 3 separate hands is

$$\frac{4 \cdot 16\binom{48}{18}}{19\binom{52}{20}} = \frac{6,144}{54,145}.$$

The probability that out of 9 hold'em hands there is a pair of aces in a hand and a single ace in another hand is

$$\frac{4 \cdot 3\binom{48}{15}}{17\binom{52}{18}} = \frac{288}{15,925}.$$

The probability that out of 9 hold'em hands there are 3 aces in 3 separate hands is

$$\frac{4 \cdot 14\binom{48}{15}}{17\binom{52}{18}} = \frac{192}{2,275}.$$

This brings us to the case that there are 4 aces among the hands. There are $\binom{48}{16}$ ways to choose 20 cards containing all 4 aces, and there are $\binom{48}{14}$ ways to choose 18 cards containing all the aces. For a given set of 20 cards containing the 4 aces, there are $3 \cdot 15!!$ ways to partition the cards with 2 pairs of aces. There are $6 \cdot 16 \cdot 15!!$ ways to partition the cards so that there is exactly 1 pair of aces. Finally, there are $16 \cdot 14 \cdot 15!!$ ways to partition the cards so that the aces occur in 4 separate hands. This implies that $3/323$ of the deals have a pair of aces, $96/323$ of the deals have exactly 1 pair of aces, and $224/323$ of the deals have the 4 aces sperad over 4 hands.

The preceding then gives the following probabilities for 10-handed hold'em. The probability that there are 2 pairs of aces is

$$\frac{3\binom{48}{16}}{323\binom{52}{20}} = \frac{9}{54,145}.$$

The probability that there is a pair of aces and 2 other players with an ace is

$$\frac{96\binom{48}{16}}{323\binom{52}{20}} = \frac{288}{54,145}.$$

Finally, the probability that there are 4 aces spread over 4 hands is

$$\frac{224\binom{48}{16}}{323\binom{52}{20}} = \frac{96}{7,735}.$$

For a given set of 18 cards containing the 4 aces, there are $3 \cdot 13!!$ ways to partition the cards with 2 pairs of aces. There are $6 \cdot 14 \cdot 13!!$ ways to partition the cards so that there is exactly 1 pair of aces. Finally, there are $14 \cdot 12 \cdot 13!!$ ways to partition the cards so that the aces occur in 4 separate hands. This implies that 3/255 of the deals have a pair of aces, 84/255 of the deals have exactly 1 pair of aces, and 168/255 of the deals have the 4 aces spread over 4 hands. We then obtain the following probabilities for 9-handed hold'em.

The probability that there are 2 pairs of aces is

$$\frac{3\binom{48}{14}}{255\binom{52}{18}} = \frac{36}{270,725}.$$

The probability that there is a pair of aces and 2 other players with an ace is

$$\frac{84\binom{48}{14}}{255\binom{52}{18}} = \frac{144}{38,675}.$$

Finally, the probability that there are 4 aces spread over 4 hands is

$$\frac{168\binom{48}{14}}{255\binom{52}{18}} = \frac{288}{38,675}.$$

We now put the preceding information in an easy to read table. In the preceding text, we worked out the exact probabilities for the various events. This enables us to check the work for correctness. We convert the exact rational values into decimal form which means the answers are rounded off. Thus, the probabilities in the table may not sum exactly to 1. Do not worry as this results from roundoff errors.

Aces	10-handed	9-handed
0	.1328	.1713
1	.3664	.3979
1,1	.3298	.2984
2	.0183	.0186
1,2	.0213	.0181
1,1,1	.1135	.0844
2,2	.000166	.000133
1,1,2	.00532	.00372
1,1,1,1	.0124	.00745

Let's say a few words about reading the table. The numbers in the left column tell you how the aces are distributed. For example, 2 means there is a single pair of aces, whereas, 1,2 means one player has a pair of aces and another player has a single ace. The corresponding numerical values are the probabilities of having the pattern of aces dealt for 9-handed and 10-handed hold'em.

We now consider the same problem from the standpoint of a fixed player. Suppose we have a player holding A-x, where the card of rank x is not another ace. The remaining hands are being chosen from 50 cards of which 3 are aces and 47 are not aces. This sets the parameters for the calculations that follow. Everything is similar to the work above.

There are $\binom{47}{18}$ ways to choose 18 cards without an ace and $\binom{47}{16}$ ways to choose 16 cards without an ace. Dividing by $\binom{50}{18}$ and $\binom{50}{16}$, respectively, gives the probabilities that none of the remaining 9 or 8 players, respectively, has an ace.

There are $3\binom{47}{17}$ ways to have exactly 1 ace among 18 cards and $3\binom{47}{16}$ ways among 16 cards. This leads to the corresponding probabilities for exactly 1 ace among the other players.

There are $3\binom{47}{16}$ ways to have exactly 2 aces among 18 cards. Of these, 1/17 have the aces as a pair and 16/17 spread the aces over 2 hands. There are $3\binom{47}{14}$ ways to have exactly 2 aces among 16 cards. Of these, 1/15 have the aces as a pair and 14/15 place the aces in 2 different hands.

There are $\binom{47}{15}$ ways to have all 3 remaining aces in the other 9 hands for the 10-handed case. We have 3/17 of the hands with a pair of aces and 14/17 of the hands with the aces spread over 3 hands. There are $\binom{47}{13}$ ways to have the 3 remaining aces in the the other 8 hands for the 9-handed case. Of these, 1/5 have a pair of aces and 4/5 have the aces spread over 3 hands.

We display the information just obtained in the following table. The table also contains more information so let's say a few words about reading it. The notation for the distribution of the aces is the same as that used for the earleir table. The other column headings tell us whether the game is 9-handed or 10-handed, and the columns headed 'yes' and 'no' tell us whether or not the fixed player has an ace in her hand. Thus, looking across the row for 1,1, we find a probability of .2351 that a player holding a single ace in a 10-handed hold'em game is facing two other players each with a single ace, whereas, we find a probability of .2728 that a player not holding an ace in 9-handed hold'em is facing two other players each with a single ace.

Aces	10-handed		9-handed	
	yes	no	yes	no
0	.2531	.1561	.3053	.2014
1	.4555	.3877	.458	.4157
1,1	.2351	.3101	.1943	.2728
2	.01469	.01938	.01388	.01949
1,2	.00735	.02001	.00571	.01653
1,1,1	.03429	.09337	.02286	.06614
2,2	-	.000156	-	.000122
1,1,2	-	.00438	-	.00292
1,1,1,1	-	.00875	-	.00486

We now determine the probabilities for the preceding table when our fixed player does not have an ace. Now there are 4 aces and 46 non-aces in the 50 cards left over. Thus, in 10-handed hold'em, there are $\binom{46}{18}$ ways none of the other players has, and in 9-handed hold'em there are $\binom{46}{16}$ ways to choose 16 cards without an ace. This leads to the entries in the row labelled 0.

There are $4\binom{46}{17}$ ways to choose 18 cards with exactly 1 ace, and $4\binom{46}{15}$ ways to choose 16 cards with exactly 1 ace. We divide by $\binom{50}{18}$ and $\binom{50}{16}$, respectively, to obtain the probabilities for the row labelled 1.

There are $6\binom{46}{16}$ ways to choose 18 cards with precisely 2 aces among them. Of these, 1/17 form hands with the 2 aces in a single hand, while 16/17 of the hands spread the aces over 2 hands.

In a similar manner, there are $6\binom{46}{14}$ ways to choose 16 cards containing exactly 2 aces. There are 1/15 of the hands formed with the aces paired and 14/15 with the aces not paired.

There are $4\binom{46}{15}$ ways to choose 18 cards with exactly 3 aces among them. Of these, 3/17 of the hands formed have a pair and 14/17 of the hands formed do not have a pair. We divide by $\binom{50}{18}$ to get the probabilities in the table.

There are $4\binom{46}{13}$ ways to choose 16 cards with exactly 3 aces among them. Of these, 1/5 of the hands formed have a pair and 4/5 of the hands formed do not have a pair. We divide by $\binom{50}{16}$ to get the probabilities in the table.

There are $\binom{46}{14}$ ways to choose 18 cards that contain all 4 aces. For a given set of 18 cards containing all 4 aces, there are 3 ways the aces can be split into 2 hands and 13!! ways to partition the remaining 14 cards into 7 hands. This gives us $3 \cdot 13!!$ ways to form the cards into hands with 2 pairs of aces. There are 6 ways to choose a single pair of aces and $14 \cdot 13$ ways to choose cards to go with each of the other aces. This leaves 11!! ways to partition the remaining cards. Altogether we then have $84 \cdot 13!!$ ways to form 9 hands with a single pair of aces. Finally, there are $14 \cdot 13 \cdot 12 \cdot 11$ ways to choose cards to go with the aces. This gives us $168 \cdot 13!!$ ways to form hands with the aces spread over 4 hands. Note that these 3 numbers sum to 17!! as they should. In particular, 3/225 of the hands have a pair of aces, 84/225 of the hands have a single pair of aces, and 168/225 of the hands have no pairs of aces.

There are $\binom{46}{12}$ ways to choose 16 cards containing all 4 aces. Using an analysis similar to the preceding paragraph, we find that 1/65 of the hands formed have a pair of aces, 24/65 have a single pair of aces, and 40/65 have no pair of aces.

We complete this file by working out the probabilities that a player holding A-x, where x is not another ace, is facing at least one other player with an ace and a bigger kicker. To determine these probabilities we use the standard version of inclusion-exclusion for probabilities.

Here is a basic description of how we do this. We let the other players be labelled 1,2,...,9, when the game is 10-handed, and 1,2,...,8, when the game is 9-handed. We assume that a given player is holding an A-x, where x is a card of rank 2 through K. We let $p(i)$ denote the probability that player i has an A-y, where y is a rank bigger than x. We allow y to be another ace. Similarly, we let $p(i, j)$ denote the probability that both player i and player j have aces with bigger kickers. Finally, we let $p(i, j, k)$ denote the probability that the 3 players i, j, k all have aces with bigger kickers.

The principle of inclusion-exclusion then tells us that the probability at least one player has an ace with bigger kicker is given by

$$\sum_i p(i) - \sum_{i,j} p(i, j) + \sum_{i,j,k} p(i, j, k), \tag{1}$$

where the sums are over all i from 1 through 9, all distinct unordered pairs i, j from 1 through 9, and all distinct unordered triples i, j, k from 1 through 9 when the game is 10-handed hold'em. When the game is 9-handed, we use 1 through 8.

In fact, we may use equation (1) to obtain an exact formula in terms of the number of ranks larger than x. Let $n + 1$ be the number of ranks larger than x, where we include rank A in counting $n + 1$. For example, if the player has A-Q, then $n = 1$ as both A and K are larger than Q.

For a given player i, there are $\binom{50}{2} = 1,225$ possible hands. There are 3 ways i can have A-A, and there are $12n$ ways that i can have a hand of the form A-y, where y is a rank different from A that is bigger than x. Thus,

$$p(i) = \frac{12n + 3}{1,225}.$$

For two given players i, j, there are $\binom{50}{2}\binom{48}{2} = 1,381,800$ ways they may be dealt hold'em hands. Note that if x = K, then not both i and j may be dealt A-A as one ace already is in the given player's hand. Thus, $p(i, j) \neq 0$ only when $n \geq 1$. When $n \geq 1$, player i may have A-A in 3 ways and player j may then have A-y in $4n$ ways. The same is true with the roles of i and j switched. This gives $24n$ ways that one of them has A-A. If both have hands of the form A-y, where y is not A and is bigger than x, there are 3 choices for the ace for i and $4n$ choices for the kicker. There are now 2 choices for the ace for j and $4n - 1$ choices for the kicker. Altogether we have $96n^2$ combinations for which both i and j have aces with bigger kickers. Thus,

$$p(i, j) = \frac{96n^2}{1,381,800}.$$

We now consider three players i, j, k. Again we have $p(i, j, k) = 0$ unless $n \geq 1$. We now must share the 3 remaining aces among the 3 players. There are 3 choices for the ace to go to player i and $4n$ choices for his kicker. There are 2 choices for the ace to go to player j and $4n - 1$ choices for his kicker. Player k has no choices for his ace and $4n - 2$ choices for the kicker. We then have $384n^3 - 288n^2 + 48n$ ways that all 3 have an ace with better kickers. This yields

$$p(i, j, k) = \frac{384n^3 - 288n^2 + 48n}{1,430,163,000},$$

since there are $\binom{50}{2}\binom{48}{2}\binom{46}{2} = 1,430,163,000$ ways to deal hold'em hands to 3 fixed players.

Consider 10-handed hold'em first. In looking at equation (1), we see there are 9 equal terms in the first sum, $\binom{9}{2} = 36$ equal terms in the second sum, and $\binom{9}{3} = 84$ equal terms in the last sum. Therefore, the probability that a player holding A-x is facing at least one player with an ace and bigger kicker is given by

$$\frac{108n + 27}{1,225} - \frac{3,456n^2}{1,381,800} + \frac{32,256n^3 - 24,192n^2 + 4.032n}{1,430,163,000} \tag{2}$$

for 10-handed hold'em. We plug in the corresponding values of n to get the entries in the table below.

kicker	10-handed	9-handed
K	.022	.0196
Q	.1077	.096
J	.1885	.1686
10	.2645	.2375
9	.3359	.3027
8	.4027	.3644
7	.4653	.4226
6	.5236	.4775
5	.5778	.529
4	.628	.5774
3	.6745	.6227
2	.7172	.6649

To obtain the entries for 9-handed hold'em, we work with an equation similar to equation (2). The difference in the coefficients arises from the fact we are choosing combinations from 8 opponent hands instead of 9 opponent hands. The corresponding equation is

$$\frac{96n + 24}{1,225} - \frac{2,688n^2}{1,381,800} + \frac{21,504n^3 - 16,128n^2 + 2,688n}{1,430,163,000}. \tag{3}$$

We substitute the values of n to complete the above table.