MAXIMUM NORM ERROR ESTIMATES FOR DIFFERENCE SCHEMES FOR FULLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. This article establishes error bounds for finite difference schemes for fully nonlinear parabolic Partial Differential Equations (PDEs). For classical solutions the global error is bounded by a known constant times the truncation error of the exact solution. As a corollary, this gives a convergence rate of 1 or 2 for first or second order accurate schemes, respectively. Our results also apply for schemes where the local truncation error depends on multiple parameters.

1. Introduction

This article establishes error bounds for finite difference schemes for fully nonlinear parabolic Partial Differential Equations (PDEs). These equations arise in classical areas such as geometry and fluid dynamics as well as modern applications such as Image Processing, Stochastic Control, and Mathematical Finance.

The well-known Lax Equivalence Theorem [23, 15] states

> A consistent finite difference scheme for a partial differential equation for which the initial-value problem is well posed is convergent if and only if it is stable.

The theorem, originally stated for linear equations, can be modified to apply in more general settings. Paraphrasing from [3]

> A consistent finite difference scheme for a fully nonlinear parabolic partial differential equation for which the initial-value problem is well posed is convergent if it is monotone.

A monotone scheme is one which satisfies the Discrete Comparison Principle (a nonlinear version the Maximum Principle). A characterization of such schemes was provided in [21], and will be recalled below.

1.1. The class of equations.

Our results concern the fully nonlinear elliptic partial differential operator

$$F[u](x) \equiv F(D^2u(x), Du(x), u(x), x). \tag{1}$$

Here, Du and D^2u denote the gradient and Hessian of u, respectively. The function $F(X, p, r, x)$ is defined on $\mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$, where Ω is a domain in $\mathbb{R}^n$, and $\mathbb{S}^n$ is the space of symmetric $n \times n$ matrices.

We study numerical solutions of the nonlinear parabolic PDE,

$$u_t(x,t) + F[u](x,t) = 0, \tag{PDE}$$

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for (x,t) in $\Omega \times [0,T)$, along with initial and boundary conditions

$$\begin{cases} u(x,t) = g(x,t), & \text{for } (x,t) \text{ on } \Omega \times \{0\} \\ u(x,t) = h(x,t), & \text{for } (x,t) \text{ on } \partial\Omega \times (0,T). \end{cases}$$

Example. We list several examples of equations which fit into the class. More can be found in the first few pages of [8]. Example schemes can be found in [21] and the other references.

(1) The heat equation $u_t - \Delta u = 0$.

(2) Linear parabolic equations, where $F[u] = L[u] = -\sum_{ij} a_{ij} u_{x_i x_j}$, provided the matrix of coefficients a_{ij} is nonnegative definite.

(3) Parabolic Obstacle problems, for example

$$F[u] = \min(u - f(x), -\Delta u).$$

(4) Time dependent free boundary problems, such as the Stefan Problem.

(5) The reaction diffusion equation $u_t = \Delta u - f(u)$, where $f'(u) \le 0$. (If the sign condition on f is violated, the equation is not in the class, although these cases can be handled by similar techniques.)

(6) The Hamilton-Jacobi equation, $u_t - |u_x| = 0$, and the general HJ equation

$$u_t = H(Du, u, x).$$

(7) Geometric equations involving curvature, for example, the equation for motion of level sets by mean curvature

$$u_t = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

(8) The fully nonlinear equation parabolic equation which arises when

$$F[u] = \max(L_1[u], L_2[u])$$

where L_1, L_2 are linear elliptic operators.

(9) More generally, the Pucci Maximal and Minimal operators [22]

$$F^M[u] = \max_\alpha L_\alpha[u], \qquad F^m[u] = \min_\alpha L_\alpha[u],$$

where L_α is a family of linear elliptic operators.

(10) The value function for a stochastic control or game problem.

Since there is no need to use monotone schemes for linear or divergence structure equations, and the theory is more developed for first order equations, our main interest is in fully nonlinear or degenerate second order equations.

In this article, the goal is to relate the global error of the scheme to the local truncation error. Since this truncation error is most meaningful when the solutions are regular, we assume that we are working with classical solutions. For many cases of interest, fully nonlinear equations support regular solutions. For example, solutions are regular when the equation is convex (or concave) and uniformly elliptic [11, 4] (though stated for elliptic equations, the same, or stronger, results usually hold for the parabolic version of the equation). Thus the regularity result applies to examples 8 and 9, provided the linear operators are uniformly elliptic.

On the other hand for Hamilton-Jacobi equations or for Motion by Mean Curvature, example 7, the regularity of the solutions breaks down in general, and so do the error estimates. Even when classical solutions don't exist, monotone schemes converge to the viscosity solution [3]. (For a reference on viscosity solutions, see [8]).

But where the solution is not differentiable, the local truncation error blows up. As a consequence, the error estimates that can be obtained in that setting are much weaker. Nevertheless, even for equations which do not support regular solutions in general, an error estimate of this kind is still useful in the regions where the solution is smooth, and we can rely on existing convergence results near singularities.

An optimistic expectation is that the global error of the numerical solution is of the same order as the local truncation error. We show that this is indeed the case. In fact, we prove an even sharper result.

> The maximum norm of the global error after time t is at most (t times) the maximum norm of the truncation error for the exact solution.

As a simple consequence we obtain second order convergence for second order schemes, with *known constants*.

1.2. **Monotone schemes.** Monotone schemes are a very restrictive class of schemes, and they have the additional shortcoming of limited accuracy. However they are the right class for obtaining convergence results, and in some cases of interest, they are the only schemes which can be shown to converge. It is also true that for some equations, monotone schemes are *necessary* for convergence to viscosity solutions. For example there is a linear, degenerate equation for which non-monotone schemes fail to converge [21].

In many cases, upwinding is used for first order terms, and centered differences are used for second derivatives. Higher order interpolation violates the comparison principle so this limits the accuracy to first order for first order terms, and second order for second order terms.

The article [21] gives many example of monotone scheme using a standard stencil. Linear schemes which satisfy the maximum principle are characterized by a simple sign condition on the coefficients, and are called schemes of positive type. There is no monotone finite difference scheme for the term u_{xy}, so operators which contain that term must be expressed as a combination of second order derivatives in directions which may not line up with the grid. As a result, even for linear equations, it may not always be possible to build monotone schemes on a given grid [17].

Crandall and Lions [9] attempted to get around this difficulty by building very wide stencil schemes. This works for linear equations, but the schemes they built for quasilinear equations are not monotone (see [19] for a discussion).

There are no known general methods for building monotone schemes for quasilinear equations. However, for uniformly elliptic quasilinear equations, there is no need for monotone schemes: it is always possible to build convergent Galerkin approximations. However, in the degenerate elliptic case, the bilinear forms are no longer definite so the problem becomes ill-posed.

However, monotone schemes have been built for the equation for motion of level sets by Mean Curvature [19] and for the Infinity Laplace equation [20], both degenerate, quasilinear equations. These schemes were tailor-made for the equations. A more general class of schemes was developed for PDEs which are functions of the eigenvalues of the Hessian [22]. These schemes require successively wider stencil grids in order to converge, and so the local truncation error depends on both the spatial resolution, ϵ, and the directional resolution, $d\theta$. (In practice, schemes no wider than 3 or 4 grid points are used, and good results were obtained). The error estimates we prove here apply in this case.

Terminology. The term *monotone* is overused in the literature. For example, in convex analysis, a monotone mapping is the subgradient of a convex function. This definition is related to a mapping being a contraction in the ℓ^2 rather than the maximum norm, as is our case. Our monotone maps need not be monotone in the sense of convex analysis.

Even more confusing, in the context of difference schemes for conservation laws there is the concept of a scheme *montonicity preserving* maps. This means that non-decreasing functions (defined on an interval) are taken to non-decreasing functions. This is related to the comparison principle (since no new local maxima or minima are created by such maps), but not equivalent (since $x \mapsto 2x$ is order preserving).

To eliminate this confusion, from now on we use the term *nonlinear elliptic* for our schemes. Solutions of nonlinear elliptic schemes satisfy the comparison principle, whether or not they are given as explicit mappings. In this context, we reserve the term monotone mappings for order preserving mappings. See [21] for more details.

Related results. There is a substantial literature on convergence rates with no regularity assumption on the solutions. The earliest works [7, 24] on convergent numerical schemes for viscosity solutions provided a convergence rate of 1/2, for schemes for Hamilton-Jacobi (HJ) equations.

Later work by a number of authors established convergence rates for second order elliptic and parabolic equations, without any regularity assumptions (see for example Krylov [10], Kuo and Trudinger [14], Barles and Jacobsen [2], and Caffarelli and Souganidis [5] and the references therein). The rates obtained are small fractions, with successive papers yielding better fractions. The analysis is technical: the techniques involved come from regularity theory for nonlinear elliptic PDEs.

Stronger results can be obtained for convex HJ equations. Tadmor and Lin [16] obtained first order convergence rates in the L^1 norm: The L^1 norm of the error is estimated by the L^1 norm of the truncation error. (This weaker norm Their results are applied to general approximation schemes (e.g. regularization with the Laplacian) as well as Godunov type finite difference schemes. The analysis exploits the connection between HJ equations and conservation laws. The first result of this type in the nonlinear framework was for conservation laws [18].

Proof techniques. In the works mentioned above, the solution of the scheme is regarded as a perturbation of the solution of the PDE, and the error estimates are quite difficult because they are performed in a continuous setting. Here we regard the solution of the PDE as a perturbation of the solution of the scheme, and make estimates in the discrete setting. These estimates are much easier to obtain. The resulting proofs are short, non-technical, and self-contained (with the exception of one result which we cite from [21]).

In fact, the proof can be regarded as an adaptation of standard methods from numerical analysis to the fully nonlinear setting. To highlight this, we begin by recalling the proof of error estimates for the heat equation using the maximum principle. We then establish the result for a concrete nonlinear example (or two) before moving on to the general case.

The parabolic case is easier because errors propagate forward in time. In a future work we will present results for a class of elliptic equations.

1.3. **Nonlinear elliptic equations.** The class of equations we study are defined by a local structure condition which follows. It is the fully nonlinear version of the usual condition that the coefficient matrix of a linear elliptic equation be positive definite. Because the theory allows for even non-negative definite coefficient matrices, the condition is usually referred to as degenerate ellipticity. But we prefer the term nonlinear elliptic.

Definition. The differential operator (1) is *nonlinear* or *degenerate* elliptic if

$$F(N, p, r, x) \leq F(M, p, s, x) \text{ whenever } r \leq s \text{ and } M \leq N. \tag{2}$$

Here $M \leq N$ means that $M - N$ is a nonnegative definite symmetric matrix. The corresponding parabolic operator (PDE) is called *nonlinear* or *degenerate* parabolic.

1.4. **Approximation schemes and the truncation error.** Finite difference schemes will be defined below. We also allow for approximation schemes to be defined in the continuous setting, for example, the standard viscosity approximation $F^\epsilon \equiv F - \epsilon\Delta u$.

Definition. Consider the approximation scheme F^ϵ for the equation (1). The *truncation error* $\delta^\epsilon[u]$ for the function $u(x)$ is the function

$$\delta^\epsilon[u] \equiv F^\epsilon[u](x) - F[u](x). \tag{3}$$

(for discrete schemes it is a grid function). When the forward Euler method (7) is used, the truncation error also includes the error from the time discretization.

$$\delta^{\epsilon,\rho}[u] \equiv \delta^\epsilon[u] + \delta^\rho[u], \quad \delta^\rho[u]_j^n \equiv \frac{u_j^n - u_j^{n-1}}{\rho} - (u_t)_j^n \tag{4}$$

Remark. While forward Euler is only first order accurate, higher accuracy in time can be achieved without compromising the monotonicity using strong stability preserving (SSP) schemes [13]. However in most cases the overall accuracy is still limited by the spatial terms.

1.5. **Error estimates.** An important property of viscosity solutions is their stability in L^∞ to perturbations, not only of the data, but also of the equation itself. In fact, the convergence of approximation schemes can be regarded as an interpretation of this result.

We begin by recording the error estimates in the continuous setting. The first result is standard, see Lemma 4 (also [12, §10.4 Problem 2]).

Lemma 1 (L^∞ stability). Let u_1, and u_2 be two different exact viscosity solutions of (PDE), subject to difference initial conditions. Then

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_{L^\infty} \leq \|u_1(\cdot, 0), -u_2(\cdot, 0)\|_\infty$$

It is also a standard result that solutions of perturbations in the class converge to the unperturbed solution, see for example [1, §8]. An *a priori* error estimate does more: it measures the difference between the two solutions in terms of a known quantity.

In the theorems that follow, similar error estimates are stated for approximation schemes which move from continuous in time and space, to discrete in space, continuous in time, and then fully discrete. Proofs appear in §5.

Theorem (Error estimate)**.** *Let u be the viscosity solution of* (PDE)*, and let u^ϵ be the solution of a family of approximate equations, $u^\epsilon_t + F^\epsilon[u^\epsilon] = 0$, where F^ϵ are elliptic operators, subject to initial condition $u(x, 0)$. Then*

$$\|u(\cdot, t) - u^\epsilon(\cdot, t)\|_{L^\infty} \le t\|\delta^\epsilon[u]\|_{L^\infty(x,t)}.$$

The next corollary follows immediately.

Corollary (Perturbed equations)**.** *Let u be the viscosity solution of* (PDE)*, and let u^ϵ be the viscosity solution of $u_t + F[u] + \epsilon F_1[u]$, where F_1 is an elliptic operator. Then*

$$\|u(\cdot, t) - u^\epsilon(\cdot, t)\|_{L^\infty} \le Ct\|F_1[u^\epsilon]\|_{L^\infty(x,t)}.$$

Theorem (Error estimates for the method of lines.)**.** *Let $u(x, t)$ be a classical solution of* (PDE)*. Let F^ϵ be an elliptic scheme for* (1) *and let $U^\epsilon(t)$ be the solution of the method of lines* (ODE)*. Then the scheme converges and the global error is bounded by (t times) the truncation error for the solution,*

$$\max_j |U^\epsilon_j(t) - u_j(t)| \le t \max_{0\le s\le t} \max_j |\delta^\epsilon[u(s)]_j|.$$

Theorem (Error estimates for Forward Euler)**.** *Let $u(x, t)$ be a classical solution of* (PDE)*. Let F^ϵ be an elliptic scheme for* (1) *and let U^ϵ be the solution of the forward Euler method* (17)*. Suppose* (CFL) *holds. Then the scheme converges and the global error is bounded by ($t = n\rho$ times) the truncation error for the solution,*

$$\max_j |(U^\epsilon)^n_j - u^n_j| \le n\rho \max_{m\le n} \max_j |\delta^{\epsilon,\rho}[u]^m_j|.$$

Definition. The scheme F^ϵ is *accurate to order k* if $\delta^\epsilon[u] = O(\epsilon^k)$ for every smooth function $u(x)$.

Corollary. *Let $u(x, t)$ be a twice continuously differentiable in t, and four times continuously differentiable in x solution of* (PDE)*. Let F^ϵ be a second order accurate scheme for* (PDE)*, and let U^ϵ the solution of* (7)*. If* (CFL) *holds, then*

$$\max_j |(U^\epsilon)^n_j - u^n_j| \le n\rho C(\epsilon^2 + \rho).$$

Proof. The hypotheses guarantee that $\delta^{\epsilon,\rho}[u] \le C(\epsilon^2 + \rho)$. □

Remark. Notice the CFL condition is not needed for the method of lines. This shows that both stability and monotonicity are ensured by the ellipticity condition. The CFL condition arises from the time discretization.

2. Concrete Examples

In this section we present self-contained basic examples, before moving on to the general case. We obtain the expected error estimates $O(\epsilon^2)$ and $O(\epsilon)$ for the schemes, which are second and first order accurate, respectively.

Later, in the proof of the theorems, we keep better track of the error terms in order to get error bounds in terms of the local truncation error for the solution.

We begin by reviewing the the familiar results in the case of the heat equation, $u_t = u_{xx}$. Next we show how the proof is modified to apply to a first order nonlinear equation $u_t = |u_x|$. Finally, we move on to the model case of example 8,

$$u_t = \max(2u_{xx} + u_{yy}, u_{xx} + 2u_{yy}).$$

2.1. **Error estimates for the heat equation.** This section is classical, see [23] or [25, p18] for example. Let $u(x,t)$ be the exact solution of the heat equation

$$u_t(x,t) = u_{xx}(x,t), \qquad \text{for } (x,t) \in \mathbb{R} \times (0,T) \tag{5}$$

with the initial values $u(x,0) = h(x)$. Divide the real line into intervals of length ϵ and the time interval into intervals of length ρ and use the standard notation $u_j^n = u(j\epsilon, n\rho)$. Then by standard Taylor series approximations,

$$(u_{xx})_j^n = \frac{1}{\epsilon^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + O(\epsilon^2), \tag{6}$$

$$(u_t)_j^n = \frac{1}{\rho}\left(u_j^{n+1} - u_j^n\right) + O(\rho). \tag{7}$$

Substituting the last two equations into the PDE (5), gives

$$\frac{1}{\rho}\left(u_j^{n+1} - u_j^n\right) = \frac{1}{\epsilon^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + O(\epsilon^2 + \rho)$$

for $n \in \mathbb{N}, j \in \mathbb{Z}$, which simplifies to

$$u_j^{n+1} = (1-2\alpha)u_j^n + \alpha(u_{j-1}^n + u_{j+1}^n) + O(\rho\epsilon^2 + \rho^2), \tag{8}$$

where we have defined $\alpha \equiv \rho/\epsilon^2$.

Now let $U \equiv U^{\epsilon,\rho}$ (using the notation of capital letters for functions defined only on the grid points) be the solution of the finite difference scheme:

$$U_j^{n+1} = (1-2\alpha)U_j^n + \alpha U_{j-1}^n + \alpha U_{j+1}^n, \qquad n \in \mathbb{N}, j \in \mathbb{Z} \tag{9}$$

along with matching initial conditions $U_j^0 = u_j^0, j \in \mathbb{Z}$. Define $Z_j^n \equiv U_j^n - u_j^n$ to be the error at each grid point. Then by subtracting (8) from (9) we obtain

$$Z_j^{n+1} = (1-2\alpha)Z_j^n + \alpha Z_{j-1}^n + \alpha Z_{j+1}^n + O(\rho\,\epsilon^2 + \rho^2). \tag{10}$$

For stability we first require that the coefficients of the scheme be non-negative,

$$\alpha \le \frac{1}{2}, \quad \text{equivalently,} \quad \rho \le \frac{\epsilon^2}{2}, \tag{11}$$

which is the well-known CFL condition [6]. Using this last requirement, we can conclude

$$|(1-2\alpha)Z_j^n + \alpha Z_{j-1}^n + \alpha Z_{j+1}^n| \le \max\{|Z_j^n|, |Z_{j-1}^n|, |Z_{j+1}^n|\},$$

which can be expressed as

the mapping (9) *is a contraction in the maximum norm, provided* (11) *holds.*

Using this last result in (10), we obtain

$$|Z_j^{n+1}| \le \max_j |Z_j^n| + O(\rho\epsilon^2 + \rho^2),$$

Since the initial error is zero, we can induct and conclude $|Z_j^n| \le n\rho O(\epsilon^2)$. We thus obtain the error bound

$$|u_j^n - (U^\epsilon)_j^n| \le n\rho O(\epsilon^2) \le tO(\epsilon^2).$$

provided (11) holds. Note that $n\rho = t$. In the sequel, we will keep track of the truncation error, to obtain the bound (with no unknown constants) in the theorem.

2.2. **Error estimates for a prototypical nonlinear equation.** Next consider the one dimensional level set equation $u_t = |u_x|$, which grows level sets at unit speed in the normal direction. Use the upwind scheme,

$$(|u_x|)_j^n = \frac{1}{\epsilon}\max(u_{j+1} - u_j, u_{j-1} - u_j, 0) + O(\epsilon) \tag{12}$$

which can easily be verified by a Taylor series expansion. Also use the explicit Euler method (7) as before, to arrive at

$$\begin{aligned} u_j^{n+1} &= u_j^n + \frac{dt}{dx}\max(u_{j+1} - u_j, u_{j-1} - u_j, 0) + \rho\, O(\epsilon + \rho) \\ &= (1-\alpha)u_j^n + \alpha\max(u_{j+1}, u_{j-1}, u_j) + \rho\, O(\epsilon + \rho) \end{aligned}$$

where $\alpha \equiv \rho/\epsilon$. As before, let $U \equiv U^\epsilon$ be the solution of the finite difference scheme,

$$U_j^{n+1} = (1-\alpha)U_j^n + \alpha\max(U_{j+1}^n, U_{j-1}^n, U_j^n)$$

Again define the error at each grid point, $Z_j^n = U_j^n - u_j^n$, which satisfies

$$Z_j^{n+1} = (1-\alpha)Z_j^n + \alpha(\max(U_{j+1}^n, U_{j-1}^n, U_j^n) - \max(u_{j+1}^n, u_{j-1}^n, u_j^n)) + \rho\, O(\epsilon + \rho)$$

along with zero initial conditions.

From now on, assume

$$\alpha \le 1, \qquad \text{equivalently,} \qquad \rho \le \epsilon. \tag{13}$$

Use the easily verified fact

$$|\max_i(X_i) - \max_i(Y_i)| \le \max_i(|X_i - Y_i|)$$

to compute

$$\begin{aligned} |Z_j^{n+1}| &\le (1-\alpha)|Z_j^n| + \alpha\max(|Z_{j+1}|, |Z_{j-1}|, |Z_j|) + \rho\, O(\epsilon) \\ &\le \max(|Z_{j+1}|, |Z_{j-1}|, |Z_j|) + \rho O(\epsilon). \end{aligned}$$

Thus

$$\max_j |Z_j^{n+1}| \le \max_j |Z_j^n| + \rho O(\epsilon).$$

The remainder of the proof is similar to the proof for the heat equation. The final result is

$$|Z_j^n| \le n\rho O(\epsilon) \le tO(\epsilon).$$

provided (13) holds. Note that $n\rho = t$. In the sequel, we will keep track of the truncation error, to obtain the bound (with no unknown constants) in the theorem.

2.3. **Error estimates for a second order fully nonlinear equation.** Finally, consider the operator

$$F[u] = -\max(2u_{xx} + u_{yy}, u_{xx} + 2u_{yy}).$$

which represents the equation for a toy stochastic control problem.

The operator, which is non-differentiable, is fully nonlinear and uniformly elliptic. So solutions to (1) with F with smooth boundary data are regular.

Use centered finite differences for the u_{xx} and u_{yy} terms. Then we can obtain second order error estimates, assuming the regularity of solutions which is known to hold. The proof goes through as in the previous example: the key step is to obtain an equation for the error Z, but a similar estimate on the difference of maxima applies. The rest of the proof follows in a similar fashion. The details can be worked out as an exercise.

3. Elliptic Finite Difference Schemes

3.1. **Definition of the schemes.** Let G^ϵ be a suitable finite difference grid on the domain Ω. Let the grid points be indexed by x_i, $i = 1, \dots, N$. For a function $u(x,t)$ defined on $\Omega \times [0,T]$, write $u_i^n = u(x_i, n\rho)$. For a given grid point i, let $i' = i_1, \dots, i_k$ be the list of neighboring grid points. A grid function is a vector $U = (U_1, \dots, U_N)$ of values at the grid points, and a finite difference scheme is a nonlinear function which maps grid functions to grid functions. (A solution of the scheme is a grid function which satisfies $F^\epsilon[U] = 0$, the zero grid function). Write the scheme F^ϵ at the grid point i as

$$F^i[U] \equiv F^i\left(U_i, U_i - U_{i_1}, \dots, U_i - U_{i_k}\right) \equiv F^i(U_i, U_i - U_{i'}),$$

where $U_i - U_{i'}$ is shorthand for the same expression repeated for each of the neighbors. (When the context is clear, we will drop the superscript ϵ from the scheme and the grid functions).

Definition. The nonlinear scheme F^ϵ is *elliptic* if each component F^i is nondecreasing in each variable, i.e.

$$X \le Y \text{ imples } F^i(X) \le F^i(Y) \tag{14}$$

Remark. This form emphasizes the fact that the terms are first order finite differences in the direction of the grid. So for example the scheme u_{xx}, (6) has this form, as does the scheme for $|u_x|$, (12).

3.2. **Interpreting the definition.** The nonlinear elliptic structure condition on schemes is the discrete version of the same conditions for PDEs. In both cases, the local structure condition implies that the solution operator is a contraction, as we will demonstrate next.

Lemma 2. The function $F(X, p, r, x)$ is degenerate elliptic if and only if

$$F[u](x) \ge F[v](x), \qquad \text{whenever } x \text{ is a nonnegative local maximum of } u - v, \tag{15}$$

for twice differentiable functions u, v.

Proof. By assumption,

$$u(x) \ge v(x), \quad Dv(x) = Du(x), \quad D^2u(x) \le D^2v(x)$$

Then using the definition of degenerate ellipticity 2

$$F(x, u, Du, D^2u) = F(x, u, Dv, D^2u) \ge F(x, v, Dv, D^2u) \ge F(x, v, Dv, D^2v). \quad \square$$

Lemma 3. The scheme F is elliptic if and only if

$$F^i[U] \ge F^i[V], \qquad \text{whenever } i \text{ is a nonnegative local maximum of } U - V. \tag{16}$$

Proof. Let i be an index for which

$$U_i - V_i = \max_{j \in i'} \{U_j - V_j\} \ge 0,$$

so that $U_i - U_j \ge V_i - V_j$, for $j \in i'$. Then compute

$$F^i[U] = F^i\left(U_i, U_i - U_{i'}\right) \ge F^i\left(V_i, U_i - U_{i'}\right) \ge F^i\left(V_i, V_i - V_{i'}\right) = F^i[V].$$

where we have used (14). $\square$

3.3. **The method of lines.** When we discretize (PDE) in space, using a finite difference scheme, the result is the method of lines, a system of ODEs.

$$\frac{d}{dt}U(t) + F^\epsilon[U(t)] = 0, \quad t > 0, U \in \mathbb{R}^N \tag{ODE}$$

where $F^\epsilon[U]$ incorporates the boundary conditions as well. Recall that if F^ϵ is Lipschitz continuous, then there exist unique solutions to (ODE).

3.4. **The forward Euler method.** When we also discretize time, using the forward Euler method (7), in (ODE), we get the following the explicit method

$$U^{n+1} = U^n - \rho F^\epsilon[U^n]. \tag{17}$$

The method consists of iteratively applying the explicit map, which we record below.

Definition (The explicit Euler map). For $\rho > 0$, define the map S_ρ which takes grid functions to grid functions, by

$$S_\rho(U) = U - \rho F[U]. \tag{18}$$

Definition (Nonlinear CFL condition). Let $F^\epsilon[U]$ be an elliptic scheme, and suppose that it is Lipschitz continuous with constant K^ϵ. The nonlinear Courant-Friedrichs-Lewy condition [6] for the Euler map S_ρ is

$$\rho \le \frac{1}{K^\epsilon} \tag{CFL}$$

Remark. We can also consider locally Lipschitz continuous schemes, in which case $K^\epsilon = K^\epsilon(U)$. In some cases we still get global existence of solutions and $K^\epsilon(U)$ does not decrease in time. For example, this is the case for the scheme for the equation $u_t = u_x^2$.

4. Contraction properties

We now establish the contraction properties. These result from the local structure condition that the operators or schemes be elliptic.

Lemma 4. If the differential operator $F[u]$ is elliptic, then the solution operator of (PDE) is a contraction in the maximum norm. In other words,

$$N(t) = \max_{x\in\Omega} |u(x,t) - v(x,t)| \quad \text{is a decreasing function of time.}$$

whenever $u(x,t), v(x,t)$ are solutions of (PDE).

Proof. Let $x^+(t) \in \arg\max_{x\in\Omega} u(x,t) - v(x,t)$. Without loss of generality, we can assume that $N(t)$ is achieved at $x^+(t)$. Compute

$$\begin{aligned}
&\frac{d}{dt}u(x^+(t),t) - v(x^+(t),t) \\
&\qquad = u_t(x^+(t),t) - v_t(x^+(t),t) + (Du(x^+(t),t) - Dv(x^+(t),t))\frac{d}{dt}x^+(t) \\
&\qquad\qquad = -F(D^2u, Du, u, x) + F(D^2v, Dv, v, x) \mid_{x=x^+(t)} \le 0.
\end{aligned}$$

where we have used the fact that $Du = Dv$ at a local max of $u - v$, in the first step, and (15) in the second step. (The argument can be made valid for viscosity solutions, by replacing u, v in the calculation by smooth test functions touching above or below, as necessary). □

Lemma 5. If $F^\epsilon[U]$ is an elliptic scheme, then the solution operator of (ODE) is a contraction in the maximum norm. In other words

$$N(t) = \max_j |U_j(t) - V_j(t)| \text{ is a decreasing function of time.}$$

whenever $U(t), V(t)$ are solutions of (ODE).

Proof. Let $k \in \arg\max_j U_j(t) - V_j(t)$. Without loss of generality, we can assume that $N(t)$ is achieved at k. Then since the scheme is elliptic, (16) holds, which gives

$$\frac{d}{dt}\left(U_k(t) - V_k(t)\right) = F^k(U) - F^k(V) \le 0. \qquad \square$$

Lemma 6. Let F be a Lipschitz continuous, degenerate elliptic scheme. Then the Euler map (18) is a contraction in $\mathbb{R}^N$ equipped with the maximum norm, provided (CFL) holds.

Proof. Refer to [21]. □

5. Proofs of Error Estimates

5.1. Approximations in the continuous setting.

Theorem (Error estimate). *Let u be the viscosity solution of* (PDE)*, and let u^ϵ be the solution of a family of approximate equations, $u^\epsilon_t + F^\epsilon[u^\epsilon] = 0$, where F^ϵ are elliptic operators, subject to initial condition $u(x,0)$. Then*

$$\|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^\infty} \le t\|\delta^\epsilon[u]\|_{L^\infty(x,t)}.$$

Proof. Compute

$$\begin{aligned}\frac{d}{dt}(u - u^\epsilon) &= F^\epsilon[u^\epsilon] - F[u]\\ &= F^\epsilon[u^\epsilon] + F^\epsilon[u] - F^\epsilon[u] - F[u]\\ &= F^\epsilon[u^\epsilon]_j - F^\epsilon[u] + \delta^\epsilon_j\end{aligned}$$

where we have used the equation satisfied by u^ϵ, (PDE), and the definition of the truncation error (3). As in Lemma 4, the first two terms have a favorable sign at a local maximum, or minimum, of $u^\epsilon - u$. As a result,

$$\frac{d}{dx}\|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{L^\infty} \le \|\delta^\epsilon\|_{L^\infty(\cdot,t)}$$

and the result follows. □

5.2. The method of lines.

Theorem (Error estimates for the method of lines.)**.** *Let $u(x,t)$ be a classical solution of* (PDE)*. Let F^ϵ be an elliptic scheme for* (1) *and let $U^\epsilon(t)$ be the solution of the method of lines* (ODE)*. Then the scheme converges and the global error is bounded by (t times) the truncation error for the solution,*

$$\max_j |U^\epsilon_j(t) - u_j(t)| \le t \max_{0\le s\le t} \max_j |\delta^\epsilon[u(s)]_j|.$$

Proof. Compute

$$\begin{aligned}\frac{d}{dt}(u_j(t) - U_j^\epsilon(t)) &= F^\epsilon[U^\epsilon]_j - F[u]_j \\ &= F^\epsilon[U^\epsilon]_j + F^\epsilon[u]_j - F^\epsilon[u]_j - F[u]_j \\ &= F^\epsilon[U^\epsilon]_j - F^\epsilon[u] + \delta_j^\epsilon\end{aligned}$$

where we have used (ODE), (PDE), and the definition of the truncation error (3).

Having set up the equation above, we reiterate the proof that the solution mapping is a contraction, carrying the inhomogeneous term to arrive at the conclusion. Let $N(t) = \max_j |U_j(t) - u_j(t)|$. Assume that $N(t) = \max_j U_j(t) - u_j(t)$. (A similar computation will work if the sign is reversed). Compute for any $k(t) \in \arg\max_j U_j(t) - u_j(t)$,

$$\frac{d}{dt}N(t) = F^\epsilon[U]_k - F^\epsilon[u]_k + \delta_k^\epsilon(t) \leq \delta_k^\epsilon(t)$$

where we have used the fact that F^ϵ is elliptic (16). Since $N(0) = 0$, we have $N(t) \leq t|\delta^\epsilon|_\infty$ as desired. □

5.3. **Forward Euler.**

Theorem (Error estimates for Forward Euler)**.** *Let $u(x,t)$ be a classical solution of* (PDE)*. Let F^ϵ be an elliptic scheme for* (1) *and let U^ϵ be the solution of the forward Euler method* (17)*. Suppose* (CFL) *holds. Then the scheme converges and the global error is bounded by ($t = n\rho$ times) the truncation error for the solution,*

$$\max_j |(U^\epsilon)_j^n - u_j^n| \leq n\rho \max_{m\leq n} \max_j |\delta^{\epsilon,\rho}[u]_j^m|.$$

Proof. We first show that

$$u^{n+1} = S_\rho(u^n) + \rho\, \delta^{\epsilon,\rho}[u]^n$$

This is simply a matter of collecting the definitions of the truncation errors and plugging them into the equations,

$$\begin{aligned}u^{n+1} &= u^n + \rho\,(\delta^\rho[u]^n + (u_t)^n) && \text{from (4)} \\ &= u^n + \rho\,(\delta^\rho[u]^n - F[u]^n) && \text{from (PDE)} \\ &= u^n + \rho\,(\delta^\rho[u]^n - F^\epsilon[u]^n + \delta^\epsilon[u]^n) && \text{by (3)} \\ &= S_\rho(u^n) + \rho\,(\delta^\rho[u]^n + \delta^\epsilon[u]^n) && \text{by (18)} \\ &= S_\rho(u^n) + \rho\,\delta^{\epsilon,\rho}[u]^n && \text{by (4)}\end{aligned}$$

From (17), dropping the ϵ superscript,

$$U^{n+1} = S_\rho(U^n)$$

So subtracting

$$U^{n+1} - u^{n+1} = S_\rho(U^n) - S_\rho(u^n) - \rho\,\delta^{\epsilon,\rho}[u]^n$$

By Lemma 6, the Euler map is a contraction, thus

$$\|U^{n+1} - u^{n+1}\|_\infty \leq \|U^n - u^n\|_\infty + \rho\|\delta^{\epsilon,\rho}[u]^n\|_\infty$$

Since $U^0 = u^0$, the result follows by induction. □

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