

THE DIRICHLET PROBLEM FOR THE CONVEX ENVELOPE

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ABSTRACT. This work studies the Dirichlet problem for the Convex Envelope. While the convex envelope is a natural object of study, recent work has linked this object with nonlinear elliptic Partial Differential equations. We further explore this connection by applying nonlinear PDE methods to the convex envelope. We show that the convex envelope provides a subsolution to a class of nonlinear elliptic PDEs. We reformulate the convex envelope as the value function of a stochastic control problem. Finally, the main result is an interior regularity estimate for the solution, which is a viscosity solution of a degenerate elliptic Partial Differential Equation (PDE).

1. INTRODUCTION

In this article we study a problem which is at the interface of convex analysis and nonlinear elliptic Partial Differential Equations. The object of study is the Convex Envelope. In a previous work by one of the authors [11], a Partial Differential Equation in the form of an obstacle problem for the convex envelope was obtained. In this article we further explore the connection between the convex envelope and nonlinear elliptic PDEs, by studying the Dirichlet Problem for the convex envelope. By this we mean the convex envelope of a data given on the boundary of a domain. The Dirichlet problem is a natural setting for regularity question for the underlying PDE. While there are simple counter-examples to regularity for this problem, we prove an optimal regularity result in two dimensions, given regular boundary data.

We also explain how the convex envelope of Dirichlet data is a natural subsolution of a class of degenerate elliptic DPEs, and give an interpretation of the solution as the value function of a stochastic control problem. The solution can be written as the supremum of supporting hyperplanes. While the solution may not always be differentiable (or even continuous up to the boundary) the PDE nevertheless is well-posed when we use the notion of viscosity solutions [6].

1.1. The convex envelope. The convex envelope of a given function $g(x)$ is a natural mathematical object which has been the subject of study for many years. It was recently observed [11] that the convex envelope is the solution of a partial differential equation. The convex envelope of the function $g(x)$ is defined as the supremum of all convex functions which are majorized by g ,

$$u(x) = \sup\{v(x) \mid v \text{ convex, } v(y) \leq g(y) \text{ for all } y \in \mathbb{R}^n\}.$$

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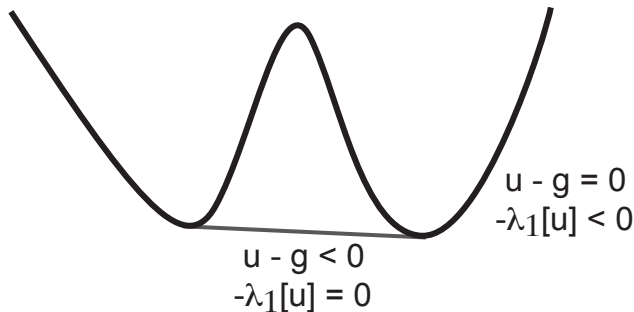


FIGURE 1. Illustration of the equation

The equation for the convex envelope, u , of the function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, is

$$(1) \quad \max \{u(x) - g(x), -\lambda_1[u](x)\} = 0$$

here $\lambda_1[u](x)$ is the smallest eigenvalue of the Hessian $D^2u(x)$. See Figure 1.

The equation (1) is a combination of a fully nonlinear second order PDE, $\lambda_1[u](x)$, and an obstacle term. In one dimension, this equation is the classical obstacle problem. But the differential operator is fully nonlinear in dimensions two and higher, and the problem of computing the convex envelope is correspondingly more challenging.

1.2. Related results. The possibility of an equation for the convex envelope was suggested by [9]. A computational method for computing the convex envelope which used a related equation was performed in [15]. Methods for enforcing convexity constraints in variational problems have been studied as well [4]. For more references on computational work, see [12].

For the convex envelope problem, when the envelope function $g(x)$ is $C^{1,\alpha}$, the convex envelope is as well [10, 9]. See also [1]. The analysis for the envelope problem is somewhat easier, since supporting hyperplanes will touch the function g at some point where it is differentiable with matching derivatives. For the Dirichlet problem, the analysis is more difficult, and hence the restriction of our result to two dimensions.

2. THE DIRICHLET PROBLEM FOR THE CONVEX ENVELOPE

In this article we consider the Dirichlet problem for the fully nonlinear degenerate elliptic equation

$$(PDE) \quad -\lambda_1[u](x) = 0,$$

for x in a bounded domain Ω in \mathbb{R}^n , with $u(x) = g(x)$ for $x \in \partial\Omega$. This equation describes a convex function which is nowhere strictly convex.

We refer to the solution of (PDE) as the convex envelope of the boundary data $g(x)$. In fact, we define the convex envelope of the boundary data $g(x)$ to be

$$(CE) \quad u(x) = \sup\{v(x) \mid v \text{ convex, } v(y) \leq g(y) \text{ for all } y \in \partial\Omega\}$$

and below we show that these two definitions coincide. Analogous results for concave functions can be made by replacing λ_1 with λ_n , the largest eigenvalue.

The purpose of this article is to deepen the analysis of the equation, which was derived in [11] and studied numerically in [12]. We investigate regularity properties of solutions to the equation, in the setting of Dirichlet data.

Solutions of (PDE) need not be differentiable; for example $u(x, y) = |x|$ is a solution. Neither must they be continuous up to the boundary. For example, when $f(x, y) = x^2 - y^2$, the solution of (PDE) is $u(x, y) = x^2 - 1$, when Ω is a square in the plane, centered at the origin.

2.1. Convexity. For basic definitions of convexity, see [3] or the appendix of [7]. For more on convex analysis see the textbooks [2][14]. The function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ and $0 \leq t \leq 1$

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y).$$

If $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then for each $x_0 \in \mathbb{R}^n$ there exists a *supporting hyperplane* to u at x_0 . In other words, there exists an affine function $P(x) \equiv u(x_0) + p \cdot (x - x_0)$ such that

$$\begin{cases} u(x) \geq P(x) & \text{for all } x \in \mathbb{R}^n \\ u(x_0) = P(x_0) \end{cases}$$

If u is differentiable at x , then $\nabla u(x_0) = \nabla P(x_0)$ and P is unique; if not, there may be more than one supporting hyperplane.

For twice-differentiable functions, convexity can be characterized by the local condition that the Hessian of the function be nonnegative definite,

$$D^2u(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \quad \text{if and only if} \quad u \text{ is convex.}$$

Note the first condition is equivalent to $\lambda_1[u](x) \geq 0$. The characterization is valid even for continuous functions, when the equation is interpreted in the viscosity sense, see below.

2.2. Viscosity Solutions. The theory of viscosity solutions is a powerful tool for proving existence, uniqueness and stability results for fully nonlinear elliptic equations. The standard reference is the User's Guide [6]. A readable introduction is the Primer, by Crandall [5]. The theory applies to scalar equations of the form $F[u] \equiv F(D^2u(x), Du(x), u(x), x)$, which are nonlinear and elliptic, i.e. nondecreasing in the first argument. Solutions are *stable* in the sense that if $F^\epsilon \rightarrow F$, the corresponding solutions $u^\epsilon \rightarrow u$, uniformly on compact subsets. Uniqueness is a consequence of the *comparison principle*: if $F[u] \geq F[v]$ in Ω , with $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω . Viscosity solutions can be constructed using Perron's method,

$$u(x) = \sup\{\phi(x) \mid \phi \text{ is a subsolution of } F[\phi] = 0\}.$$

This last construction will coincide in our case with the definition of the convex envelope (CE). The definition of viscosity solutions for (PDE) follows.

Definition 2.1. *The upper semicontinuous function u is a viscosity subsolution of (PDE) if*

$$u(x) \leq g(x), \quad \text{for } x \in \partial\Omega$$

and if for every twice-differentiable function $\phi(x)$,

$$(2) \quad -\lambda_1[\phi](x) \leq 0, \quad \text{when } x \text{ is a local max of } u - \phi.$$

The lower semicontinuous function u is a viscosity supersolution of (PDE) if

$$u(x) \geq g(x), \quad \text{for } x \in \partial\Omega$$

and if for every twice-differentiable function $\phi(x)$,

$$(3) \quad -\lambda_1[\phi](x) \geq 0, \quad \text{when } x \text{ is a local min of } u - \phi.$$

The function u is a viscosity solution of (PDE) if it is both a subsolution and a supersolution.

2.3. Derivation of the equation. The following theorem was proved in [12]

Theorem 2.2. *The continuous function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if it is a viscosity subsolution of (PDE).*

An immediate consequence of the previous theorem and Perron's method is the characterization which follows.

Theorem 2.3. *The convex envelope of the boundary data is a viscosity solution of (PDE).*

Proof. Compare the Perron formula

$$u(x) = \sup\{v(x) \mid v(x) \text{ is a subsolution of (PDE)}\}.$$

and the definition (CE). Use the definition of viscosity solutions and Theorem 2.2 so see that the supremum is over the same set of functions. \square

2.4. Estimators for Elliptic Partial Differential Equations. We next discuss how the convex envelope provides a subsolution for a class of elliptic PDEs. Consider the nonlinear elliptic PDE

$$(4) \quad F(D^2u) = f,$$

in the domain Ω along with Dirichlet boundary conditions $u = g$ on $\partial\Omega$.

Suppose that F is nonlinear elliptic (i.e. a monotone function of the Hessian D^2u) so that viscosity solutions of the equation are well-defined. Suppose also that we have the one-sided condition

$$F(D^2u) \geq \alpha\lambda_1[u]$$

Let u^c be the solution of

$$\alpha\lambda_1[D^2u^c] = f$$

in Ω along with $u = g$ on $\partial\Omega$. Then, since $F(D^2u^c) \geq \lambda_1[D^2u^c] = f$, by the Comparison Principle, u^c is a subsolution of (4), so

$$u^c \leq u.$$

2.5. Stochastic Control Interpretation. In [11] the convex envelope was reinterpreted as the value function of a stochastic control problem. For the Dirichlet problem (PDE), an even simpler interpretation is available. Our derivation is formal but can be made rigorous; we refer to [8] for a rigorous derivation of related equations. For readers not familiar with stochastic control problems, an introduction to optimal control and viscosity solutions of Hamilton-Jacobi equations can be found in [7].

Consider the controlled diffusion

$$(5) \quad \begin{cases} d\mathbf{x}(t) = \sqrt{2}\theta(t)dw(t), \\ \mathbf{x}(0) = x_0. \end{cases}$$

where w is a one-dimensional Brownian motion, and the control, $\theta(\cdot)$, is a mapping into unit vectors in \mathbb{R}^n . The process stops when it reaches the boundary of the

domain, at which point we incur a cost $g(x_\tau)$, where τ is the time when it reaches the boundary. The objective is to minimize the expected cost

$$J(x, \theta(\cdot)) \equiv E^x[g(x_\tau)]$$

over the choice of control $\theta(\cdot)$. The value function is

$$u(x) = \min_{\theta(\cdot)} J(x, \theta(\cdot)).$$

We formally apply the DPP. One strategy is to fix $\theta(\cdot) = \theta$ to be constant, and let the diffusion proceed for time t , thereafter following the optimal strategy. This strategy costs $E^{x_0, \theta}[u(\mathbf{x}(s))] \equiv E^{x_0}[u(\mathbf{x}(s)) \mid \theta(\cdot) = \theta]$. Minimizing over θ gives

$$u(x_0) = \min_{\theta} E^{x_0, \theta}[u(\mathbf{x}(t))] + o(t).$$

Using the definition of infinitesimal generator corresponding to the diffusion (5), with θ fixed, (see e.g. [13]), gives

$$(6) \quad \lim_{t \rightarrow 0} \frac{E^{x, \theta}[u(\mathbf{x}(t))] - u(x)}{t} = \theta^T D^2 u(x) \theta.$$

Using (6) in the preceding equation and taking the limit $t \rightarrow 0$ yields

$$- \inf_{|\theta|=1} \theta^T D^2 u(x) \theta = 0,$$

along with $u = g$ on the boundary. Finally, using the Courant-Hilbert characterization of the eigenvalues recovers (PDE).

3. C^1 REGULARITY IN TWO DIMENSIONS

Next we prove an optimal regularity result in two dimensions.

We begin with an example to show that fact that the interior regularity result is optimal in the sense that it cannot be continued up to the boundary.

Example 3.1. *Consider the unit ball in two dimensions and the function*

$$u(x, y) = -(1 + x)^{1-\epsilon}$$

for $\epsilon > 0$ small. This function is convex. Writing

$$u(x, y) = f(\theta) = -(1 + \cos \theta)^{1-\epsilon}$$

we see that u is $C^{1,1-2\epsilon}$ on the unit circle (since near $\theta = \pm\pi$ the function behaves like $\theta^{2-2\epsilon}$).

$$f'(\theta) = (2 - \epsilon) \sin \theta (1 + \cos \theta)^{1-\epsilon}$$

which has a singularity about $\theta = \pm\pi$ like $\theta^{2-2\epsilon}$]

However the function is not $C^{1,\alpha}$ up to the boundary. for any $\alpha > 0$ because of the singularity at $(-1, 0)$.

For a $C^{1,\alpha}$ function $g : S^1 \rightarrow \mathbb{R}$, let us consider the two dimensional problem:

$$\begin{aligned} u &= g && \text{on } \partial B_1 = S^1 \\ u(x) &= \max \{L(x) : L(x) = A \cdot x + b, L \leq g \text{ on } \partial B_1\} \end{aligned}$$

Theorem 3.2. *If g is $C^{1,\alpha}$ in S^1 , then u is $C^{1,\alpha}$ in $B_{1/2}$.*

Preliminaries We wish to prove:

- (i) $Du(x)$ exists and is continuous for all $x \in B_{1/2}$
- (ii) Du is Hölder for some α , for $x \in B_{1/2}$, i.e.

$$|Du(x_1) - Du(x_2)| \leq C|x_1 - x_2|^\alpha$$

It is enough to show the following estimate.

Given x_1 and x_2 in $B_{1/2}$ and the corresponding supporting planes $L_1(x) = A_1 \cdot x + b_1$ and $L_2(x) = A_2 \cdot x + b_2$.

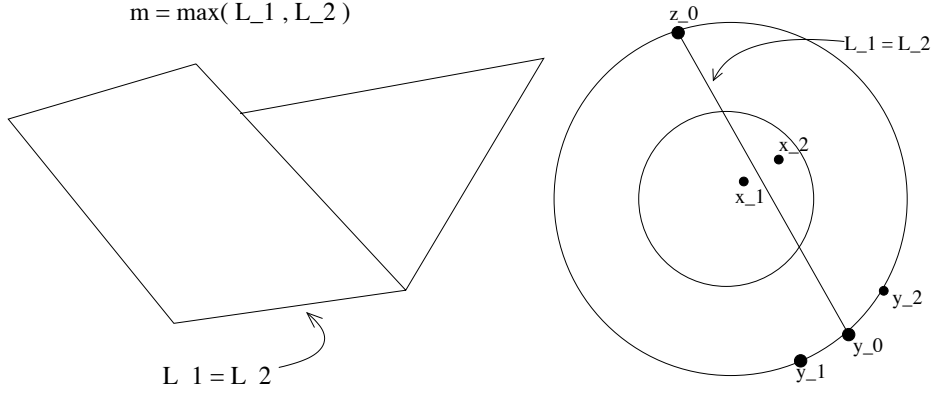
$$(7) \quad |A_1 - A_2| \leq C|x_1 - x_2|^\alpha$$

for some constant C .

Proof. By subtracting an appropriate plane to u , we can assume $A_1 + A_2 = 0$. Let us also assume that $A_1 \neq A_2$, otherwise there would be nothing to prove.

Let $M(x) = \max(L_1(x), L_2(x))$. Then $u \geq M$ in \bar{B}_1 . Consider the line where $L_1 = L_2$, this line separates x_1 and x_2 and meets ∂B_1 at two points that we will call y_0 and z_0 . Since the line separates x_1 and x_2 , it crosses $B_{1/2}$ somewhere, and therefore it hits the boundary ∂B_1 with at least a minimum angle $\theta_0 > 0$. Notice that this line is orthogonal to $(A_1 - A_2)$.

FIGURE 2. The function m and the points



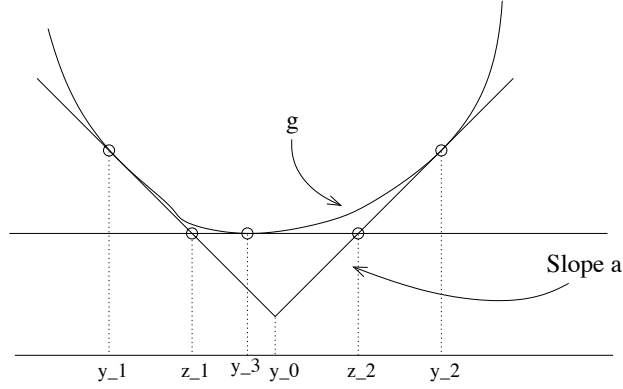
The function u is always above m and touches at some points. It cannot touch at y_0 or z_0 since that would imply that g is not C^1 at those points.

Now, let y_1 and y_2 be the closest points in ∂B_1 to y_0 such that $u(y_1) = L_1(y_1)$ and $u(y_2) = L_2(y_2)$. At each y_i , the slope of g coincides with A_i in the direction tangent to the boundary ∂B_1 . That means that $|y_1 - y_2| \geq c|A_1 - A_2|^{1/\alpha}$ for some small constant c , since $g \in C^{1,\alpha}$ and the tangent space of ∂B_1 at y_0 is at least an angle θ_0 from being orthogonal to $A_1 - A_2$.

Now we will focus for a while on the piece of the boundary $\partial B_1 = S^1$ in between y_1 and y_2 . We will make the pictures and computations as if it was straight. That means that all the computations have a quadratic error, that is ok for us as long as we want to prove $C^{1,\alpha}$ with $\alpha < 1$. Maybe to prove a $C^{1,1}$ estimate we would have to be more careful.

So, now we have two points y_1 and y_2 where the $C^{1,\alpha}$ function g touches the max of two lines from above. These two lines have slopes $-a$ and a (recall we assumed $A_1 = -A_2$), and $a \geq c|A_1|$ for some constant c .

FIGURE 3. At the boundary



Let y_3 be the point (strictly) in between y_1 and y_2 where g attains its minimum. So that at y_3 there is a horizontal supporting plane for g . Without loss of generality we can assume that y_3 is in between y_1 and y_0 . Since g is $C^{1,\alpha}$, we know that $|g'(x) - g'(y)| \leq C_0|x - y|^\alpha$, which implies

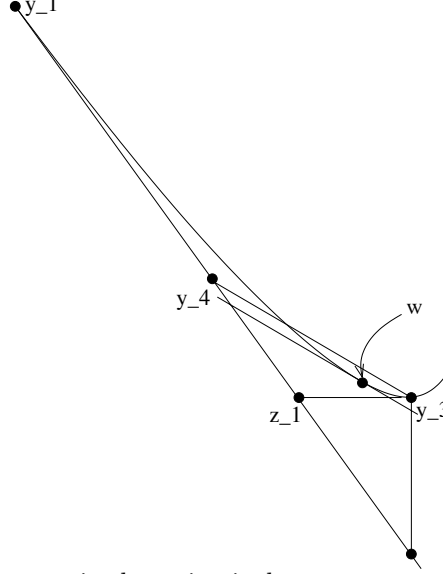
$$(8) \quad |y_3 - y_1|^\alpha \geq C_0^{-1}a$$

Let $z_1 \in (y_1, y_0)$ and $z_2 \in (y_0, y_2)$ be the points such that $m(z_1) = m(z_2) = g(y_3)$. Now we want to show that $|z_1 - z_2| \geq ca^{1/\alpha}$ for some small constant c depending only on the $C^{1,\alpha}$ estimate for g .

As we said, we know $|y_3 - y_1|^\alpha \geq C_0^{-1}a$. Suppose $|z_1 - y_3| < c_1a^{1/\alpha}$ for a small c_1 , then $g(y_3) - m(y_3) < c_1a^{1+1/\alpha}$.

Since $m \leq g$, for $y_4 = y_3 - (C_0^{-1}a/3)^{1/\alpha}$ there is a point $w \in (y_4, y_3)$ such that

$$\begin{aligned} g'(w) &= -a + \frac{g(y_3) - m(y_3)}{y_3 - y_4} \\ &\leq -a + \frac{c_1a^{1+1/\alpha}}{(C_0^{-1}a/3)^{1/\alpha}} \\ &\leq a(c_1C_0^{1/\alpha}C - 1) \end{aligned}$$



w is the point in between y_3 and y_4 that has as a supporting plane parallel to the line that goes through y_3 and y_4

At the same time, we know that since $g \in C^{1,\alpha}$, $|g'(w) - g'(y_3)| \leq C_0|w - y_3|^\alpha$. Putting it all together

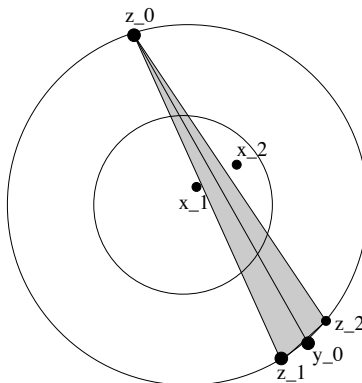
$$\begin{aligned} (1 - c_1 C_0^{1/\alpha} C)a &\leq -g'(w) = |g'(w) - g'(y_3)| \\ &\leq C_0|w - y_3|^\alpha \\ &\leq C_0 C_0^{-1} a/3 = a/3 \end{aligned}$$

But if we choose c_1 small enough so that $(1 - c_1 C_0^{1/\alpha} C) > 1/3$, we arrive to a contradiction. Therefore we conclude that $|z_1 - y_3| \geq c_1 a^{1/\alpha}$, and thus also $|z_1 - z_2| \geq c_1 a^{1/\alpha}$

Now we look at the whole disk B_1 again. Let L be the affine function that agrees with g at z_0 , z_1 and z_2 . Then L is below m (and thus also u) everywhere on ∂B_1 except between z_1 and z_2 . But from construction of z_1 and z_2 , L is below $g = u$ in the piece of ∂B_1 in between z_1 and z_2 . Therefore L has to be below u everywhere in B_1 . This implies that at the points $x \in B_1$ where $L(x) > m(x)$, also $u(x) > m(x)$, and thus neither x_1 nor x_2 belongs to $\{L > m\}$. The set $\{L > m\}$ is a triangle that has width greater than $ca^{1/\alpha}$ inside $B_{1/2}$. The line $L_1 = L_2$ is inside $\{L > m\}$, and thus this set separates x_1 from x_2 . Then $|x_1 - x_2| \geq ca^{1/\alpha} \geq c|A_1 - A_2|^{1/\alpha}$, which finishes the proof. \square

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FIGURE 4. The strip separates x_1 from x_2 

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