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CONVERGENT DIFFERENCE SCHEMES FOR DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS: HAMILTON–JACOBI EQUATIONS AND FREE BOUNDARY PROBLEMS*

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Abstract. Convergent numerical schemes for degenerate elliptic partial differential equations are constructed and implemented. Simple conditions are identified which ensure that nonlinear finite difference schemes are monotone and nonexpansive in the maximum norm. Explicit schemes endowed with an explicit CFL condition are built for time-dependent equations and are used to solve stationary equations iteratively. Explicit and implicit formulations of monotonicity for first- and second-order equations are unified. Bounds on orders of accuracy are established. An example of a scheme which is stable, but nonmonotone and nonconvergent, is presented. Schemes for Hamilton–Jacobi equations, obstacle problems, one-phase free boundary problems, and stochastic games are built and computational results are presented.

Key words. finite difference schemes, partial differential equations, monotone schemes, viscosity solution, Hamilton–Jabobi equation, free boundary problems

AMS subject classifications. 65N06, 65N12, 65M06, 65M12, 35B50, 35J60, 35R35, 35K65, 49L25

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1. Introduction.

We devise practical techniques for building convergent numerical schemes for a class of nonlinear partial differential equations. This is the class of *degenerate elliptic* (in the sense of Crandall, Ishii, and Lions [11]) partial differential equations, for which unique viscosity solutions exist. The class includes Hamilton–Jacobi equations, which are nonlinear first-order equations; elliptic equations which may be degenerate; and fully nonlinear second-order equations. It also includes free boundary problems and the equation for the value function from control and game theory.

The approximation theory developed by Barles and Souganidis [5] provides the following criteria for the convergence of approximation schemes: monotone, consistent, and stable schemes converge to the unique viscosity solution of a degenerate elliptic equation. Despite the clear requirements of the theory, building monotone schemes remains a challenge for many important equations. The finite difference method is the natural method for building monotone schemes, but conditions which ensure monotonicity are different for first- and second-order equations, and for explicit and implicit schemes.

For linear elliptic equations, Motzkin and Wasow [28] introduced the notion that a scheme is of "positive type." These linear schemes respect the discrete maximum principle. This formulation of monotonicity was further studied in [7] and later generalized to nonlinear elliptic equations by Kuo and Trudinger [23, 24, 25, 26]. Related notions for linear parabolic equations have been studied [1, 21, 35].

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For conservation laws, monotonicity[1] is associated with entropy solutions. In this setting, monotone schemes are contractions in ℓ^1 [13] and are at most first-order accurate [18]. Higher-order accuracy is achieved by essentially nonoscillatory (ENO) or weighted ENO (WENO) schemes [19], which are not monotone: they selectively use high-order (nonmonotone) interpolation in smooth regions of the solution and monotone schemes in nonsmooth regions.

For Hamilton–Jacobi equations, monotonicity is necessary for convergence. Early numerical papers studied explicit schemes for time-dependent equations on uniform grids [10, 33]. A number of methods have since been developed, which include fast marching [32], fast sweeping [34], semi-Lagrangian [16], central [27], and ENO [31].

For certain second-order equations, which include fully nonlinear equations and degenerate linear elliptic equations, monotonicity is necessary for convergence. An early result of Motzkin and Wasow illustrated difficulties associated with monotone schemes: even for linear elliptic equations, in general it is not possible to build monotone schemes using a narrow stencil [28]. Very large stencil schemes for quasi-linear equations were studied by Crandall and Lions [12]. Wide stencil schemes have been used to solve certain degenerate second-order equations [29, 30].

We identify a class of nonlinear finite difference schemes which we call *degenerate elliptic*. Degenerate elliptic schemes are monotone. They also enjoy a strong form of stability: they are nonexpansive in the maximum norm. The class includes implicit or explicit schemes for first- or second-order equations on structured or unstructured grids.

Degenerate elliptic schemes are built in simple ways from building blocks consisting of schemes for basic equations. They begin with an implicit scheme for the spatial part of the equation. This scheme may then be extended to an explicit scheme for the time-dependent equation, or equivalently, to an iterative method for the stationary equation. The explicit scheme is endowed with a nonlinear CFL condition which is easily calculated.

A guiding principle of this work is that, in order to build effective numerical schemes, it is essential to have a thorough understanding of the underlying equations. In this manner, schemes can be built that inherit desirable properties from the equations.

THEOREM 1. *The solution operator of a degenerate elliptic partial differential equation is monotone and nonexpansive in the maximum norm, provided mild analytic conditions hold so that it is well defined.*

THEOREM 2. *The solution operator of a degenerate elliptic finite difference scheme is monotone and nonexpansive in the maximum norm, provided mild analytic conditions hold so that it is well defined.*

THEOREM 3. *A scheme is monotone and nonexpansive in the ℓ^∞ norm if and only if it is degenerate elliptic.*

Remark 1. Monotonicity by itself does not ensure stability. For example, $u_j^{n+1} = 2u_j^n$ is unstable; examples with worse growth rates are easily constructed.

While degenerate ellipticity is stronger than monotonicity for abstract schemes, the condition occurs naturally for schemes built using the finite difference method; see section 2.3. For these schemes, the two conditions are equivalent.

Remark 2. Theorem 3 recalls a theorem from [14], which states that monotonicity is equivalent to nonexpansivity in ℓ^∞, for mappings which are invariant under

[1] Not to be confused with "monotonicity preserving," which means that increasing functions on the line remain increasing.

translation by a constant.

THEOREM 4. *The accuracy of a monotone finite difference scheme is at most first order for first-order equations and at most second order for second-order equations.*

Remark 3. Useful numerical solutions can be obtained with first- or second-order schemes. Despite the fact that singularities occur, the accuracy requirements are not as high in this setting are they are for conservation laws.

Contents. Section 1.1 demonstrates the equivalence of the explicit and implicit formulations of monotonicity. Section 1.2 contains basic examples to illustrate the definitions. Section 1.3 contains an example which shows monotonicity is necessary for convergence.

Section 2 is the bulk of the theory. Section 2.1 summarizes relevant aspects of degenerate elliptic equations. Section 2.2 defines the class of degenerate elliptic *equations*, followed by section 2.3, which defines the class of degenerate elliptic *schemes*. Section 2.4 provides the nonlinear CFL-type condition for explicit schemes. Section 2.5 establishes properties of the solution operator. Section 2.6 contains proofs of Theorems 1, 2, 3, and 4.

A technique is developed in section 3 to build schemes for complicated equations using building blocks consisting of schemes for simpler equations. This technique is used to build schemes for various equations, including Hamilton–Jacobi equations, obstacle problems, one-phase free boundary problems, and stochastic games. In section 4 computational results are presented.

1.1. Equivalent formulations of monotonicity. In its most general formulation, monotonicity means that the comparison principle holds. This global property was used in [5] to prove convergence of nondiscrete approximation schemes.

For the purpose of building schemes, it is useful to have an easily verified local condition which guarantees monotonicity. The condition comes in two forms. The explicit formulation, usually seen for time-dependent equations, is

$$u_i = H^i\left(u|_{j=N(i)}\right), \tag{1}$$

where $N(i)$ is the list of neighbors of u_i. For example, it appears as $u_i^{n+1} = H(u_{j-1}^n, u_j^n, u_{j+1}^n)$ in the case of three-point explicit schemes [10]. The explicit formulation (1) is monotone if H^i is a nondecreasing function of each variable. The implicit formulation, usually seen for stationary elliptic equations, is

$$F^i\left(u_i, u|_{j=N(i)}\right) = 0. \tag{2}$$

For example, linear schemes $\sum_{i=0}^n a_i u(x + idx)$ are monotone (of "positive type") if $a_0 \geq 0$ and $a_i \leq 0$ for $i \neq 0$ [28]. For nonlinear equations, schemes are monotone if F^i is nondecreasing in the first variable and nonincreasing in the remaining variables [24].

Remark 4. The two formulations are formally equivalent. To put the explicit form into an implicit form is trivial. To go from the implicit form to the explicit form, differentiate implicitly to obtain $D_{u_i} F^i du_i + \sum_{j=N(i)} D_{u_j} F^i du_j = 0$. Fixing all but neighbor u_k, we obtain $du_i/du_k = -D_{u_k} F^i / D_{u_i} F^i \geq 0$. Use the implicit function theorem to solve for u_i as a nondecreasing function of the neighbors.

1.2. Illustration of the definitions. Consider the standard centered difference scheme for $-u_{xx}$, $\left(u_i - u_{i-1} + u_i - u_{i+1}\right)/dx^2$. The scheme is degenerate elliptic, since, as in Definition 2, it is a nondecreasing function of the differences between the reference variable and its neighbors, $u_i - u_{i-1}$ and $u_i - u_{i+1}$. Solving for u_i gives

$u_i = \frac{1}{2}(u_{i+1} + u_{i-1})$. This puts the scheme in the explicit form of monotonicity, since the righthand side is a nondecreasing function of its arguments.

The implicit Euler scheme for the heat equation, $u_t - u_{xx} = 0$, is also degenerate elliptic,

$$\frac{1}{dt}(u_i^n - u_i^{n-1}) + \frac{1}{dx^2}\left(u_i^n - u_{i-1}^n + u_i^n - u_{i+1}^n\right) = 0.$$

On the other hand, the explicit Euler scheme,

$$u_i^n = (1 - 2dt/dx^2)u_i^{n-1} + (dt/dx^2)(u_{i-1}^{n-1} + u_{i+1}^{n-1}),$$

is monotone if and only if $0 \le dt \le dx^2/2$. In this case, the scheme is also degenerate elliptic,

$$\left(1 - 2dt/dx^2\right)\left(u_i^n - u_i^{n-1}\right) + dt/dx^2(u_i^n - u_{i-1}^{n-1} + u_i^n - u_{i+1}^{n-1}) = 0.$$

Remark 5. The restriction on the time step coincides with the usual CFL condition [9], which is a condition for stability in ℓ^2. In general, these conditions do not coincide.

Remark 6. The example generalizes naturally to nonlinear schemes. As an exercise for the reader, repeat the example with $|u_x|$ instead of $-u_{xx}$. Use the discretization $\max\{u_i - u_{i-1}, u_i - u_{i+1}, 0\}/dx$. The resulting nonlinear CFL condition is $0 \le dt \le dx$.

1.3. A stable but nonconvergent scheme. In this section, we give an example of a difference scheme which is stable but nonmonotone and nonconvergent. The example involves the linear, but degenerate, second-order elliptic equation,

$$-(u_{xx} + 2u_{xy} + u_{yy}) = -\frac{d^2u}{dv^2} = 0, \qquad v = (1,1),$$

along with Dirichlet boundary conditions on the unit square. Continuous functions of the form $f(x-y)$, whose level sets are straight lines in the direction of v, are viscosity solutions. The equation is degenerate: it has a zero eigenvalue in the direction perpendicular to v.

Two discretizations. We present two consistent, second-order accurate difference schemes on a uniform grid with spacing h. For the first scheme, simply use the centered second difference in the diagonal direction,

$$\frac{1}{h^2}\left(u(x+h, y+h) - 2u(x,y) + u(x-h, y-h)\right).$$

For the second scheme, use centered differences for u_{xx} and u_{yy}, and a symmetric centered difference for u_{xy}, to obtain

$$\begin{aligned}\frac{1}{h^2}&\left(2u(x+h,y) + 2u(x,y+h) + 2u(x-h,y) + 2u(x,y-h)\right.\\ &\left.\quad - 6u(x,y) - u(x+h,y-h) - u(x-h,y+h)\right).\end{aligned}$$

The first scheme is degenerate elliptic. The second scheme is not, since the coefficients of the values at grid points $(x+h, y-h)$ and $(x-h, y+h)$ are negative.

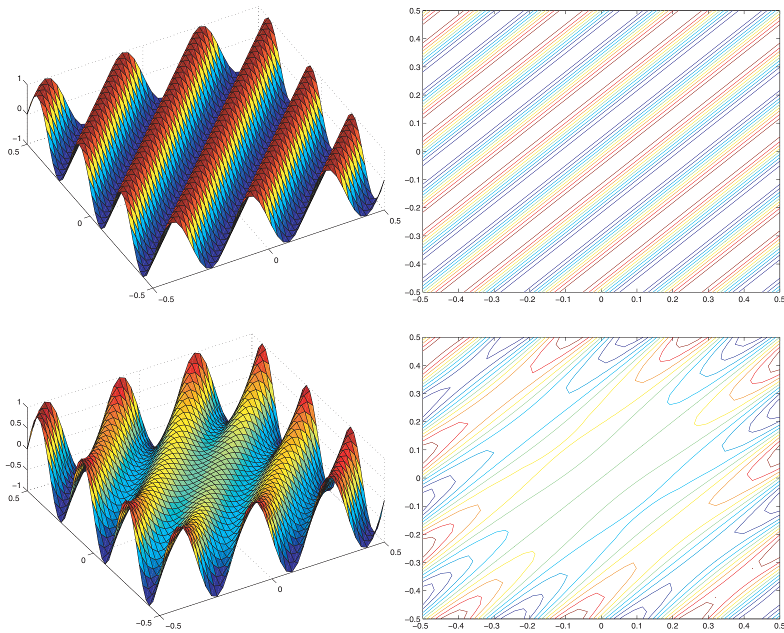

FIG. 1. *Solution and level sets, computed using the first (top) and second (bottom) scheme.*

Numerical experiments. The first scheme converges. The second scheme gives errors of a size comparable to the data, independent of the grid spacing. The solutions were found by an explicit, stable iteration scheme. Computational results using Dirichlet boundary values $\sin(6\pi(x-y))$ are presented in Figure 1. Note that the level sets fail to be straight lines for the second scheme.

Stability analysis. We verify stability in ℓ^2 directly. Consider for the sake of analysis a periodic, 2×2 grid. The grid functions

$$v_1 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}, \qquad v_2 = \begin{pmatrix} + & - \\ + & - \end{pmatrix}, \qquad v_3 = \begin{pmatrix} + & + \\ - & - \end{pmatrix}, \qquad v_4 = \begin{pmatrix} + & - \\ - & + \end{pmatrix},$$

which consist of horizontal, vertical, and diagonal stripes, form a simultaneous set of eigenvectors for the schemes, with eigenvalues $\{0,-4,-4,0\}$ and $\{0,-4,-4,-8\}$, respectively. Thus the operators are stable. The explicit scheme with time step dt corresponds to adding the identity to dt times the linear map, so it has the same eigenvectors, with eigenvalues $\lambda \mapsto 1 + dt\lambda$. Taking $dt \le 1/2, 1/4$, respectively, gives a scheme with eigenvalues in the unit circle, and that is thus stable in ℓ^2.

Conclusion. Despite the stability of the second scheme, it is nonconvergent. For this equation, monotonicity is necessary for convergence. We offer a heuristic explanation: while the equation is sensitive to data only in the diagonal direction, the second scheme uses data from grid points in other directions.

2. Theory.

2.1. Viscosity solutions. We have endeavored to make this article accessible to readers who are not familiar with the theory of viscosity solutions. The standard reference is [11]. An introduction to the first-order case, with applications to control theory, is [15]. A readable introductory article with valuable exercises is the contribution by Crandall [3]. The complete first-order theory can be found in [2, 4].

Viscosity solutions are weak solutions defined for the class of degenerate elliptic and parabolic equations. In this class, under mild analytic assumptions [20], there exist unique viscosity solutions. These solutions are stable in the maximum norm under perturbations: a perturbation to the data of size ϵ results in an error in the solution of size at most ϵ. Solutions are also stable under perturbations of the *equation*, as long as the resulting equation is still in the class. For example, adding ϵ times the Laplacian regularizes the equation (this is where the term *viscosity solutions* comes from). It is also common to add ϵ time u to the equation. In addition, replacing the equation with a finite difference approximation is a valid perturbation, as long as the approximation is monotone [5]. In this case, ϵ may represent the grid spacing.

Remark 7 (nonsmooth functions). Although solutions need not be smooth (or even differentiable), the definition of viscosity solutions requires only verifying inequalities for smooth test functions. In particular, when verifying consistency for numerical schemes, we may work freely with smooth functions.

2.2. Degenerate elliptic equations. Let Ω be a domain in $\mathbb{R}^n$, Du and D^2u denote the gradient and Hessian of u, respectively, and $F(x, r, p, X)$ be a continuous real valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, with $\mathbb{S}^n$ being the space of symmetric $n \times n$ matrices. Write $F[u](x) \equiv F(x, u(x), Du(x), D^2u(x))$. Consider the nonlinear, degenerate elliptic partial differential equation with Dirichlet boundary conditions,

$$\begin{cases} F[u](x) = 0 & \text{for } x \text{ in } \Omega, \\ u(x) = g(x) & \text{for } x \text{ on } \partial\Omega, \end{cases}$$

or the initial-boundary value problem for the degenerate parabolic partial differential equation,

$$\begin{cases} u_t(t, x) = -F[u](t, x) & \text{for } (t, x) \text{ in } \Omega \equiv [0, t) \times \Omega, \\ u(t, x) = g(t, x) & \text{for } (t, x) \text{ on } \partial\Omega \equiv \{t = 0\} \times \Omega \cup [0, t) \times \partial\Omega. \end{cases}$$

In both cases, $\partial\Omega$ is the correct set on which boundary conditions are set for the equation, not the topological boundary.

Definition 1. *The equation F is* degenerate elliptic *if*

$$F(x, r, p, X) \leq F(x, s, p, Y) \text{ whenever } r \leq s \text{ and } Y \leq X,$$

where $Y \leq X$ means that $Y - X$ is a nonnegative definite symmetric matrix.

Example. The obstacle problem, $\min(-u_{xx}, u - g(x)) = 0$, is degenerate elliptic. The Hamilton–Jacobi equation, $u_t - |u_x| = 0$, is degenerate parabolic.

Given a degenerate elliptic equation, F, consider the solution mapping, S, which takes continuous boundary data, g, to the continuous solution, u, assuming it is well-defined. We say that S is *monotone* if for all continuous functions g, h on $\partial\Omega$,

$$g(x) \leq h(x) \text{ for all } x \in \partial\Omega \text{ implies } S(g)(x) \leq S(h)(x) \text{ for all } x \in \Omega. \tag{3}$$

Likewise, the solution mapping is nonexpansive in the maximum norm if

$$\max_{x \in \Omega} |S(g)(x) - S(h)(x)| \leq \max_{x \in \partial\Omega} |g(x) - h(x)|. \tag{4}$$

These conditions generalize the maximum principle, with equivalence when constants (or zero) are solutions.

2.3. Degenerate elliptic schemes. We begin with the definition of a finite difference scheme on an unstructured grid. We regard a scheme as an *equation* which holds at each grid point, and thereby study monotonicity and stability properties of the solution operator.

For the purpose of convergence, we implicitly assume the existence of an interpolation operator, which takes grid functions to functions on the domain. We also require a sequence of grids indexed by a small parameter. Typically, the small parameter is dx, the maximum distance between neighboring grid points, but we might want to allow for $d\theta$, the directional resolution [29, 30]. The interpolation operator and the sequence of approximations puts us in the framework of the convergence theory in [5].

Define an unstructured grid on the domain Ω as a directed graph consisting of a set of points, $x_i \in \Omega, i = 1, \dots, N$, each endowed with a list of neighbors, $N(i)$. A *grid function* is a real valued function defined on the grid, with values $u_i = u(x_i)$. The scheme is represented at each grid point by an equation of the form

$$F^i[u] \equiv F^i\left(u_i, \left.\frac{u_i - u_j}{|x_i - x_j|}\right|_{j=N(i)}\right), \quad i = 1, \dots, N. \tag{5}$$

A finite difference scheme is local: it depends only on the value at the reference points, and on the first-order approximations to the derivatives in the direction of the neighbors. Higher-order approximations are obtained by taking linear combinations of the first-order derivatives.

From now on, we suppress the explicit dependence on $|x_i - x_j|$ and write

$$F^i[u] \equiv F^i\left(u_i, u_i - u_j|_{j=N(i)}\right) \equiv F^i(u_i, u_i - u_j),$$

where u_j is shorthand for the list of neighbors $u_j|_{j=N(i)}$.

A *boundary* point is a grid point with no neighbors. Dirichlet boundary conditions are imposed at boundary points by setting $F^i[u] = u_i - g(x_i)$. A *solution* is a grid function which satisfies $F[u] = 0$. If, for arbitrary boundary data g, there exists a unique solution u, we write $u = S(g)$ for the *solution operator*. We regard the solution operator as a mapping from the Dirichlet data on the boundary points to grid functions.

We now define degenerate elliptic schemes.

DEFINITION 2. *The scheme F is* degenerate elliptic *if each component F^i is nondecreasing in each variable.*

Remark 8. We emphasize that the scheme is a nondecreasing function of u_i and the differences $u_i - u_j$.

2.4. The nonlinear CFL condition. Write $\|x\|_\infty$ for the maximum norm, $\max_i |x_i|$.

While schemes may be nonlinear and nondifferentiable, we assume that they are globally Lipschitz continuous, with constant K. The resulting restriction on the time step is simply that $dt \leq K^{-1}$. We can also allow for schemes which are only locally Lipschitz continuous by allowing for the time step to depend on the data, $dt = K(u)^{-1}$. Higher-order time-stepping methods which still maintain monotonicity and nonexpansivity may also be used [17].

DEFINITION 3 (Lipschitz continuity). *The finite difference scheme, F, is Lipschitz continuous if there is a constant K such that for all $i = 1, \dots, N$, $x, y \in \mathbb{R}^{|N(i)|+1}$,*

$$|F^i(x) - F^i(y)| \leq K\|x - y\|_\infty. \tag{6}$$

Definition 4 (the explicit Euler map). *For $\rho > 0$, define $S_\rho : \mathbb{R}^N \to \mathbb{R}^N$ by*

$$S_\rho(u) = u - \rho F[u]. \tag{7}$$

This map is the explicit Euler discretization, with time step ρ, of the ordinary differential equation $du/dt + F[u] = 0$.

Definition 5 (nonlinear CFL condition). *Let F be a Lipschitz continuous, degenerate elliptic scheme, with Lipschitz constant K. The nonlinear CFL condition for the Euler map S_ρ is*

$$\rho \leq \frac{1}{K}. \tag{CFL}$$

2.5. Existence, uniqueness, and comparison for schemes. Given $u, v \in \mathbb{R}^N$, define $u \vee v = \max(u, v), u^+ = \max(u, 0)$, $u^- = \min(u, 0)$, componentwise and define $u \leq v$ to mean $u_i \leq v_i$ for $i = 1, \dots, N$

Definition 6 (proper schemes). *The finite difference scheme is* proper *if there exists $\delta > 0$ such that for $i = 1, \dots, N$ and for all $x \in \mathbb{R}^{|N(i)|}$ and $x_0, y_0 \in \mathbb{R}$,*

$$x_0 \leq y_0 \text{ implies that } F^i(x_0, x) - F^i(y_0, x) \leq \delta(x_0 - y_0). \tag{8}$$

Remark 9. If a scheme is not proper, we can consider instead $F[u] + \epsilon u$. By taking ϵ to be small enough (for example, smaller than the discretization error), we can assume the scheme is proper without any loss of generality.

Remark 10. This property is introduced to simplify the existence proof. It can be relaxed for the proof of comparison. An alternative approach would be to generalize the "marching to the boundary" argument of [28].

Theorem 5 (comparison of sub- and supersolutions). *Let F be a proper, degenerate elliptic finite difference scheme. If $F[u] \leq F[v]$, then $u \leq v$. In particular, solutions are unique.*

Proof. Suppose $u \not\leq v$ and let i be an index for which

$$u_i - v_i = \max_{j=1,\dots,N} \{u_j - v_j\} > 0, \tag{i}$$

so that

$$u_i - u_j \geq v_i - v_j, \quad j = 1, \dots, N. \tag{ii}$$

(See Figure 2.) Then we obtain a contradiction as follows:

$$\begin{aligned} F[u]^i = F^i(u_i, u_i - u_j) &\geq F^i(u_i, v_i - v_j) && \text{by (ii) and Definition 2,} \\ &> F^i(v_i, v_i - v_j) = F[v]^i && \text{by (i) and (8).} \end{aligned}$$

Uniqueness follows, since if u, v are solutions, then $F[u] = F[v] = 0$, so $u \geq v$ and $u \leq v$, and thus $u = v$. ∎

The next result combines the Lipschitz continuity property with the degenerate elliptic property of the scheme to give an ordered Lipschitz continuity property.

Lemma 1 (ordered Lipschitz continuity property). *Let F be a Lipschitz continuous, degenerate elliptic scheme, with Lipschitz constant K. Then for all $i = 1, \dots, N$ and $x, y \in \mathbb{R}^{|N(i)|+1}$,*

$$-K\|(x-y)^-\|_\infty \leq F^i(x) - F^i(y) \leq K\|(x-y)^+\|_\infty. \tag{9}$$

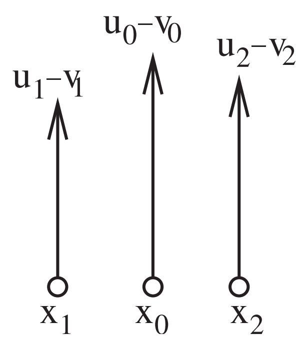

FIG. 2. *Discrete local maximum of $u_i - v_i$ at $i = 0$.*

Proof. Given x, y, use Definition 2 and (6) to compute

$$F(x) - F(y) \le F(x \vee y) - F(y) \le K\|x \vee y - y\|_\infty = K\|(x-y)^+\|_\infty.$$

The other inequality is similar. ∎

THEOREM 6 (the Euler map is monotone). *Let F be a Lipschitz continuous, degenerate elliptic scheme. Then the Euler map* (7) *is monotone provided (CFL) holds.*

Proof. Suppose $u \le v$. Compute for an arbitrary index i,

$$\begin{aligned} S_\rho^i(u) - S_\rho^i(v) &= u_i - v_i + \rho\left(F^i(v_i, v_i - v_j) - F^i(u_i, u_i - u_j)\right) \\ &\le u_i - v_i + \rho K\|(v_i - u_i, v_i - u_i + u_j - v_j)^+\|_\infty && \text{by (9)} \\ &\le (1-\rho K)(u_i - v_i) && \text{since } u \le v \\ &\le 0 && \text{by (CFL).} \end{aligned}$$

∎

THEOREM 7 (the Euler map is a contraction). *Let F be a Lipschitz continuous, degenerate elliptic scheme. Then the Euler map* (7) *is a contraction in $\mathbb{R}^N$ equipped with the maximum norm, provided* (CFL) *holds. If, in addition, F is proper, and strict inequality holds in* (CFL)*, then the Euler map is a strict contraction.*

Proof. We will show that

$$\text{(i)} \qquad \|S_\rho(u) - S_\rho(v)\|_\infty \le r\|u - v\|_\infty$$

for $r = \max(1 - \rho\delta, \rho K)$. We assume without loss of generality that $\rho\delta, \rho K < 1/2$.

We proceed to find upper and lower bounds on $S_\rho^k(u) - S_\rho^k(v)$ for an arbitrary index k. The lower bound will follow easily, while the upper bound will rely on careful application of the ordered Lipschitz continuity property.

1. Assume $u_k \ge v_k$. The alternative will follow by a similar argument.
2. For the lower bound, use (9) in the definition of the Euler map (7) to obtain

$$\begin{aligned} S_\rho^k(u) - S_\rho^k(v) &\ge u_k - v_k - \rho K\|\left(u_k - v_k, u_k - v_k - (u_j - v_j)\right)^-\|_\infty \\ \text{(ii)} \qquad &\ge -\rho K\|u - v\|_\infty, \end{aligned}$$

since $u_k \ge v_k$.

3. For the upper bound, add and subtract $\rho F^k(v_k, u_k - u_j)$ to $S^k_\rho(u) - S^k_\rho(v)$,

$$\text{(iii)} \qquad \begin{aligned} S^k_\rho(u) - S^k_\rho(v) = u_k - v_k - \rho \left(F^k(u_k, u_k - u_j) - F^k(v_k, u_k - u_j)\right) \\ + \rho \left(F^k(v_k, v_k - v_j) - F^k(v_k, u_k - u_j)\right). \end{aligned}$$

Use (8) to estimate the second to last term in (iii),

$$\text{(iv)} \qquad -\left(F^k(u_k, u_k - u_j) - F^k(v_k, u_k - u_j)\right) \le -\delta(u_k - v_k).$$

Next use (9) to estimate the last term in (iii),

$$\text{(v)} \qquad \begin{aligned} F^k(v_k, v_k - v_j) - F^k(v_k, u_k - u_j) &\le K\|((u_j - v_j) - (u_k - v_k))^+\|_\infty, \\ &\le K\left(\|u - v\|_\infty - (u_k - v_k)\right), \end{aligned}$$

since $u_k \ge v_k$. Combining (iv) and (v) gives

$$\text{(vi)} \qquad \begin{aligned} S^k_\rho(u) - S^k_\rho(v) &\le (1 - \rho\delta - \rho K)(u_k - v_k) + \rho K\|u - v\|_\infty \\ &\le (1 - \rho\delta - \rho K)\|u - v\|_\infty + \rho K\|u - v\|_\infty \\ &\le (1 - \rho\delta)\|u - v\|_\infty. \end{aligned}$$

4. Combining (ii) and (vi) gives (i) as desired. □

THEOREM 8. *A proper, Lipschitz continuous degenerate elliptic scheme has a unique solution. The iterates of the Euler map converge to the solution for arbitrary initial data, provided strict inequality holds in* (CFL).

Proof. By Theorem 7, S_ρ is a strict contraction on $\mathbb{R}^N$, equipped with the maximum norm. Thus by Banach's fixed point theorem, iterates of S_ρ converges to a unique fixed point from arbitrary initial data. Such a fixed point is a solution. □

Remark 11. Since the error tolerance is on the order of the spatial discretization error, the number of iterations need not be prohibitive. Experimentally, the number of iterations is on the order of the diameter of the graph, when the time step is optimal.

2.6. Proofs. We begin by establishing a link between the degenerate ellipticity condition and the comparison principle.

LEMMA 2 (exercise in [3]). *The function $F(x, r, p, X)$ is degenerate elliptic if and only if whenever x is a nonnegative local maximum of $u - v$, for $u, v \in C^2$, $F[u](x) \ge F[v](x)$.*

Proof. If x is a local maximum, $u \ge v$, $Dv = Du$, and $D^2u \le D^2v$, at x. Then $F(x, u, Du, D^2u) = F(x, u, Dv, D^2u) \ge F(x, v, Dv, D^2u) \ge F(x, v, Dv, D^2v)$. □

LEMMA 3. *The scheme F is degenerate elliptic if and only if whenever x_i is a nonnegative maximum of $u - v$, for u, v grid functions, $F^i[u] \ge F^i[v]$.*

Proof. Let i be an index for which $u_i - v_i = \max_{j=1,\dots,N}\{u_j - v_j\} \ge 0$, so that $u_i - u_j \ge v_i - v_j$, $j = 1, \dots, N$. Then $F^i[u] = F^i(u_i, u_i - u_j) \ge F^i(v_i, u_i - u_j) \ge F^i(v_i, v_i - v_j) = F^i[v]$. □

Proof of Theorem 1. The proof is formal, but can be made rigorous. Let $\epsilon > 0$, set $F^\epsilon[u] = F[u] + \epsilon u$, and let S^ϵ be the corresponding solution operator. Let $u^\epsilon = S^\epsilon(g), v^\epsilon = S^\epsilon(h)$. If x is a strict local max of $u^\epsilon - v^\epsilon$, then as in Lemma 2, $F^\epsilon[u](x) > F^\epsilon[v](x)$, which contradicts $F^\epsilon[u] = F^\epsilon[v] = 0$. So the maximum of $u^\epsilon - v^\epsilon$ occurs on the boundary. Likewise, the minimum of $u^\epsilon - v^\epsilon$ occurs on the boundary. Stability of viscosity solutions implies that $u^\epsilon \to u, v^\epsilon \to v$, and thus sending $\epsilon \to 0$ allows the same conclusions to hold for u, v. Thus (4) follows; assuming $g \le h$ gives (3). □

Proof of Theorem 2. We can assume without loss of generality that F is proper. Then as in Lemma 3, we can show that the max and min of $S(g), S(h)$ occur on the boundary. The conclusion follows as in the proof of Theorem 1. ∎

Proof of Theorem 3. We have already shown that degenerate elliptic schemes are monotone and nonexpansive. Now, suppose a scheme given in explicit form (1) is monotone and nonexpansive. Then H^i is a nondecreasing function for each i. Nonexpansivity means

$$|H^i(x) - H^i(y)| \le \|x - y\|_\infty$$

for all x, y. Estimate $|H^i(x) - H^i(y)| \le \|DH^i\|_1 \|x - y\|_\infty$. Since equality may hold for some x, y, we require $\|DH^i\|_1 \le 1$. Locally define $F^i = (1 - \sum_{j=N(i)} D_j H^i) u_i + \sum_{j=N(i)} D_j H^i (u_i - u_j)$. Applying the implicit function theorem gives F^i in the required form. ∎

Remark 12. The linear scheme $u^{n+1} = Mu^n$ is monotone if and only if $m_{ij} \ge 0$ and nonexpansive in the ℓ^∞ norm if and only if $\sum_j |m_{ij}| \le 1$ for each i. The differentiable scheme, $u^{n+1} = F(u^n)$, is monotone (respectively, nonexpansive) if the gradient $DF(u)$ is monotone (nonexpansive) for every u. In the linear case, nonexpansivity in ℓ^∞ does not imply nonexpansivity in ℓ^2, or in ℓ^1, as simple examples illustrate.

Accuracy. Given the equation $F(x, u(x), Du(x), D^2u(x))$ and the scheme $F^i(u_i, u_i - u_j|_{j=N(i)})$, fix $x = x_i$ and set $h_j = |x_i - x_j|$, $j = N(i)$. Assume the h_j are of the same order so that the expression $O(h)$ is meaningful. The order of accuracy of the scheme is the best possible number r in the expression

$$F(x_i, u(x_i), Du(x_i), D^2u(x_i)) - F^i(u(x_i), u(x_i) - u(x_j)|_{j=N(i)}) = O(h^r),$$

over all functions u which have all derivatives defined in a neighborhood of x_i.

Proof of Theorem 4. It is sufficient to show that higher-order accuracy is impossible for functions of a particular form. By considering functions of the form $u(x) = g(n \cdot x)$, where n is a direction vector, we reduce to an equation in one space dimension. Considering functions with constant values $u(x_i)$ and constant first or second derivatives further reduces the equation to the form $H(u_x)$ or $H(u_{xx})$. While some reductions yield trivial equations, any nontrivial equation will give a nontrivial reduction for some choice of the direction n, $u(x_i)$, and the derivatives.

Redefine $h_j = x_j - x_i$, and expand u in Taylor series, $u(x_j) = \sum_{k=0}^\infty \frac{h_j^k}{k!} \frac{\partial^k u}{\partial x^k}(x_i)$. Apply the series to each u_j in the expression for $F^i[u]$, dropping the u_i dependence to give

$$\text{(i)} \qquad F^i[u] = F^i\left(- \sum_{k=1}^\infty \frac{h_j^k}{k!} \frac{\partial^k u}{\partial x^k}(x_i) \Bigg|_{j=N(i)} \right).$$

Consider first the case when both the scheme and the equation are linear. Write $F^i[u] = \sum_{j=1}^{|N(i)|} a_j (u_i - u_j)$, $a_j > 0$, and insert the Taylor expansion into (i) to obtain $-\sum_{j=1}^{|N(i)|} \sum_{k=1}^\infty \frac{a_j h_j^k}{k!} \frac{\partial^k u}{\partial x^k}(x_i)$. Observe that the coefficients of $\partial^k u / \partial x^k$ have the same sign for even values of k, and thus no cancellation is possible. In addition, the coefficients are homogeneous of order k in h. After dividing by the leading coefficient, a scheme for u_x will have a nonzero coefficient of u_{xx} of $O(h)$, and a scheme for u_{xx} will have a nonzero coefficient of u_{xxxx} of $O(h^2)$.

We now treat the general case. Since F is nondecreasing, there is still no cancellation among the terms with even values of k. An expression containing u_x to $O(1)$ appears with a u_{xx} term of $O(h)$, and similarly u_{xx} appears with a u_{xxxx} term of $O(h^2)$. Since the higher derivatives do not appear at all in the expression for H, the error is $O(h)$ and $O(h^2)$ for the first-order and second-order equations, respectively. □

3. Building elliptic schemes. In this section we construct examples of degenerate elliptic schemes. Using simple schemes as building blocks, we build schemes for nontrivial equations. We begin with parallel observations for equations and schemes which, when taken together, give a technique for building schemes.

Many types of operations (addition, min, max, nondecreasing transformations) may be used to combine schemes. On the other hand, selection criteria ("if" statements) generally do not preserve ordering properties and must be used with care.

Observation 1 (see [11, p. 8]). Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a nondecreasing function. If F_1 and F_2 are degenerate elliptic functions, then so is $F = g(F_1, F_2)$.

Observation 2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a nondecreasing function. If F_1 and F_2 are degenerate elliptic finite difference schemes, then so is $F = g(F_1, F_2)$.

Example (order preserving operations). The constant scheme $F[u] = u - g$ is degenerate elliptic. If F, F_1, F_2 are degenerate elliptic, then so are $F^+ = \max(F, 0)$, and $F^- = \min(F, 0)$ as well as $\min(F_1, F_2), \max(F_1, F_2)$, and $aF_1 + bF_2$ for $a, b \in \mathbb{R}^N, a, b \geq 0$.

Example ("if" statements). If $F_1[u], F_2[u]$ are degenerate elliptic, and $G[u]$ is an equation, then

$$F[u] = \begin{cases} F_2[u] = 0 & \text{if } G[u] > 0, \\ F_1[u] = 0 & \text{otherwise} \end{cases}$$

is not usually degenerate elliptic. If, however, G is degenerate elliptic, and $F_2[u] \geq F_1[u]$ for all u, then F is degenerate elliptic. This follows by using Lemma 3. If $u - v$ has a nonnegative local max at i, then $G^i[u] \geq G^i[v]$, and thus $F^i[u] \geq F^i[v]$.

Example (distance function and eikonal equation). Starting from the upwind schemes $u_x = (u_j - u_{j-1})/dx$ and $-u_x = (u_j - u_{j+1})/dx$, which are degenerate elliptic, write $|u_x| = \max(u_x, -u_x)$, $-|u_x| = \min(u_x, -u_x)$ and apply the observations to build the schemes

$$|u_x| = \frac{1}{dx} \max\left(u_j - u_{j-1}, u_j - u_{j+1}, 0\right), \quad -|u_x| = \frac{1}{dx} \min\left(u_j - u_{j-1}, u_j - u_{j+1}, 0\right),$$

accurate to $O(dx)$. Next write $u_x^2 = |u_x|^2$ to obtain the scheme

$$u_x^2 = \frac{1}{dx^2} \max\left(u_j - u_{j-1}, u_j - u_{j+1}, 0\right)^2,$$

which is accurate to $O(dx)$. Schemes for $|Du|$ and $|Du|^p$ in higher dimensions are easily built.

Example (obstacle problems). Let F_1 be a degenerate elliptic scheme for $F[u]$. The obstacle problem

$$\min(F[u], u - g(x)) = 0$$

is degenerate elliptic, and the scheme $\min(F_1, u - g)$, is consistent and degenerate elliptic. This example can be generalized to double obstacle problems.

Example (finite maxima and minima of schemes [5]). A degenerate elliptic scheme for $\min_{\gamma\in\Gamma}\max_{\beta\in B}\{F_{\gamma\beta}\}$, where the index sets are finite, can be built from schemes for each $F_{\gamma\beta}$ by taking the corresponding finite minima and maxima over the schemes.

Example (nonlinear one-dimensional equations). The degenerate elliptic equation $F(x, u_{xx})$ is nonincreasing in u_{xx}, and thus the scheme $F^i[u] = F(x_i, (2u_i - u_{i+1} - u_{i-1})/dx^2)$ is degenerate elliptic. If $H(x, u_x)$ is increasing in u_x, then the scheme $F^i[u] = H(x_i, (u_i - u_{i-1})/dx)$ is also degenerate elliptic. Likewise, if $H(x, u_x)$ is decreasing in u_x, then $F^i[u] = H(x_i, (u_i - u_{i+1})/dx)$ is degenerate elliptic. Simply combining the previous two schemes with an if statement will not yield a degenerate elliptic scheme for general H.

Example (one-phase free boundary problems). Consider

$$\begin{cases} F[u] = 0 & \text{in } \{u > 0\}, \\ H(x, Du) = 0 & \text{on } \partial\{u = 0\}, \end{cases}$$

where $F[u] = F(x, Du, D^2u)$ is degenerate elliptic. Time-dependent versions may also be considered. This one-phase free boundary problem (see [8, 22] for examples) is degenerate elliptic when the boundary condition is interpreted in the viscosity sense,

$$\min(F, H) \le 0 \text{ and } \max(F, H) \ge 0 \quad \text{on } \partial\{u = 0\}.$$

Eliminate the free boundary by extending to a computational domain large enough to contain $\{u = 0\}$, and consider instead

$$\begin{cases} F[u] = 0 & \text{in } \{u > 0\}, \\ \min(F[u], H(x, Du)) = 0 & \text{on } \{u \le 0\}. \end{cases}$$

Given F_1, F_2, degenerate elliptic schemes for F and H, respectively, the following scheme is consistent and degenerate elliptic:

$$F^i[u] = \begin{cases} F_1^i[u] & \text{if } u_i > 0, \\ \min(F_1^i[u], F_2^i[u]) & \text{if } u_i \le 0. \end{cases}$$

4. Computations.

Example (front propagation). For $u_t = |u_x|$, (CFL) gives $\rho \le dx$. Setting $\rho = dx$ gives the exact solution for piecewise linear initial data. For $|u_x|^2$, the function $F(x, y, 0) = \max(x, y, 0)^2$ is locally, but not globally, Lipschitz, with constant $K = \max(x, y, 0)$. (Simply differentiating overestimates the constant by a factor of 2.) This leads to $dt \le dx^2 / \max_j\{|u_j - u_{j-1}|\}$. The solution was computed using sinusoidal initial data and periodic boundary conditions, with 500 grid points. The solutions are displayed in Figure 3.

Homogenization of Hamilton–Jacobi equations. Next we consider one- and two-dimensional nonconvex Hamilton–Jacobi homogenization problems. The problem involves solve $H(x, Du) = \overline{H}$, for the function u and the constant $\overline{H}$, using periodic boundary conditions. The value $\overline{H}$ is unique, although the function u is not. The solution can be obtained by solving the time-dependent problem $u_t = H(x, Du)$ for a long time, because (see [6]) $u_t \to \overline{H}, u(t, \cdot) \to u + \overline{H}t$ as $t \to \infty$.

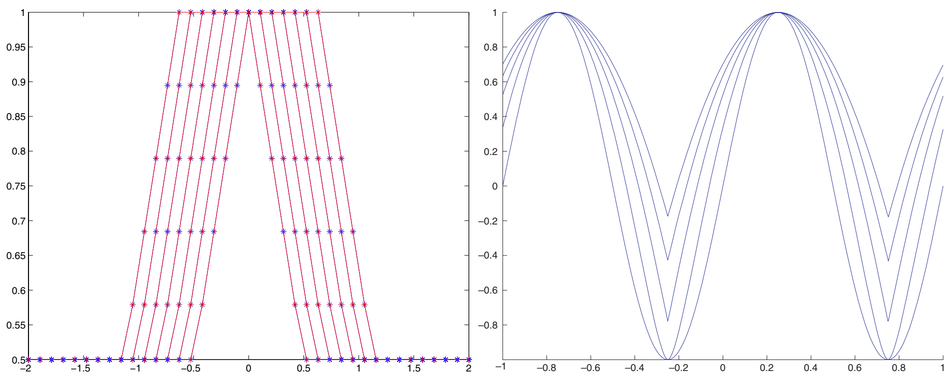

FIG. 3. *Snapshots of the solution of* $u_t = |u_x|$*, and* $u_t = |u_x|^2$.

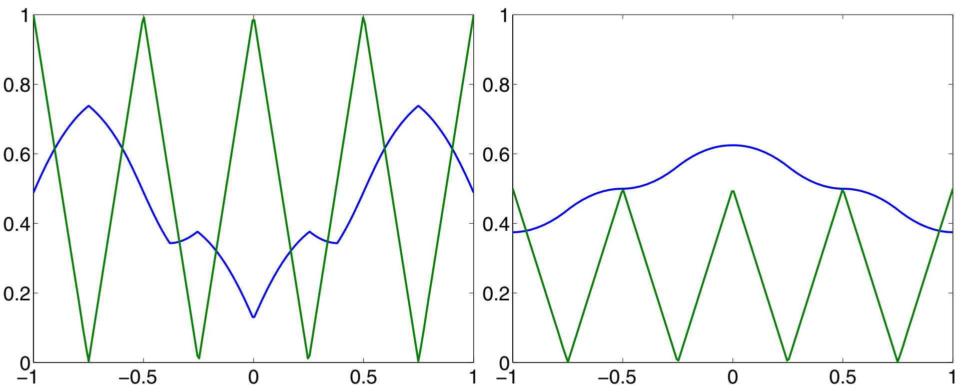

FIG. 4. *Solutions* u *plotted with the zig-zag functions.*

Example (one-dimensional nonconvex Hamilton–Jacobi equation). Set $H(x, u_x) = \max(|u_x|, 1 - |u_x|) + f(x)$, where $f(x)$ is periodic on $[-1, 1]$. With $f(x)$ the "zigzag" function $f(x) = 2|2x \pmod 1) - .5|$, we found $\overline{H} \approx .48$. The solutions were also computed with $f(x) = |2x \pmod 1) - .5|$. The solutions, along with the functions $f(x)$, are displayed in Figure 4, plotted so that the average of the solution is $\overline{H}$. In the second case, with the hindsight afforded by the numerical solution, we found an exact piecewise quadratic solution, with $\overline{H}(u) = .5$. Modifying the solution by reflecting the portion between $-.5$ and $.5$ (where $u_x = 0$) in the line $y = .5$ gives another solution with $\overline{H} = .5$.

Example (two-dimensional nonconvex Hamilton–Jacobi equation). Set $H(u_x, u_y) = u_x^2 - u_y^2$ in a periodic domain, with periodic boundary conditions. With initial data $\sin(x)\sin(y)$, a nontrivial steady solution was computed, shown in Figure 5. Examination of the numerical solution reveals an exact solution, which is piecewise quadratic, with u_x, u_y as piecewise linear functions with slopes ± 1.

Free boundary problems.

Example (two-dimensional double obstacle problem). We have the equation

$$-\max\left(u - h, \min\left(u_{xx} + u_{yy}, u - g\right)\right) = 0,$$

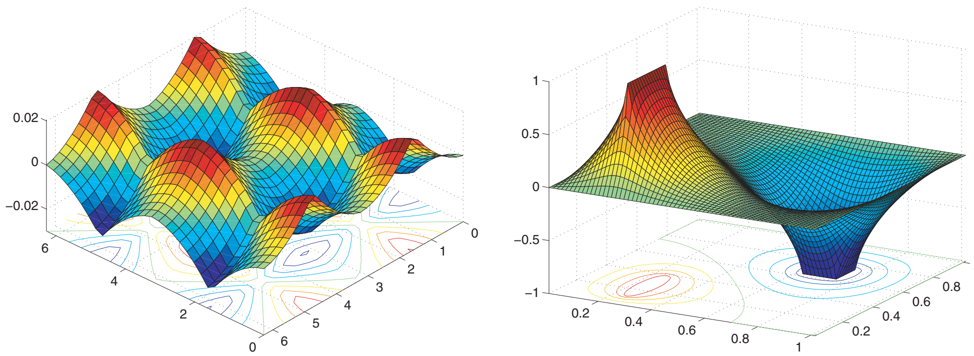

FIG. 5. *Piecewise quadratic solution of $u_x^2 - u_y^2 = 0$, solution of the double obstacle problem.*

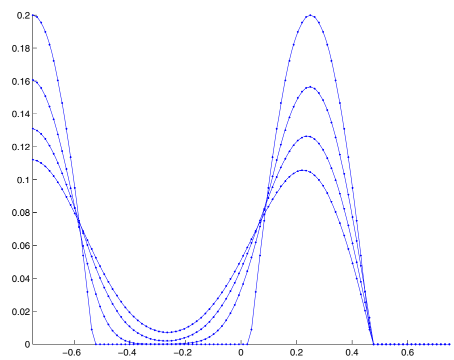

FIG. 6. *Snapshots in time of the solution to the Stefan problem. Two bumps move together and merge.*

where the obstacle functions h, g are characteristic functions of a square and a line on different parts of the domain. The solution is displayed in Figure 5.

Example. The one-phase Stefan problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u = 0\} \end{cases}$$

is solved in one dimension with sinusoidal initial data. Snapshots of the solution are shown in Figure 6.

Example (a nonconvex, fully nonlinear second-order equation). The fully nonlinear, uniformly elliptic second-order equation $-\max(\min(L^1u, L^2u)L^3u) + 1 = 0$, where $L^1u = u_{xx} + u_{yy}$, $L^2u = \frac{1}{2}u_{xx} + 2u_{yy}$, $L^3u = \frac{1}{2}u_{xx} + u_{yy}$, is solved in the unit square with Dirichlet boundary values $\frac{1}{2}\max(\min(x^2+y^2, \frac{1}{2}x^2+2y^2), \frac{1}{2}x^2+y^2)$. The solution is displayed in Figure 7. The boundary shown is composed of two parts: the dotted lines correspond to the boundary of the set $\{L^1u \leq L^2u\}$. The heavy lines correspond to the boundary of the set $L^3u \geq \min(L^1u, L^2u)$.

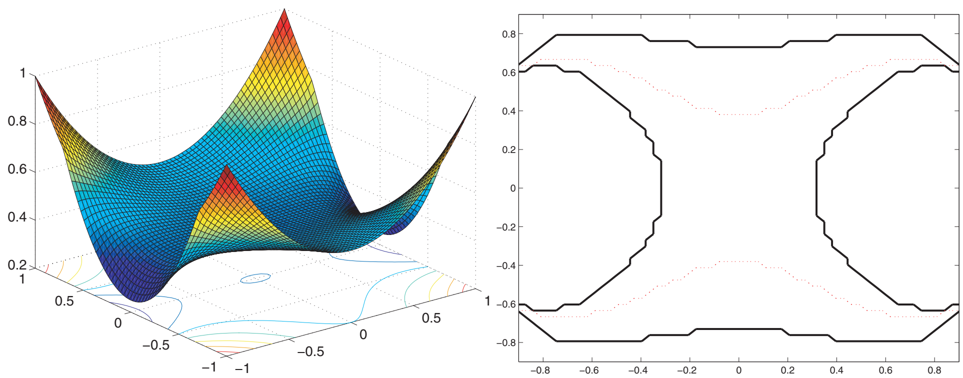

FIG. 7. *Solution and free boundary for the fully nonlinear second-order equation* $-\max(\min(L^1 u, L^2 u) L^3 u) + 1 = 0$.

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