Pricing early exercise contracts in incomplete markets

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May 2003, typographical corrections November 7, 2003

Abstract

We present a utility-based methodology for the valuation of early exercise contracts in incomplete markets. Incompleteness stems from non-traded assets on which the contracts are written. This methodology takes into account the individual's attitude towards risk and yields nonlinear pricing rules. The early exercise indifference prices solve a quasilinear variational inequality with an obstacle term. They are also shown to satisfy an optimal stopping problem with criterion given by their European indifference price counterpart. A class of numerical schemes are developed for the variational inequalities and a general approach for solving numerically nonlinear equations arising in incomplete markets is discussed.

Key words: Non-traded assets, early exercise contracts, utility maximization with discretionary stopping, Hamilton-Jacobi-Bellman equations, quasilinear variational inequalities, nonlinear asset pricing.

JEL Classification: C61, G11, G13

Mathematics Subject Classification (1991): 93E20, 60G40, 60J75

Acknowledgement: The second author acknowledges partial support from NSF Grants DMS 0102909 and DMS 0091946.

1 Introduction

This paper is a contribution to the valuation and risk management in incomplete market environments. Incompleteness comes from the fact that the contracts are written on assets that are not traded. Such situations arise for example on options on commodities or funds when one can at best trade another correlated asset. In other situations, as in the case of basket options, even when one can trade the basket components, for efficiency reasons one may still prefer to use a correlated index for pricing and risk management. In a more general picture, such situations arise often in the area of real options (see Dunbar (2000)).

The classical approach towards contingent claim pricing consists of dynamically replicating a future liability by trading the assets on which the liability is written. This is the well known arbitrage free theory which yields the derivative prices as (discounted) expectations of future payoffs. Expectation is taken under the so-called risk neutral measure which is unique. When nontraded assets are present, the traditional valuation approach cannot be applied and alternatives to the arbitrage pricing must be developed in order to specify the appropriate price concept and define the related risk management.

If it is not possible to hedge all risk, there are multiple risk neutral measures. It then seems natural to extend the classical valuation approach and obtain the prices as expectations of future payoffs with respect to one of these measures, chosen under certain optimality criteria. One may also consider different pricing criteria, like for example, the variance of the relevant random variables in order to quantify the additional unhedged risk. This general approach is known as mean-variance hedging. The analysis can then be based on the self-financing trading strategies with the aim of minimizing the tracking error at the terminal date only (see, for example Duffie and Richardson (1991) and Pham et al. (1998)). Alternatively, one can start by enlarging the class of trading strategies to allow for an additional transfer of funds. This means that the usual assumption that a trading strategy should be self-financing is simply abandoned. The aim of this approach is to focus on the minimization of the future risk exposure at any time, and not only at the terminal date. This method of hedging in incomplete markets originates from work by Follmer and Sondermann (1986). Both ideas draw on the concept of arbitrage based pricing and generalizations of the classical Black-Scholes model. There is extensive literature on the topic, we refer the interested reader to Musiela and Rutkowski (1997) and the references therein.

A very different approach to pricing and risk management is based on *utility maximization*. The underlying idea aims at incorporating an investor's attitude towards the risk that cannot be eliminated. From this perspective, this utility-based pricing method has traditionally been the approach of pricing static actuarial risks. In stochastic dynamic market environments, the utility approach borrows many characteristics and a lot of insight from the seminal work of Merton (1969) on stochastic models of expected utility maximization. The concept of utility-based derivative price that takes into account transaction costs was introduced by Hodges and Neuberger (1989) for the case of European type (fixed maturity) instruments. It was further extended and analyzed by a number of authors; see, among others, Davis et al. (1993), Davis and Zariphopoulou (1995), Barles and Soner (1998), Constantinides and Zariphopoulou (1999), (2001).

Departing from models with transaction costs, in a market environment similar to ours, Davis (1999), (2000) used the utility-based method to formulate the pricing and hedging problem of European options, considering the basis risk as the source of market incompleteness. He analyzed the underlying optimization problem via its dual that, in turn, gives rise to another nonlinear problem for which no explicit solution was given.

From a different direction and in a more general setting, Frittelli (see Frittelli (200a),(200b)) analyzed the connection between entropy measures and utility-based prices when utilities are exponential. He exposed the fundamental idea of an emerging pricing measure which turns out to have the minimal relative entropy with respect to the real (historical) one. These measures were also studied by Rouge and El Karoui (2000) who produced a pricing formula in terms of an alternative nonlinear optimization problem with criteria involving the payoff and relative entropic terms. For related results see also Becherer (2002), Delbean et al. (2002) and Follmer and Schied (2002a),(2002b).

Despite the generality of the results obtained in the above works and the use of the powerful duality method, no intuitive price formulae were produced that would extend the arbitrage free prices in a natural way. We recall that arbitrage free prices are given as expectation of the payoff under the risk neutral measure, a formula that is at the same time elegant and universal with two fundamental pricing ingredients, namely, a linear pricing rule and a specific statistical vehicle (risk neutral measure). Up to now, the incomplete market analogues of these two crucial valuation components are still lacking.

Assuming, as in Davis (2000), exponential risk preferences and European claims but following a different path Musiela and Zariphopoulou (2002a) produced closed form expressions for the indifference price (see (15) and (17)), as a nonlinear transformation of a solution to a second order linear partial differential equation. It turns out that the price has two interesting features. First, it is given in terms of a *nonlinear pricing* rule that has certainty equivalent characteristics. However, this nonlinear pricing functional is not the static analogue of the certainty equivalent corresponding to the exponential preferences. It is a distorted certainty equivalent with the distortion depending only on the correlation of the traded and nontraded asset. The second interesting ingredient is the measure under which the indifference price is computed. It turns out that it is a measure under which the stock price is a martingale. Moreover, this martingale measure has the minimal entropy with respect to the historical one.

The results in Musiela and Zariphopoulou (2002a) indicate that the utility-based pricing approach yields the indifference price as a *nonlinear expectation* under a martingale measure that minimizes the relative entropy with respect to the historical one. In a sense, the formulae in Musiela and Zariphopoulou provide an intriguing extension of the arbitrage free prices that are given in terms of a (linear) expectation under the risk neutral measure. The risk neutral measure is replaced by the minimal, relative to the historical, entropy martingale measure and the linear expectation by a nonlinear pricing operator. A new nonlinear but, in many aspects, universal pricing concept seems to emerge. Recent results of Musiela and Zariphopoulou (2002b) show, always in the context of European claims, that this pricing mechanism preserves many of the appealing characteristics of the arbitrage free prices: numeraire independence, coherence and projection properties.

Given the recent advances in the area and the ever increasing number of more complex and the same time realistic applications, it is desirable to generalize the existing results. In many situations, the claims to be priced do not have a fixed maturity and/or are path dependent. Early exercise claims arise often in situations in which a certain project is undertaken or abandoned (Smith and Nau (1998), Smith and McCardle (1995)), executives decide when to exercise their employee stock options, household owners refinance their mortgages or sell certain property. Path dependent claims arise in non traditional employee stock option models (Johnson and Tian (2000)) and development of R&D venture projects.

The scope herein is to price claims of *early exercise* that are written on non-traded assets. As the analysis will indicate, allowing for early exercise gives rise to stochastic optimization models of expected utility with discretionary stopping. In complete markets, expected utility problems with discretionary stopping were studied by Karatzas and Wang (2000) who focused on optimal portfolio management rather than derivative pricing. In the case of incomplete markets but in an infinite horizon setting, similar problems were analyzed by MacNair and Zariphopoulou (2000). Early exercise claims were priced for the first time by Davis and Zariphopoulou (1995) under the assumption that the claims are written on traded assets but with proportional transaction costs.

We assume a market environment in which the traded assets are a riskless bond and a risky stock. An early exercise claim is written on a third asset which is correlated with the stock. The stock is assumed to follow a lognormal process and the non-traded asset is modelled as a diffusion process with general coefficients. We derive the early exercise indifference price, called also *American indifference* price, as the solution to a quasilinear variational inequality with an obstacle constraint. The part of the variational inequality that is of second order is quasilinear and resembles the one we recover in the (quasilinear) equation of the European counterpart. The obstacle term is given by the claim's payoff.

We next characterize the indifference early exercise price as the solution to an optimal stopping problem of a nonlinear expectation criterion. The latter has certainty equivalent characteristics but does not coincide with the classical static insurance-type pricing rule. The (nonlinear) expectation is taken under a new measure that is on one hand a martingale measure for the price of the traded asset and, on the other, has the minimal relative entropy with respect to the historical measure. We see that the two main characteristics highlighted in the analysis of Musiela and Zariphopoulou (2002a) for the case of European derivatives, are preserved in the case of early exercise instruments.

Looking further at the optimal stopping problem that the indifference early exercise price solves, we recover another desirable property of the utility-based pricing mechanism. Namely, the *American indifference price turns out to be the optimally stopped value of its European indifference counterpart.* This fact, albeit not at all obvious in such a nonlinear framework, is consistent with what we observe in complete markets.

Under natural assumptions on the market coefficients, we deduce that the quasilinear variational inequalities that the indifference early exercise prices solve have a unique solution. However, one may not in general obtain explicit solutions for the prices, the optimal exercise boundary and the risk monitoring strategies. Numerical approximations are then needed in order to produce results of practical interest and ultimately even test the validity of the pricing methodology. The second part of the paper is dedicated to the development of a class of pricing schemes for the equations at hand.

The overall goal however is not to produce numerical results for a specific class of applications, but rather to set the framework for a broader computational analysis of nonlinear pricing models arising in incomplete markets. Generally speaking, these models give rise to high dimensional fully nonlinear equations that do not have in general smooth solutions. This is an immediate consequence of not only the specific degeneracy of the model but also the potential discontinuity of the equation itself as function of its arguments. This is, for example, the case herein since the price solves a quasilinear variational inequality with an obstacle term.

As our analysis demonstrates, one needs to go beyond the classical approximation schemes and work with a weaker notion of solutions, namely, the *viscosity solutions.* Viscosity solutions were introduced by Crandall and Lions (1983) and by Lions (1983) for second order equations. For a general overview of the theory we refer to Ishii and Lions (1990), the User's Guide by Crandall, Ishii and Lions (1992) and the book of Fleming and Soner (1993). Viscosity solutions were used for the first time by the second author in stochastic optimization models in markets with frictions and have by now become a standard tool of analysis in Markovian models of asset pricing and portfolio optimization (see for example the review article by Zariphopoulou (2001)). It is this class of solutions that we use throughout our analysis both for the theoretical as well as the numerical part of our work. First, the value functions turn out to be the unique viscosity solutions of their associated Hamilton-Jacobi-Bellman equations. This, in turn, yields, through the appropriate pricing equality, the indifference price as the unique viscosity solution of the associated quasilinear variational pricing inequality. Sensitivity analysis is then performed using the comparison properties in the viscosity sense. However, the most important contribution of the viscosity theory for the problems of interest lies in the convergence of a large class of numerical schemes. Barles and Souganidis (1991) established that schemes that are *stable*, *consistent* and *monotone* converge to the viscosity solution of the nonlinear equation at hand, provided that the latter admits a strong comparison result in the viscosity sense. Even though to establish these properties might not be a formidable task, it is not always straightforward to actually construct such schemes. This is our contribution herein. We build a scheme in which the nonlinear terms and the obstacle are treated in a monotone and consistent way. The scheme is explicit and the method captures the free boundary in direct and natural steps, without requiring for the optimal boundary to be tracked. We provide an explicit condition on the time step that is used to establish the scheme's convergence.

The paper is organized as follows. In Section 2 we introduce the concept of early exercise indifference price and we review the existing results on European prices. In Section 3 we produce the pricing equation and we produce sensitivity results in terms of the two important model indices, namely, the correlation, that measures incompleteness and the risk aversion parameter, that characterizes the nonlinear input in the pricing methodology. In Section 4 we relate the early exercise indifference prices to solutions of optimal stopping and we provide representation results of the American prices in terms of their European analogues. In Section 5 we build the approximation scheme and we establish its convergence. We also present the numerical results. We provide conclusions and directions for further research in Section 6.

2 The model and pricing methodology

We assume a dynamic market setting with two risky assets, namely, a stock that can be traded and a nontraded asset. We model their prices as diffusion processes, denoted by S and Y, respectively.

The traded asset's price satisfies a diffusion process with lognormal dynamics, namely,

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dW_s^1, & t \leq s, \\ \\ S_t = S > 0 \end{cases} \tag{1}$$

with μ and σ being positive constants.

The level of the nontraded asset is given by

$$\begin{cases} dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s, & t \leq s, \\ \\ Y_t = y \in R. \end{cases} \tag{2}$$

The processes W_s^1 and W_s are standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$,where $\mathcal{F}_s$ is the augmented σ-algebra generated by $\left(W_u^1, W_u, \ \ 0 \leq u \leq s\right)$. The Brownian motions are correlated with correlation $\rho \in (-1, 1)$. Assumptions on the drift and diffusion coefficients b and a are such that the above equations have a unique strong solution.

We also assume that a riskless bond B with maturity T is available for trading, yielding constant interest rate r. Throughout the analysis, it is assumed that $r = 0$. The results for $r > 0$ follow from standard rescaling arguments and are not presented.

We now introduce a contract of early exercise time written on the non-traded asset. Its payoff $g(Y_\tau)$, at discretionary exercise time τ, is taken to be bounded. A larger class of payoffs can be considered for more specific choices of the traded and nontraded asset dynamics. A standing assumption is that the payoff does not depend on the traded asset.

The valuation method used herein is based on the comparison of maximal expected utility payoffs corresponding to investment opportunities with and without involving the derivative. In both situations, trading occurs in the time horizon $[t, T]$, $0 \leq t \leq T$, and only between the two traded assets, i.e., the riskless bond B and the risky asset S. The investor starts, at time t, with initial endowment x and rebalances his portfolio holdings by dynamically choosing the investment allocations , say π_s^0 and π_s, $t \leq s \leq T$, in the bond and the stock, respectively. It is assumed throughout that no intermediate consumption nor infusion of exogenous funds are allowed. The current wealth, defined by $X_s = \pi_s^0 + \pi_s$, $t \leq s \leq T$, satisfies the controlled diffusion equation

$$\begin{cases} dX_s = \mu \pi_s ds + \sigma \pi_s dW_s^1, \quad t \leq s \leq T, \\ \\ X_t = x, \quad x \in R \end{cases} \tag{3}$$

which is derived via (1) and the assumptions on the bond dynamics (see, for example, Merton (1969)). It is worth noticing that the price of the traded asset does not appear in the wealth state equation because of the linearity assumption on stock dynamics. The control policy π_s, $t \leq s \leq T$, is deemed admissible if it is $\mathcal{F}_s$-progressively measurable and satisfies the integrability condition $E \int_t^T \pi_s^2 ds < \infty$. The set of admissible controls is denoted by $\mathcal{Z}$.

The individual risk preferences are modelled via an exponential utility function

$$U(x) = -e^{-\gamma x}, \, x \in R \tag{4}$$

with the risk aversion parameter $\gamma > 0$.

Next, we introduce two stochastic optimization problems via which the indifference price will be constructed. The first problem arises in the classical Merton model of optimal investment, namely

$$V(x, t) = \sup_{\mathcal{Z}} E \left(U(X_T) / X_t = x \right). \tag{5}$$

In this model, the investor seeks to maximize the expected utility of terminal wealth without taking into account the possibility of employing the contract.

It is now assumed that at time t, a contract (claim) is *bought*. The latter yields payoff $g(Y_\tau)$ at the random exercise time τ. In the time interval $[t, T]$ no trading of the asset Y nor of the derivative is allowed. Following the investment policy π, the buyer trades up to (discretionary) time τ in $[t, T]$ at which he decides to exercise the claim. At exercise, the buyer's wealth X_τ increases to $X_\tau + g(Y_\tau)$, due to the contract proceeds. After time τ, the buyer faces the same investment opportunities as the plain investor and continues trading between the stock and the bond till the end of the trading horizon T.

The Dynamic Programming optimality principle yields that at time τ the buyer's expected utility payoff is given by

$$J^b(x, y, t; \pi) = E \left(V(X_\tau + g(Y_\tau), \tau) / X_t = x, Y_t = y \right).$$

The latter equality reflects the fact that the value function (dynamic utility) in the absence of the claim can be viewed as the utility functional of the buyer at the discretionary exercise time τ.

The buyer's value function, denoted by u^b, is then defined for $0 \leq t \leq T$, as

$$u^b(x, y, t) = \sup_{\mathcal{A}} J^b(x, y, t; \pi)$$

$$= \sup_{\mathcal{A}} E\left(V(X_\tau + g(Y_\tau), \tau)/X_t = x, Y_t = y\right). \tag{6}$$

$\mathcal{A}$ is the set of admissible strategies defined by $\mathcal{A} = \{(\pi, \tau) : \pi_s$ is $\mathcal{F}_s-$progressively measurable, $E\int_t^T \pi_s^2 ds < \infty$ and $\tau \in \mathcal{T}_{[t,T]}$ $\}$ where $\mathcal{T}_{[t,T]}$ is the set of stopping times of filtration $\mathcal{F}$. This stochastic optimization problem combines optimal investment with discretionary stopping. It is important to observe that the optimal exercise time is not exclusively defined by the early exercise (American) claim, but rather it is directly related to the buyer's investment portfolio which combines the proceeds both from trading and exercising the claim. At the end of the next section, we show that as the model reduces to the one of complete market, the optimal exercise time of (6) converges to the optimal exercise time of the American claim when priced by arbitrage.

We are now ready to provide the definition of the early exercise buyer's indifference price. It is a natural extension of the one used by Musiela and Zariphopoulou (2002a) for European claims written on nontraded assets.

Definition 1 *The buyer's indifference price of the early exercise contract $g(Y)$, is defined as the function $h^b \equiv h^b(x, y, t)$, such that the investor is indifferent towards the following two scenarios: optimize the utility payoff without employing the contract and optimize his utility payoff taking into account, from one hand, the cost $h^b(x, y, t)$ at time of inscription t and, on the other, the contract proceeds at execution. The indifference price h^b must then satisfy for all (x, y, t)*

$$V(x, t) = u^b\left(x - h^b(x, y, t), y, t\right), \tag{7}$$

where V and u^b are defined respectively in (5) *and* (6)*.*

It is important to recall that it is the buyer of the claim who decides when the contract is exercised. The writer of the derivative does not have this opportunity and, therefore, he will have to maximize his expected utility contingently on the buyer's optimal actions. In a sense, the valuation problem of the writer reduces to a European type one (fixed exercise time) with expiration given by the buyer's optimally chosen exercise time. This asymmetry is not observed in complete markets where there is a unique price. However, in incomplete markets such asymmetries naturally emerge and give rise to realistic price spreads.

Next, we introduce a measure that will play an important role in all pricing formulae herein.

Definition 2 *Let $\mathbb{P}$ be the historical measure and E the expectation with respect to it. Define the measure $\tilde{\mathbb{P}}$ given by*

$$\tilde{\mathbb{P}}(A) = E(\exp(-\rho\frac{\mu}{\sigma}W_T - \frac{1}{2}\rho^2\frac{\mu^2}{\sigma^2}T)I_A), \quad A \in \mathcal{F}_T^W . \tag{8}$$

Proposition 3 *i) Under the measure $\tilde{\mathbb{P}}$ the stock price is a martingale. The price of the nontraded asset solves*

$$dY_s = (b(Y_s, s) - \rho\frac{\mu}{\sigma}a(Y_s, s))ds + a(Y_s, s)d\tilde{W}_s$$

where the process

$$\tilde{W}_s = W_s + \rho \frac{\mu}{\sigma} s$$

is a Brownian motion on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \tilde{\mathbb{P}})$*.*

ii) The measure $\tilde{\mathbb{P}}$ *has the minimal entropy relative to the historical measure* $\mathbb{P}$*, with the relative entropy being defined as*

$$H(\tilde{\mathbb{P}} \,/\, \mathbb{P}) = E_{\mathbb{P}} \left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \ln \, \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right) = E_{\tilde{\mathbb{P}}} \left(\ln \, \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right).$$

(For a proof see Section 3 in Musiela and Zariphopoulou (2002a)).

We conclude this section by deriving the price of the *European* counterpart of the afore introduced contracts. This case was extensively analyzed in Musiela and Zariphopoulou (2002a) and we refer the reader to the latter work for the precise technical probabilistic and computational arguments.

When early exercise is not allowed, the optimization problem (6) reduces to

$$\tilde{u}^b(x,y,t) = \sup_{\mathcal{A}_0} E \left(V(X_T + g(Y_T), T) / X_t = x, Y_t = y \right) \tag{9}$$
$$= \sup_{\mathcal{A}_0} E \left(U(X_T + g(Y_T)) / X_t = x, Y_t = y \right)$$

with U given in (4). The set $\mathcal{A}_0$ is the restriction of $\mathcal{A}$ if we take $\tau = T$. Since there are no early exercise considerations, the writer's expected utility optimization problem reduces in a similar way to

$$\tilde{u}^w(x,y,t) = \sup_{\mathcal{A}_0} E \left(V(X_T - g(Y_T), T) / X_t = x, Y_t = y \right) \tag{10}$$
$$= \sup_{\mathcal{A}_0} E \left(U(X_T - g(Y_T)) / X_t = x, Y_t = y \right).$$

The above payoff reflects the liability of the writer at the *fixed* expiration time T.

The above value functions satisfy the parity relation

$$\tilde{u}^w(x,y,t;g) = \tilde{u}^b(x,y,t;-g). \tag{11}$$

The *European writer's* (resp. *buyer's) indifference price* is defined by

$$V(x,t) = \tilde{u}^w(x + H^w(x,y,t), y, t) \tag{12}$$

and

$$V(x,t) = \tilde{u}^b(x - H^b(x,y,t), y, t). \tag{13}$$

The equality (11) yields the *parity* relation between the indifference prices, namely,

$$H^w(x,y,t;g) = -H^b(x,y,t;-g). \tag{14}$$

Due to the scaling properties of the exponential utility together with the specific assumptions on the dynamics of the traded asset, one may solve for $\tilde{u}^w$ and $\tilde{u}^b$ and produce, via (12) and (13), the relevant indifference prices.

Proposition 4 *i) The writer's indifference price of a European contract $g(Y)$, written on the nontraded asset Y that is correlated with the traded asset S, with dynamics given respectively by (2) and (1), satisfies the quasilinear partial differential equation*

$$H_t^w + \frac{1}{2}a^2(y,t)H_{yy}^w + (b(y,t) - \rho\frac{\mu}{\sigma}a(y,t))H_y^w + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)(H_y^w)^2 = 0$$

with $H^w(y,T) = g(y)$. It is given by

$$H^w(y,t) = \frac{1}{\gamma(1-\rho^2)} \ln E_{\tilde{\mathbb{P}}}(e^{\gamma(1-\rho^2)g(Y_T)}/Y_t = y) \tag{15}$$

where the measure $\tilde{\mathbb{P}}$ as defined in (8).

ii) The buyer's indifference price satisfies the quasilinear partial differential equation

$$H_t^b + \frac{1}{2}a^2(y,t)H_{yy}^b + (b(y,t) - \rho\frac{\mu}{\sigma}a(y,t))H_y^b - \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)(H_y^b)^2 = 0 \tag{16}$$

with $H^b(y,T) = g(y)$. It is given by

$$H^b(y,t) = -\frac{1}{\gamma(1-\rho^2)} \ln E_{\tilde{\mathbb{P}}}(e^{-\gamma(1-\rho^2)g(Y_T)}/Y_t = y). \tag{17}$$

The key ingredient for the derivation of (15) and (17) was a power (*distortion*) transformation that removed certain nonlinearities in the relevant HJB equations (see Section 3 in Musiela and Zariphopoulou (2002a)). Note however that such transformations cannot be applied once the assumption on the log-normality of stock dynamics and/or on the dependence of the payoff solely on the nontraded asset is removed.

The above indifference prices demonstrate two important consequences of the utility- based valuation approach: *a nonlinear asset pricing mechanism* and a specific *pricing measure*. We see that the classical linear arbitrage free pricing operator has been replaced by a pricing device that has certainty equivalent characteristics. However the presence of the conditional variance factor $(1-\rho^2)$ strongly indicates that the pricing algorithm is not given by a mere imitation of static certainty equivalent criteria but, rather, by a dynamic analogue of it that takes into account appropriate conditioning terms and distortion operators. In a sense, prices are given in terms of a *nonlinear* expectation of the contract's future payoff. The second intriguing characteristic is the measure under which this nonlinear expectation is computed. The measure $\tilde{\mathbb{P}}$ is not the risk-neutral of the nested Black and Scholes neither the historical measure $\mathbb{P}$. It is a measure under which the stock price is a martingale and, at the same time, its entropy relative to the historical one is minimized. We refer the reader to the analysis in Musiela and Zariphopoulou (2002b) for further properties and comments on this new pricing mechanism.

The apparent appeal of the above pricing formulae together with certain fundamental properties, numeraire independence, coherence and projection (see Musiela and Zariphopoulou (2002b)), indicates their potential importance in the pricing and risk management of unhedgeable risks. In what follows we explore how the results of Musiela and Zariphopoulou (2002a) can be extended to the case of early exercise contracts. We would like to caution the reader that this extension is not at all obvious given all the relevant ingredients that enter in the specification of the utility-based price, namely, optimality of investments, discretionary stopping and risk monitoring.

3 The early exercise indifference price

The scope of this section is to characterize the buyer's early exercise price and analyze its behavior with respect to the market parameters. The analysis is based on arguments from the theory of stochastic control and nonlinear partial differential equations. We first study the value functions V and u^b that will determine the price through the price equality (7). We carry out our analysis using the Hamilton-Jacobi-Bellman (HJB) equations that the value functions solve. The HJB equation satisfied by V is well known and explicit solutions are readily derived (see Merton (1969)). The value function u^b however is expected to satisfy a combination of a HJB equation and an obstacle problem. Such problems, known in the area of portfolio management, as expected utility problems with discretionary stopping, are relatively new and only special cases have been studied so far (see Karatzas and Wang (2000) and MacNair and Zariphopoulou (2000)). Generally speaking, these problems are degenerate and smooth solutions might not exist. The notion of solutions needs then to be relaxed. A class of weak solutions, the *viscosity solutions*, turns out to be the appropriate vehicle to characterize the value function as the unique weak (viscosity) solution of the HJB equation and, moreover, to obtain convergence results for a wide class of numerical schemes.

In the analysis below, we do not provide any technical arguments for the results involving the notion of viscosity solution. Moreover, for the characterization of the indifference price, we proceed as if the involved solutions were smooth. This is done only to ease the presentation since all related arguments can be proved in a rigorous matter. We revert to viscosity solutions in Section 5 where we build our numerical schemes. For more detailed arguments and key ingredients of the viscosity theory, we refer the technically oriented reader to the review article of Zariphopoulou (2001) where an overview of the use of viscosity solutions in optimization problems of mathematical finance is provided.

We show that the buyer's early exercise price satisfies a quasilinear variational inequality with an obstacle term. The nonlinearity of the operator is a direct consequence of the market incompleteness. As the markets become complete ($\rho^2 \to 1$) we show that the variational inequality converges to the classical obstacle problem of the American claims. We also provide a sensitivity analysis with respect to the risk aversion parameter γ and we study its asymptotic behavior for $\gamma \to 0$.

To simplify the presentation, we skip the b-notation. We also introduce the differential operators,

$$\mathcal{L} = \frac{1}{2}a^2(y,t)\frac{\partial^2}{\partial y^2} + b(y,t)\frac{\partial}{\partial y} \tag{18}$$

and

$$\tilde{\mathcal{L}} = \frac{1}{2}a^2(y,t)\frac{\partial^2}{\partial y^2} + (b(y,t) - \rho\frac{\mu}{\sigma}a(y,t))\frac{\partial}{\partial y}. \tag{19}$$

We start with the characterization of the value functions V and u as solutions of their HJB equations.

Proposition 5 *The value function V solves the Hamilton-Jacobi-Bellman (HJB) equation*

$$V_t + \max_{\pi}(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \mu\pi V_x) = 0 \tag{20}$$

with $V(x,T) = -e^{-\gamma x}$. It is given by

$$V(x,t) = -e^{-\gamma x}e^{-\frac{\mu^2}{2\sigma^2}(T-t)}. \tag{21}$$

Proof. The proof follows from direct substitution of the candidate solution (21) and classical verification results. For more detailed arguments, we refer the reader to Merton (1969).

Theorem 6 *The value function u is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{aligned} \min(-u_t - \max_{\pi}(\frac{1}{2}\sigma^2\pi^2 u_{xx} + \pi(\rho\sigma a(y,t)u_{xy} + \mu u_x)) - \mathcal{L}u \ , \\ u - V(x+g(y),t)) = 0 \end{aligned} \tag{22}$$

with

$$u(x,y,T) = V(x+g(y),T) = -e^{-\gamma(x+g(y))}$$

in the class of functions that are concave and increasing in the spatial argument x and bounded in y.

The proof follows closely the arguments used in Davis and Zariphopoulou (1995) and it is omitted.

We are now ready to construct the buyer's indifference price.

Theorem 7 *The buyer's early exercise indifference price is the unique bounded viscosity solution of the quasilinear variational inequality*

$$\min(-h_t - \tilde{\mathcal{L}}h \ + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2, \ h - g(y)) = 0 \tag{23}$$

with

$$h(y,T) = g(y) \tag{24}$$

and $\tilde{\mathcal{L}}$ given by (19).

Proof. Using the pricing equality (7) and the HJB equation (22), evaluated at the point $(x-h(y,t),y,t)$, we see that the latter becomes

$$\min((-V_t+\frac{\mu^2}{2\sigma^2}\frac{{V_x}^2}{V_{xx}})+V_x(-h_t-\tilde{\mathcal{L}}h+\frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2),\ V-V(x-h+g,t))=0 \tag{25}$$

where all the derivatives of V are evaluated at the point (x,t).We now observe that the first term in the second order part above coincides with the HJB equation (20), evaluated at the optimum, and therefore it vanishes, i.e.

$$-V_t+\frac{\mu^2}{2\sigma^2}\frac{{V_x}^2}{V_{xx}}=0.$$

Moreover, the exact formula for V (see (21)) yields that V_x is positive and, therefore, h satisfies

$$-h_t-\tilde{\mathcal{L}}h+\frac{1}{2}\gamma(1-\rho^2)a^2(y,t)h_y^2\geq 0, \tag{26}$$

for $(x,y,t)\in R\times R\times[0,T]$. On the other hand, the monotonicity of V with respect to the spatial argument and the form of the obstacle term in (25) yield

$$h\geq g \tag{27}$$

for $(x,y,t)\in R\times R\times[0,T]$. Combining inequalities (26) and (27) yields the desired result.

Next, we examine the behavior of the indifference price with respect to the risk aversion parameter γ. Intuitively speaking, more risk averse buyers should be willing to buy the claim at a lower price which implies that the price should be decreasing with respect to γ. We establish this results in the Proposition below.

Proposition 8 *The buyer's early exercise indifference price is decreasing with respect to the risk aversion parameter. Moreover, as $\gamma\to 0$ the early exercise price converges to the unique bounded viscosity solution of the variational inequality*

$$\min(-h_{0,t}-\tilde{\mathcal{L}}h_0\ ,\ h_0-g(y))=0 \tag{28}$$

with $h_0(y,T)=g(y)$.

Proof. The proof is based on the comparison principle for viscosity solutions (see Duffie and Zariphopoulou (1993)) which yields that subsolutions of the relevant equation are dominated by its solution.

We assume that $0\leq\gamma_1\leq\gamma_2$ and we denote by $h^{(\gamma_1)}$ and $h^{(\gamma_2)}$ the associated solutions. We note that the nonlinear term in (23) is monotone with respect to γ while the rest of the differential expression is independent of γ. This in turn yields

$$0=\min(-h_t^{(\gamma_1)}-\tilde{\mathcal{L}}h^{\ (\gamma_1)}+\frac{1}{2}\gamma_1(1-\rho^2)a^2(y,t)(h_y^{(\gamma_1)})^2,\ h^{(\gamma_1)}-g(y))$$

$$\min(-h_t^{(\gamma_1)} - \tilde{\mathcal{L}}h^{\ (\gamma_1)} + \frac{1}{2}\gamma_2(1-\rho^2)a^2(y,t)(h_y^{(\gamma_1)})^2,\ h^{(\gamma_1)} - g(y)).$$

The terminal condition does not depend on the risk aversion which implies, together with the above differential inequality, that $h^{(\gamma_1)}$is a subsolution to the variational inequality satisfied by $h^{(\gamma_2)}$. The comparison result follows easily. Next, we examine the behavior of the price as the risk aversion converges to zero. We first observe that $h^{(\gamma)}$ are uniformly bounded with respect to γ and therefore converge along subsequences. Moreover, we readily obtain that, as functions of its arguments, the price equation (23) converges, locally uniformly in γ, to the linear variational inequality

$$\min(-h_{0,t} - \tilde{\mathcal{L}}h_0\ ,\ h_0 - g(y)) = 0 \tag{29}$$

Classical results from the theory of optimal stopping (see Ishii and Lions (1990)), yield that the above problem has a unique viscosity solution in the class of bounded functions. Then, the *robustness* properties of viscosity solutions (see Lions (1983)) yields that $h \to h_0$ locally uniformly in γ and the proof is complete.

We conclude this section by looking at the behavior of the price as $\rho^2 \to 1$. In this case, the market converges to the complete one and one expects the indifference price to converge to the classical arbitrage free price that corresponds to the nested Black and Scholes model. In this case, arbitrage free arguments can be used and the utility methodology becomes redundant. Even though intuition is clear, it is not obvious that the nonlinear incomplete market pricing mechanism is robust when market incompleteness disappears. To facilitate the presentation, we denote the value function and the indifference price by $u^{(\rho)}$ and $h^{(\rho)}$ respectively. We denote by $h^{(1)}$ the limit of $h^{(\rho)}$ as the market converges to the complete one (we use the same notation for $\rho^2 \to 1$ and $\rho^2 \to -1$).

Proposition 9 *In the perfectly correlated case, and under the assumption that the excess return per unit of risk is the same for both the traded and the non-traded asset, i.e. when* $\rho^2 \to 1$ *and*

$$\frac{b(y,t)}{a(y,t)} = \rho\frac{\mu}{\sigma}, \tag{30}$$

the buyer's early exercise indifference price converges to the arbitrage free American price solving the variational inequality

$$min(-h_t^{(1)} - \frac{1}{2}a^2(y,t)h_{yy}^{(1)},\ h^{(1)} - g(y)) = 0. \tag{31}$$

Proof. The key ingredient of the proof comes from the stability properties of the viscosity solutions of (22). We present the proof for $\rho \to 1$ since the case $\rho \to -1$ follows along similar arguments. To this end, we observe that in order to obtain the limit, as $\rho \to 1$, of $h^{(\rho)}$ it suffices to pass to the limit in the pricing equality (7), rewritten below for convenience

$$V(x,t) = u^{(\rho)}(x - h^{(\rho)}(y,t), y, t). \tag{32}$$

The first step is to define the limit of $u^{(\rho)}$. To this end, we observe that, in the limit and under (30), the HJB equation (22) satisfied by $u^{(\rho)}$ converges to

$$\min(-\hat{u}_t - \max_{\pi}(\frac{1}{2}\sigma^2\pi^2\hat{u}_{xx} + \pi(\sigma a(y,t)\hat{u}_{xy} + \mu\hat{u}_x)) \qquad (33)$$
$$-\mathcal{L}\hat{u},\ \hat{u} - V(x+g(y),t)) = 0,$$

with $\hat{u}(x,y,T) = -e^{-\gamma(x+g(y))}$. Following the arguments in Davis and Zariphopoulou (1995), we deduce that the above problem has a unique viscosity solution in the class of functions that are concave and increasing in x and, bounded in y. The stability results of viscosity solutions will then yield that $u^{(\rho)} \to \hat{u}$ locally uniformly. Next, we observe that a candidate solution can be constructed for (33). In fact, let

$$\bar{u} = -e^{-\gamma x}e^{v(y,t)}$$

with $v(y,T) = g(y)$. Direct calculations in (26) yield that $\bar{u}$ is a solution to (33) provided that v solves the variational inequality

$$\min(-v_t - \frac{1}{2}a^2(y,t)v_{yy},\ v - g(y)) = 0 \ .$$

Standard results in linear optimal stopping problems yield that the above equation has a unique smooth solution. Therefore, $\bar{u}$ is smooth and thus a viscosity solution of (33). The uniqueness result of viscosity solutions implies that $\hat{u} = \bar{u}$. Passing to the limit in (32), using the form of $\hat{u}$ and the above equation we conclude.

4 Early exercise indifference prices and optimal stopping

In complete markets, the arbitrage free theory yields prices of early exercise claims as solutions of optimal stopping problems of the (discounted) expected derivative payoff under the risk neutral measure (see Musiela and Rutkowski (1997)). In this sense, one may obtain the price of an American claim by solving its European counterpart with random exercise time, aiming at the maximal price over all stopping times in the relevant filtration. In incomplete markets, such a representation would be naturally desirable but, to our knowledge, is still lacking. In what follows we show that early exercise indifference prices preserve this property and can be written as solutions of an optimal stopping problem with payoff given by its indifference price European counterpart.

Proposition 10 *The early exercise indifference price satisfies, for* $(y,t) \in R \times [0,T]$,

$$h(y,t) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \left(-\frac{1}{\gamma(1-\rho^2)} \ln E_{\tilde{\mathbb{P}}}(e^{-\gamma(1-\rho^2)g(Y_\tau)}/Y_t = y) \right) \qquad (34)$$

$$= \sup_{\tau \in \mathcal{T}_{[t,T]}} H(Y_\tau, \tau)$$

where H is the European indifference price

$$H(y,t) = -\frac{1}{\gamma(1-\rho^2)} \ln E_{\tilde{\mathbb{P}}}(e^{-\gamma(1-\rho^2)g(Y_T)}/Y_t = y)$$

(cf. (17)). The pricing measure $\tilde{\mathbb{P}}$, given in (8), is a martingale measure with the minimal entropy with respect to the historical one.

Proof. In order to show the results, it suffices to show that the candidate function

$$\hat{h}(y,t) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \left(-\frac{1}{\gamma(1-\rho^2)} \ln E_{\tilde{P}}(e^{-\gamma(1-\rho^2)g(Y_\tau)}/Y_t = y) \right) \tag{35}$$

solves the same variational inequality as h and satisfies the same terminal condition. We will then conclude using the uniqueness of viscosity solutions.

To this end, we rewrite $\hat{h}$ as

$$\hat{h}(y,t) = -\frac{1}{\gamma(1-\rho^2)} \ln f(y,t)$$

with

$$f(y,t) = \inf_{\tau \in \mathcal{T}_{[t,T]}} E_{\tilde{\mathbb{P}}}(e^{-\gamma(1-\rho^2)g(Y_\tau)}/Y_t = y)$$

with $\tilde{\mathbb{P}}$ defined in (8) and Y solving (2). We recall the Girsanov' s theorem which yields that, under the measure $\tilde{\mathbb{P}}$, the process

$$\tilde{W}_s = W_s + \rho \frac{\mu}{\sigma} s$$

is a standard Brownian motion and that, under $\tilde{\mathbb{P}}$, the dynamics of Y are given by

$$dY_s = (b(Y_s, s) - \rho \frac{\mu}{\sigma} a(Y_s, s))ds + a(Y_s, s)d\tilde{W}_s. \tag{36}$$

Classical results from the theory of optimal stopping imply that f solves the obstacle problem

$$\max(-f_t - \frac{1}{2}a^2(y,t) f_{yy} - \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right) f_y,\ f - e^{-\gamma(1-\rho^2)g(y)}) = 0.$$

Then the function $\hat{h}$ solves

$$\max(\gamma(1-\rho^2)(\hat{h}_t + \frac{1}{2}a^2(y,t)\hat{h}_{yy} + (b(y,t) - \rho\frac{\mu}{\sigma}a(y,t))\hat{h}_y - \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)\hat{h}_y^2,$$

$$e^{-\gamma(1-\rho^2)\hat{h}} - e^{-\gamma(1-\rho^2)g(y)}) = 0.$$

Taking into account that $\gamma > 0$ and $\rho^2 < 1$, the above variational inequality yields

$$-\hat{h}_t - \frac{1}{2}a^2(y,t)\hat{h}_{yy} - (b(y,t) - \rho\frac{\mu}{\sigma}a(y,t))\hat{h}_y + \frac{1}{2}\gamma(1-\rho^2)a^2(y,t)\hat{h}_y^2 \geq 0$$

and

$$\hat{h} \geq g.$$

Combining the above inequalities we deduce that $\hat{h}$ solves (23). The terminal conditions for h and $\hat{h}$ are easily verified from the properties of h and the definition of $\hat{h}$.

We conclude this section by providing a probabilistic representation of the limiting prices as $\gamma \to 0$ and as $\rho^2 \to 1$. We recall that the limits $h_0 = \lim_{\gamma\to 0} h^{(\gamma)}(y,t)$ and $h^{(1)}(y,t) = \lim_{\rho^2\to 1} h^{(\rho)}(y,t)$ solve the variational inequalities

$$\min(-h_{0,t} - \tilde{\mathcal{L}}h_0 \ , \ h_0 - g(y)) = 0$$

and

$$\min(-h_t^{(1)} - \frac{1}{2}a^2(y,t)h_{yy}^{(1)}, \ h^{(1)} - g) = 0$$

with $h_0(y,T) = h^{(1)}(y,T) = g(y)$. The results below follow from classical representation results of classical optimal stopping problems (see, for example, Musiela and Rutkowski (1997)).

Corollary 11 *As $\gamma \to 0$, the buyer's early exercise indifference price converges to the solution of the optimal stopping problem*

$$h_0(y,t) = \sup_{\tau\in\mathcal{T}_{[t,T]}} E_{\tilde{\mathbb{P}}}(g(Y_\tau)/Y_t = y)$$

where $\tilde{\mathbb{P}}$ is the martingale measure given in (8).

Corollary 12 *As $\rho^2 \to 1$ and under (30), the buyer's early exercise indifference price converges to the solution of the optimal stopping problem*

$$h^{(1)}(y,t) = \sup_{\tau\in\mathcal{T}_{[t,T]}} E_{\mathbb{P}^*}(g(Y_\tau)/Y_t = y)$$

where $\mathbb{P}^*$ is the risk neutral martingale measure of the nested Black and Scholes model.

5 Approximation schemes and numerical results

The purpose of this section is to construct a class of approximation schemes for the quasilinear variational inequality (23) whose solutions yield the early exercise indifference prices. A byproduct of the work herein and, in a sense, our overall goal is to develop useful insights for a general computational approach for (fully) nonlinear problems arising in incomplete market models. Generally speaking, such models give rise to fully nonlinear equations whose solutions represent the price of a claim, the optimal investment strategy or the risk monitoring policy. Due to market incompleteness, these equations are degenerate and thus do not have in general smooth solutions. Methodologies based on classical arguments need then to be modified and alternative criteria to be used. As it was discussed in Section 2, a class of solutions that seems to have all necessary properties for the unique characterization of solutions and the uniform convergence of numerical approximations are the so called *viscosity solutions* (see Definition 14 below).

Using the concept of viscosity solutions, in particular, the stability property, the general theory of Barles and Souganidis (1991) provides a framework for proving the uniform convergence of numerical schemes, (see also Crandall and Lions (1991) and Souganidis (1985)). We refer to Zariphopoulou (2001) and the references therein for a thorough discussion of the convergence of numerical schemes with applications to finance. See also Barles (1997), Barles et al. (1995), Hodder et al. (2001), and Tourin and Zariphopoulou (1994).

As the presentation below indicates, schemes that are *stable, consistent* and *monotone* converge to the solution of the associated equation provided that the latter has a *unique* viscosity solution. To establish these three properties and the uniqueness of viscosity solutions is not in general a formidable task, especially for value functions that, at least in a Markovian framework, are expected to be the unique solutions of their HJB equations. However, the construction of such schemes is not always straightforward and a general methodology is still lacking. Given the plethora of nonlinear equations that arise in optimization problems in asset pricing and derivative valuation, it is highly desirable to gain some insights for the construction of such schemes that take into account the specific characteristics of the underlying pricing mechanism and valuation principles.

We start with reviewing the definition of viscosity solutions and the main results of Barles and Souganidis.

Examples of nonlinear problems that can be coherently studied in the viscosity sense are the degenerate elliptic equation,

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \ , \tag{37}$$

the fully nonlinear equation, such as the Isaac's equation

$$\sup_{\alpha} \inf_{\beta} \left\{ \mathcal{L}^{\alpha,\beta} \right\} = 0 \tag{38}$$

where for each α, β in some index set, $\mathcal{L}^{\alpha,\beta}$ is a linear elliptic operator

$$\mathcal{L}^{\alpha,\beta}u = -\sum_{i,j=1}^{n} a_{ij}^{\alpha,\beta}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i^{\alpha,\beta}(x)u_{x_i} - f^{\alpha,\beta}(x),$$

and the obstacle problem

$$\min \left\{ G(x, u, Du, D^2u), u(x) - g(x) \right\} = 0, \tag{39}$$

when the equation $G = 0$ has itself unique solutions. For more examples, see Oberman (2004). All these equations can be written as

$$F(x, u, Du, D^2u) = 0$$

for some possibly non-smooth function $F : R^n \times R \times R^n \times \mathcal{N}^n \to R$, where $\mathcal{N}^n$ is the space of symmetric $n \times n$ matrices. Allowing for *non-smoothness* of the operator F is an important advantage of the approach because it allows for operations of maximum and minimum, as in (38). It also allows for possibly discontinuous operators, which in turn permits the inclusion of the boundary conditions as part of the equation.

A fundamental requirement for existence and uniqueness of viscosity solutions that the equation $F(x, r, p, X)$ is (degenerate) *elliptic*, i.e.

$$F(x, r, p, X) \leq F(x, s, p, Y) \tag{40}$$

for $r \leq s$ and $Y \leq X$ where $Y \leq X$ means that $Y - X$ is a non-negative definite symmetric matrix. A less general definition of ellipticity is that the matrix $D_{x_i x_j} F$ is positive semi-definite, however the definition (40) also allows F to be non-differentiable. Notice that we include parabolic equations in this definition, by taking the t variable to be part of the x term, and then getting an equation which is degenerate in the t variable.

The definition of viscosity solutions is motivated by the *comparison principle* which holds when the equation $F = 0$ has a unique solution. We give a particular example of the comparison principle in the parabolic case with F continuous.

Definition 13 (Comparison Principle) *Let u and v be uniformly continuous solutions of*

$$u_t - F(t, x, u, Du, D^2u) = 0, \qquad \text{for } t \in [0, T) \text{ and } x \text{ in } R^n$$

where F is continuous and (degenerate) elliptic. If $u(x, T) \leq v(x, T)$ then

$$u \leq v \text{ for } t \in [0, T] \text{ and } x \in R^n \ .$$

Definition 14 (Continuous Viscosity solutions) *The bounded, uniformly continuous function u is a viscosity solution of*

$$F(x, u, Du, D^2u) = 0 \text{ in } \Omega$$

where F is a continuous function satisfying (40) *if and only if, for all $\phi \in C^2(\Omega)$, if $x_0 \in \Omega$ is a local maximum point of $u - \phi$, one has*

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0 \ ,$$

and, for all $\phi \in C^2(\Omega)$, if $x_0 \in \Omega$ is a local minimum point of $u - \phi$, one has

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \ .$$

To allow for the inclusion of boundary conditions as part of the equation, we need to extend our definitions to allow for discontinuous equations, and sub or super solutions.

We recall the notions of the *upper semicontinuous* (usc) and *lower semicontinuous* (lsc) envelopes of a function $z : C \to R^n$, where C is a closed subset of R^n. These are

$$z^*(x) = \limsup_{\substack{y \to x \\ y \in C}} z(y) \qquad \text{and} \qquad z_*(x) = \liminf_{\substack{y \to x \\ y \in C}} z(y) \ .$$

We now set the equation in $\bar{\Omega}$ instead of Ω and include the boundary conditions as follows

$$G(x,u,p,X) = \begin{cases} F(x,u,p,X) & \text{if } x \in \Omega, \\ B(x,u,p,X) & \text{if } x \in \partial\Omega \ , \end{cases}$$

where $B(x,u,p,X)$ clearly corresponds to a (differential) description of the boundary conditions.

Definition 15 (Discontinuous Viscosity Solutions) *A locally bounded upper semicontinuous (usc) function u is a viscosity subsolution of the equation*

$$G(x,u,Du,D^2u) = 0 \ \textit{on} \ \bar{\Omega}$$

if and only if, for all $\phi \in C^2(\bar{\Omega})$, if $x_0 \in \bar{\Omega}$ is a maximum point of $u - \phi$, one has

$$G_*(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0 \ .$$

A locally bounded lower semicontinuous (lsc) function u is a viscosity supersolution of the equation

$$G(x,u,Du,D^2u) = 0 \ \textit{on} \ \bar{\Omega}$$

if and only if, for all $\phi \in C^2(\bar{\Omega})$, if $x_0 \in \bar{\Omega}$ is a local minimum point of $u - \phi$, one has

$$G^*(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0 \ .$$

A viscosity solution is a function whose usc and lsc envelopes are respectively viscosity sub and super solutions of the equation.

A *numerical scheme* approximating the equation

$$G(x,u,Du,D^2u) = 0 \text{ on } \bar{\Omega} \tag{41}$$

is written in the following way

$$S(\delta, x, u^\delta(x), u^\delta) = 0 \text{ on } \bar{\Omega}$$

where S is a real-valued function defined on $R^+ \times \bar{\Omega} \times R \times B(\bar{\Omega})$ where $B(\bar{\Omega})$ is the set of bounded functions defined pointwise on Ω. We require that the scheme satisfy the following conditions.

i) Stability: For any $\delta > 0$, the scheme has a solution u^δ. Moreover, u^δ is uniformly bounded, i.e. there exists a constant $C > 0$ such that

$$-C \le u^\delta \le C$$

for any $\delta > 0$.
ii) Consistency: For any smooth function ϕ, one has:

$$\liminf_{\substack{\delta\to 0\\ y\to x\\ \xi\to 0}} \frac{S(\delta, y, \phi(y)+\xi, \phi+\xi)}{\delta} \ge G_*(x, \phi(x), D\phi(x), D^2\phi(x))$$

and

$$\limsup_{\substack{\delta\to 0\\ y\to x\\ \xi\to 0}} \frac{S(\delta, y, \phi(y)+\xi, \phi+\xi)}{\delta} \le G^*(x, \phi(x), D\phi(x), D^2\phi(x)) \ .$$

iii) Monotonicity:

$$S(\delta, x, t, u) \le S(\delta, x, t, v) \text{ if } u \le v$$

for any $\delta > 0$, $x \in \bar{\Omega}$, $t \in R$ and $u, v \in B(\bar{\Omega})$.
iv) Strong Comparison Result: If u is an upper semicontinuous viscosity subsolution of the equation (41) and if v is a lower semicontinuous viscosity supersolution of the equation (41), then

$$u \le v \text{ on } \bar{\Omega} \ .$$

We have the following theorem (see Barles and Souganidis (1991)).

Theorem 16 *Under the above assumptions, the solution u^δ of the scheme converges, uniformly on each compact subset of $\bar{\Omega}$, to the unique viscosity solution of the equation.*

We note that a key requirement of a consistent scheme is monotonicity, which is equivalent to saying that the comparison principle holds for the scheme.

Moreover, it is worth noticing that at points where the function G is continuous, as is usually the case for the 'interior points' of Ω, the consistency requirement is equivalent to

$$\frac{S(\delta, x, \phi(x), \phi)}{\delta} \to G(x, \phi(x), D\phi(x), D^2\phi(x)) \text{ as } \delta \to 0,$$

uniformly on compact subsets of Ω, for any smooth function ϕ. This is a more standard formulation and can be checked in an easier manner.

Finally, although the scheme $S(\delta)$ is written for general maps, we will be dealing with *finite difference schemes*, which are maps from a grid to a grid. We assume implicitly that the grid functions are injected into functions in the domain, and that the injection is itself monotone. This is accomplished simply by linear interpolation from the grid onto the domain.

To facilitate the exposition and to help the unfamiliar reader to gain some insights we provide two examples.

We recall that the idea of approximating linear partial differential equations by finite difference schemes, and thereby proving convergence of consistent, monotone approximation schemes goes back to the seminal (1928) paper of Courant, Freidrichs and Lewy, (translated and reprinted in Courant, Freidrichs and Lewy (1967)). Their purpose was to derive existence results for the original problem by constructing finite-dimensional approximations of the solutions, for which the existence was clear, and then showing convergence as the dimension grows. Although the aim was not numerical, the ideas presented in this paper played a fundamental role in numerical analysis.

We use standard notation from numerical analysis with h_j^n denoting an approximation of $h(n\,dt, j\,dy)$ for $n \in N$ and $j \in Z$, where dt and dy are respectively the mesh sizes in t and y.

The first example is taken from Courant, Freidrichs and Lewy (1967), the heat equation, and the second is a linear first order equation. We note that, in what follows, we work with the *forward in time* variation of the involved terminal value problems.

Example 17 (The heat equation) *We consider the classical explicit scheme for approximating the heat equation in one dimension, namely*

$$\begin{cases} h_t - h_{yy} = 0 & in\ R \times (0,T) \\ h = h_0 & on\ R \times \{0\}\ . \end{cases}$$

Taking centered finite differences for the h_{yy} terms and forward differences for the h_t term gives the scheme

$$H_j^{n+1} - H_j^n - \frac{dt}{(dy)^2}\left(H_{j+1}^n - 2H_j^n + H_{j-1}^n\right) = 0\ .$$

The above equation should now be read as

$$S\left(\delta, (j\,dy, (n+1)\,dt), H_j^{n+1}, \{H_i^n\}_{n,i}\right) = 0\ .$$

In other words, the above equation is the equation of the scheme at the point $(j\,dy, (n+1)dt)$ with the role of the variable H^δ being played by $\{H_i^n\}_{n,i}$ (although only the nearest neighbors are needed). Taking dy and dt to be given functions of δ fixes the discretization as a function of δ. Consistency holds by the choice of discretization.

We take $dy = \delta$, $dt \leq \delta^2/2$ so that the Courant-Friedrichs-Lax (CFL) condition

$$dt \leq \frac{(dy)^2}{2}$$

holds. Then we see that in the solution map S for H_j^{n+1}, the coefficients of H_i^n, $i = j$, $j \pm 1$ are all non-positive. Thus, as a map from grid points to grid points, S is non-increasing. Stability is easily seen to hold. Moreover, solutions exist since the scheme is explicit and the uniform bound will be assured by monotonicity.

Example 18 (A first order linear equation) *We next consider the upwind explicit scheme for the linear first order equation in one dimension*

$$\begin{cases} h_t - b(y,t)h_y = 0 & \text{in } R \times (0,T] \\ h = h_0 & \text{on } R \times \{0\} \ . \end{cases}$$

We discretize the spatial term by the upwind method,

$$b(y)h_y = \begin{cases} b(y,t)\dfrac{(H_{j+1} - H_j)}{dy} & \text{if } b(y,t) \geq 0 \\ b(y,t)\dfrac{(H_j - H_{j-1})}{dy} & \text{if } b(y,t) < 0 \end{cases}$$

so that the scheme becomes

$$H_j^{n+1} - H_j^n + \frac{dt}{(dy)^2}\left(|b_j^n|H_j^n - | \pm b_j^n|^+ H_{j\pm1}^n\right) = 0$$

where $|y|^+ = \max(y,0)$*. Then, letting* $dy = \delta$*, the scheme is monotone if the CFL condition*

$$dt \leq \max_{(y,t)} \frac{dy}{|b(y,t)|}$$

holds.

From now on, we restrict our analysis to *explicit* solution schemes, and write (with a slight abuse of notation)

$$H_j^{n+1} = S\left(\delta, (j\,dy, n\,dt), \{H_i^n\}_{n,i}\right) = 0 \ .$$

The schemes we will be dealing with will only use the nearest neighbors, so, in fact, we can write

$$H_j^{n+1} = S\left(\delta, (j\,dy, n\,dt), (H_{j-1}^n, H_j^n, H_{j+1}^n)\right) = 0 \ .$$

Here dy and dt are understood to be implicitly defined as function of δ, so there is only one parameter in the scheme.

In the previous examples, we used the fact that the linear map $S : R^N \to R^N$ is monotone if the coefficients are all positive, where S is regarded as mapping grid values to grid values, and N is a large number corresponding to the number of grid points.

For nonlinear equations, we must consider nonlinear solution maps. The following result follows from the mean value theorem.

Proposition 19 (Monotone Maps) *If the map* $S : R^N \to R^N$ *is differentiable, and the gradient* $DS \geq 0$*, then the map is* monotone*, i.e.* $X \leq Y$ *implies* $S(X) \leq S(Y)$*.*

When there are free boundaries or obstacles, as is the case with early exercise contracts, the equation is only *piecewise differentiable*. Then, naturally, the solution scheme may be only piecewise differentiable as a map. It is desirable to have a useful characterization of monotone maps arising from these circumstances.

We simply note that if $S_1, S_2 : R^N \to R^N$ are monotone maps then the maps $\max(S_1, S_2)$ and $\min(S_1, S_2)$ (where the maximum and minimum are taken componentwise) are also monotone. This is trivial to check, and more generally, a maximum or minimum over collections of monotone maps is still monotone.

We are now ready to present our scheme. We first provide a convergent scheme in the case of no free boundary. Next, we include the early exercise and prove convergence of the modified scheme. We note that as a particular case, we have provided a provably convergent, explicit scheme for American options. Finally, we introduce artificial boundary conditions to take into account the finite computational domain.

The scheme we introduce is explicit: no solution of algebraic equations is required, and thus computations may be implemented easily, and solutions computed very quickly. Our computations ran in less than one second in MATLAB, and required less than one hundred lines of code, including diagnostics and plotting commands. We also provide an explicit CFL condition, which dictates the size of the time step required for convergence.

To this end, we first give an *explicit, finite difference* scheme for the equation

$$h_t = Dh_{yy} + Ah_y + Bh_y^2 \tag{42}$$

where $D = D(y,t) \geq D_0 > 0$, and where $A = A(y,t)$, $B = B(y,t)$ in the interior of the domain. We ignore for now the boundary conditions of early exercise and the artificial boundary conditions imposed by a finite computational grid. We write the equation in forward time.

We next set

$$h_{yy} = \frac{1}{dy^2}\left(H_{j-1} - 2H_j + H_{j+1}\right) + O(dy^2), \tag{43a}$$

$$Ah_y = \begin{cases} \dfrac{A}{dy}(H_{j+1} - H_j) + O(dy) & \text{if } A \geq 0 \\ \dfrac{A}{dy}(H_j - H_{j-1}) + O(dy) & \text{if } A < 0, \end{cases} \tag{43b}$$

$$Bh_y^2 = \begin{cases} \dfrac{B}{dy^2}\left(\left(|H_{j+1} - H_j|^+\right)^2 + \left(|H_j - H_{j-1}|^-\right)^2\right) + O(dy) & \text{if } B \geq 0 \\ \dfrac{B}{dy^2}\left(\left(|H_{j+1} - H_j|^-\right)^2 + \left(|H_j - H_{j-1}|^+\right)^2\right) + O(dy) & \text{if } B < 0 \ . \end{cases} \tag{43c}$$

The map S is defined by

$$H_j^{n+1} = H_j^n + dt\left(Dh_{yy} + Ah_y + Bh_y^2\right) \tag{44}$$

where the derivatives appearing on the right hand of (44) are shorthand for the finite difference expressions in (43) evaluated at time $n\,dt$.

Theorem 20 *The finite difference scheme for the partial differential equation* (42) *given by* (43) *converges, provided,*

$$dt \le \frac{dy^2}{2}\left(\max_j\{D + .5\,dy|A| + |B|(|H_{j+1} - H_j|^+ + |H_j - H_{j-1}|^-)\}\right)^{-1}. \tag{45}$$

Proof. We first note that the third term in condition (45) depends on the values H_j^n at the given time, so in general the time step may vary. We show below, that in the constant coefficient case, the time step can be determined initially and remain fixed. Consistency follows from the discretization (43). The existence part of the stability requirement is satisfied by the fact that the scheme is explicit, the boundedness part follows from monotonicity (below), and the fact that constants are solutions.

To check monotonicity of the scheme, we can, by the symmetry of (43), assume without loss of generality that $A, B \ge 0$. For this it suffices to check that the partial derivatives of the explicit map with respect to each grid point are nonnegative.

First, checking with respect to H_j gives

$$\frac{\partial S}{\partial H_j} = 1 - 2D\frac{dt}{dy^2} - A\frac{dt}{dy} - 2B\frac{dt}{dy^2}\left(|H_{j+1} - H_j|^+ + |H_j - H_{j-1}|^-\right)$$

which is nonnegative if

$$dt \le \frac{dy^2}{2}\left(\max_j\left\{D + .5\,dy|A| + |B|(|H_{j+1} - H_j|^+ + |H_j - H_{j-1}|^-)\right\}\right)^{-1}$$

which reduces to the CFL condition (45).

We now check monotonicity in the other variables:

$$\begin{aligned}\frac{\partial S}{\partial H_{j-1}} &= \frac{dt}{dy^2}\left(D + 2B|H_j - H_{j-1}|^-\right) \ge 0\\ \frac{\partial S}{\partial H_{j+1}} &= \frac{dt}{dy^2}\left(D + 2B|H_{j+1} - H_j|^+ + A\,dy\right) \ge 0.\end{aligned}$$

Therefore the scheme is unconditionally monotone in the neighboring nodes.

Corollary 21 *In the constant coefficient case $D(y,t) = D$, $B(y,t) = B$ and $A(y,t) = A$, the finite difference scheme for the partial differential equation* (42)*, given by* (43)*, converges provided that*

$$dt \le \frac{dy^2}{2}\frac{1}{D + dy|A| + dyM|B|} \tag{46}$$

where $M = \max_y |h_y(y,0)|$.

Proof. In the case where the coefficients are constants, the solutions to (42) are smooth, since the equation is uniformly elliptic. Therefore,

$$|H_{j+1} - H_j|^+ + |H_j - H_{j-1}|^- = dy(|h_y|^+ + |h_y|^-) + O(dy^2)$$

and, in fact, h_y changes sign only near a local extremum, where $h_y = O(dy)$.

Furthermore, standard maximum principle techniques (Berstein estimates) show that the maximum of $|h_y|$ does not grow, and so the time step may be determined from the initial data.

Up until now, our discussion has been for equations of the form (42). We would like to consider American contracts, and since in practice we have only a finite computational domain, we need to impose boundary conditions.

Including the early exercise component gives an equation of the concise form

$$\min\left\{-h_t + Dh_{yy} + Ah_y + Bh_y^2, u - g(y)\right\} = 0 \tag{47}$$

which is an obstacle problem and corresponds to the American Option with exercise value $g(y)$.

Extending the scheme from (42) to (47) involves no extra machinery. The scheme is given by the map

$$h_j^{n+1} = \max\{h_j^n + dt\left(Dh_{yy} + Ah_y + Bh_y^2\right), g(j\,dy)\} \tag{48}$$

where again the derivatives appearing on the right hand of (48) are shorthand for the finite difference expressions in (43).

We simply note that the scheme for the (trivial) equation $h - g(y) = 0$ is monotone, and that we also have a monotone scheme for the partial differential equation. As observed in Proposition 19, minima of monotone maps are monotone, so the scheme is monotone. Further, since both the equation (in forward time) and the scheme are written as a maximum of two equations, consistency follows automatically.

We next discuss a non-rigorous heuristic for introducing artificial boundary conditions to allow for a finite computational domain.

Restricting the computational domain to a finite grid requires introducing *artificial boundary conditions* at the boundary. In principle, for large enough domains, the boundary conditions would not affect the values in the middle. For example, examining the Gaussian kernel shows that this statement is true up to an exponentially small error for the heat equation. Taking into account a finite shift in the domain due to drift terms, we can see that it holds true for an equation with bounded drift as well. Nevertheless, it is preferable to make a good choice of boundary conditions. Based on the heuristic that solutions of (47) are approximately linear as $|y| \to \infty$, (when they are not equal to $g(y)$), we make the choice of boundary condition

$$h_{yy} = 0 \quad \text{if} \quad h(y) > g(y). \tag{49}$$

Furthermore, in the exercise region, we impose boundary conditions

$$h = g \quad \text{if} \quad h(y) \leq g(y). \tag{50}$$

The condition (50) is trivial, but for implementations it ensures that the solution does not "lift off" the free boundary due to artificial boundary data.

We remark that the boundary conditions (49) may appear somewhat unorthodox, given that standard treatments of linear second order partial differential equations consider only zeroth order or first order boundary conditions. Further, in the European case, this is natural, since on one side of the domain, solutions are approximately linear, so the usual Neumann condition $h_y = 0$ is not suitable. The equation we now solve is:

$$\begin{cases} \min\left\{-h_t + Dh_{yy} + Ah_y + Bh_y^2, h - g(y)\right\} = 0 & \text{if } y \in \Omega \\ h_{yy} = 0 & \text{if } y \in \partial\Omega \text{ and } h(y) > g(y) \\ h = g & \text{if } y \in \partial\Omega \text{ and } h(y) \leq g(y)\ . \end{cases}$$

We continue with the implementation of the numerical scheme and give computational examples.

To begin, we introduce the change of variables $y \to \log y$. Now, a uniform grid in the new coordinates gives more accuracy near $y = 1$, and eliminates the possible singularity as $y \to 0$. For the purposes of computations, we set $a = a_0 y, b = b_0 y$ which gives constant coefficients in the new coordinates, and a uniformly elliptic equation. So the equation at hand, (23) becomes

$$\min\left\{-h_t - \frac{1}{2}a_0^2 h_{yy} - \left(b_0 - \rho\frac{\mu}{\sigma}a_0 - \frac{a_0}{2}\right)h_y + \frac{1}{2}a_0^2\gamma(1-\rho^2)h_y^2, h - g(e^y)\right\} = 0.$$

Here the *Sharpe ratio* μ/σ is a constant, ρ is the correlation of the nontraded asset with the traded one, b, a are the drift and volatility of the nontraded asset and $\gamma > 0$ is the risk aversion.

We set the terminal data to be that of a *put*, $g(y) = (K - y)^+$, to give

$$h(y, T) = (K - \exp(y))^+$$

and impose the artificial boundary conditions (49) and (50).

The condition for convergence of the numerical scheme (45) becomes

$$dt \leq \frac{dy^2}{a_0^2 + dy(|b_0 - \rho\frac{\mu}{\sigma}a_0 - .5a_0| + a_0^2\gamma(1-\rho^2)M)}\ ,$$

which could be simplified to

$$dt \leq \frac{1}{3}\min\left(\frac{dy^2}{a_0^2}, \frac{dy}{|b_0 - \rho\frac{\mu}{\sigma}a_0 - .5a_0|}, \frac{dy}{a_0^2\gamma(1-\rho^2)M}\right),$$

where $M = \max_y |h_y(y, 0)|$.

We make some general observations about trends in the value of the option, using the notation of (42). The value is a decreasing function of the coefficient of the nonlinear term, B, since $h_y^2 > 0$. The initial data is convex, outside of the free boundary, so initially the values are increasing in the diffusion coefficient

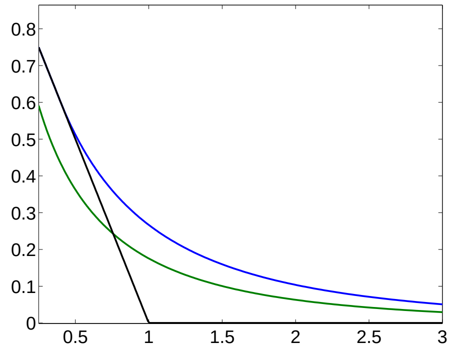

Figure 1: Comparison of European and American options after time 1, with initial data and Sharpe ratio equal to 1, $a_0 = 1, b_0 = .8, \rho = .1, \gamma = 1$

D. With the initial data of a put, the terminal data is decreasing, and remains so. Therefore, the value is decreasing as a function of the drift coefficient A. We note that when $A < 0$ the solution lifts off the free boundary instantaneously.

We implemented the code in MATLAB, using approximately 400 nodes, and domain the interval $[e^{-2}, e^2]$. In the first run, we compare the European and American option values with Sharpe ratio equal to 1, $a_0 = 1, b_0 = 0.8, \rho = .1, \gamma = 1$. The results are shown in Figure 1. The delta of the claim, and the position of the free boundary as a function of time are shown in figure 2. The position of the free boundary jumps one grid point.

Now, fixing the rest of the data, we vary γ, taking $\gamma = 0.0, 1, 5$, and keeping the other values fixed at Sharpe ratio equal to 1, $a_0 = .5, b_0 = 0.55, \rho = .1, \gamma = 1$. The solution is decreasing in γ and the position of the free boundary is increasing in γ, as shown in Figure 3.

Next, fixing the rest of the data, we vary ρ, taking $\rho = .1, .5, .8, .9$, and keeping the other values fixed at Sharpe ratio equal to 1, $a_0 = .5, b_0 = 0.55, \gamma = 1$. The solution is increasing in ρ and at $\rho = .9$ the solution moves off the free boundary, as shown in Figure 4.

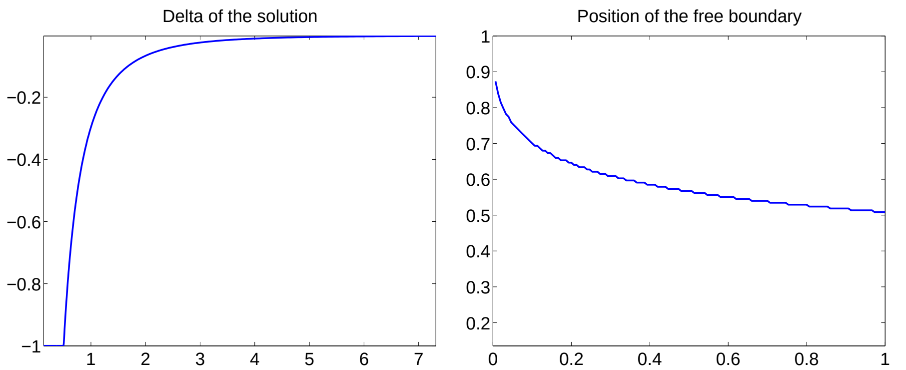

Figure 2: Delta of the option at time 1, and position of the free boundary as a function of time, Sharpe ratio equal to 1, $a_0 = 1, b_0 = .8, \rho = .1, \gamma = 1$

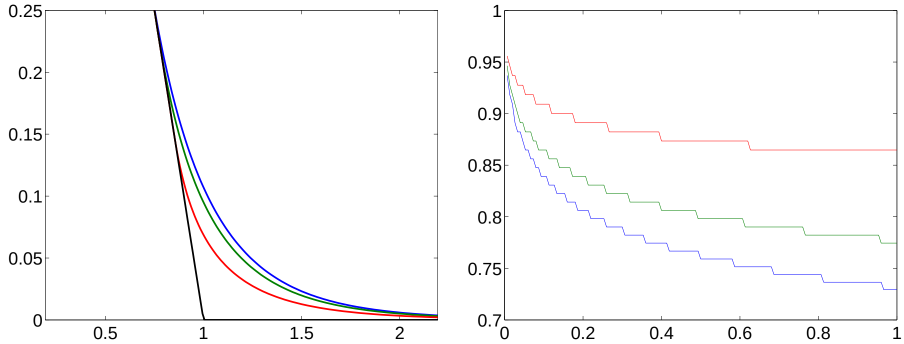

Figure 3: Value of the option (increasing as a function of γ), and position of the free boundary as a function of time (decreasing as a function of γ), for $\gamma = .0, 1, 5$, with Sharpe ratio equal to 1, $a_0 = .5, b_0 = .55, \rho = .1$

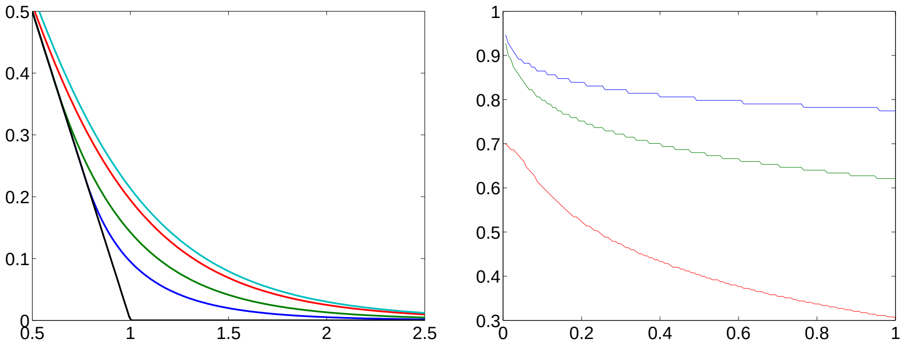

Figure 4: Value of the option (increasing as a function of ρ), and position of the free boundary as a function of time (decreasing as a function of ρ), for $\rho = .1, .5, .8, .9$, with Sharpe ratio equal to 1, $a_0 = .5, b_0 = .55, \gamma = 1$

6 Conclusions and future research directions

In this paper we extended the utility-based valuation approach to the case of early exercise contracts written exclusively on nontraded assets. We provided a differential, probabilistic and computational characterization of the so called early exercise indifference price. In a market environment with lognormal stock dynamics and general dynamics for the nontraded asset, we showed that the buyer's indifference price is the unique solution to a quasilinear variational inequality with an obstacle term, given by the contract's payoff. We established that the early exercise indifference prices are given as solutions of optimal stopping problems of their indifference European counterparts. This robustness result highlights the universality of the valuation theory by indifference. Additional robustness results and price spreads were proved as markets become complete and for limiting values of risk aversion. We also developed a general class of numerical schemes. We built a scheme that is stable, monotone and consistent and thus converges to the unique (viscosity) solution of the quasilinear pricing variational inequality. Numerical results were provided for a range of the Sharpe ratio, the risk aversion and correlation.

Having provided a complete characterization of the early exercise price, we should focus on the specification of the risk monitoring policies. One needs to construct the indifference analogue of the arbitrage free payoff decomposition that yields, via the martingale representation theorem, the correct hedging strategy. Another important task is to explore how the classical parity between the European and American prices and, early exercise premium is extended to the utility-based pricing setting.

The most challenging question however, is to understand how indifference prices can be represented in a model-independent manner, without any specific structural assumptions. In order to produce a viable incomplete market pricing mechanism, one should aim at producing prices that can be written in terms of a constitutive analogue of the risk neutral classical theory. Arbitrage free prices are represented as (discounted) expected payoffs under the (unique) risk neutral measure. As the results herein suggest, when markets become incomplete, the utility-based valuation concept produces prices that are given in terms of a nonlinear pricing functional under a new measure, namely, the one that minimizes the relative entropy with respect to the historical one. Establishing such results under very general assumptions on the market environment and the claims is of primary importance.

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