Integral Equations Arising From the Incompressible Navier-Stokes
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Abstract
An integral equation approach for solving a fourth order partial differential equation (pde) arising from the incompressible Navier-Stokes equations will be discussed.

1. The Stream Function
The usual formulation of the incompressible Navier-Stokes equations in two dimensions
\[ u_t + uu_x + v u_y = -\nabla p + \frac{\partial}{\partial x}(\Delta u) + \frac{\partial}{\partial y}(\Delta v), \quad \nabla \cdot u = 0. \]
can be posed in terms of a stream function \( \psi \) which satisfies \( u = \psi_y \) and \( v = -\psi_x \). The stream function \( \psi \) satisfies the fourth order equation
\[ \frac{\partial}{\partial x}(\Delta \psi) - \frac{1}{2} \frac{\partial}{\partial y}(\Delta \psi) + \frac{\partial}{\partial y}(\Delta \psi) = 0. \]
The Discretization
Using both implicit and explicit methods, one can discretize the time derivative which leads to a fourth order linear elliptic pde. Regardless of the time discretization, we will always be left with a system of pdes of the form
\[ (\Delta - \alpha \Delta^2)\psi^{N+1} = G(\psi^N, \psi^{N-1}, \ldots) \]
where \( \alpha \) is some positive real number and \( \Delta \) is some function depending only on the solution at previous time steps.

A Method for Solving
The solution at each time step is written as the sum of a particular solution and a homogeneous solution where
\[ (\Delta - \alpha \Delta^2)\phi^0 = -h \quad \text{and} \quad (\Delta - \alpha \Delta^2)\phi^0 = 0. \]
Integrals can be used to solve the homogeneous problem.

2. An Integral Equation Method
An integral equation can be used to solve a pde of the form
\[ (\Delta - \alpha \Delta^2)u = 0. \]
One way to treat this problem is by decoupling the full fourth order pde into the two second order pdes
\[ 1 - \alpha \Delta)u = 0, \quad \Delta u = u. \]
However, this makes it difficult to impose the boundary condition. The other option is to form an integral equation for the fourth order pde (2), but again it is difficult to impose all the boundary data. However, I am able to impose either Neumann or Dirichlet boundary data for the full fourth order problem with an integral equation.

A solution to the pde (2) with Dirichlet boundary data \( f \) is
\[ u(x) = \int_{\Omega} \frac{1}{2\pi} \int_{\Omega} \left( -\log(|x-y|) + K_0 \left( \frac{|x-y|}{r} \right) \right) \mu(y)dydx \]
where
\[ f(x) = -\mu(x) \int_{\Omega} \left( -\log(|x-y|) + K_0 \left( \frac{|x-y|}{r} \right) \right) \phi(y)dydx. \]

Solving For the Density Function \( \mu \)
To solve the integral equations (3) and (4), we discretize \( f \) and \( \mu \) as
\[ f = (f_1, f_2, \ldots, f_N) \quad \text{and} \quad \mu = (\mu_1, \mu_2, \ldots, \mu_N). \]
Then, a linear system is setup where the \( i^{\text{th}} \) equation of the system is
\[ f_1 = \frac{\partial}{\partial \theta} \mu_1 + \sum_{k=2}^{N} K(x_1, y_1) \mu_k \Delta \theta x_k \]
where \( K \) is the kernel of the integral equation. The only troublesome point in the linear equation is when \( i = n \) as the integral operator has a singularity at the origin. This problem is resolved by using the identity
\[ \lim_{\theta \to 0} \frac{\partial}{\partial \theta} K(x, \theta) = -K(x) \]
where \( K(x) \) is the curvature of the boundary of the domain at \( x \).

3. Numerical Results

Convergence of Integral Equations
Here I will look at the rate of convergence of the integral equation. The domain taken is the unit disc and the boundary data is \( f(\theta) = 2 \cos(\theta) - 4 \sin(\theta) \). When the boundary data is Neumann, the rate of convergence is of order four. For the Dirichlet problem, the same boundary data is taken. The slope of the line of best fit is -3.6133.

Fluid Simulations
A similar scheme to that of an integral equation has been implemented to solve (1) in the unit disc. The particular solution \( \phi^0 \) is solved using a second order finite difference scheme in polar coordinates and the homogeneous solution \( \phi^0 \) is solved using a spectral code. The time derivative is discretized with forward Euler.

Example Here, the stream function with boundary data \( \frac{\partial u}{\partial n} = 1 + \cos(\theta) \) is computed. The code was run until the solution was in steady state.

Example The boundary data for this example corresponds to a boundary rotating at constant velocity with two small holes. The fluid flows in one hole at constant velocity and out the other at the same velocity. A Reynolds number of 100 is taken.

4. Future Work
- Construct an integral equation that can handle boundary conditions.
- Generalize the code to arbitrary geometries.
- Increase the order of accuracy of the particular problem.
- Increase the speed of the entire scheme using the Fast Multipole Method.

5. Conclusions
The Navier-Stokes equations in two dimensions leads to interest in studying the operator \( \Delta - \alpha \Delta^2 \). This operator can be casted as an integral equation. Although boundary data is not yet fully understood, the integral equation method does have promise in finding solutions to pdes involving this operator.

Figure 1: Rate of convergence for Neumann and Dirichlet boundary data