Math 539 :: Analytic Number Theory

Fall 2005

Lecture Notes

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December 9, 2005

First version :: December 2nd, 2005

(Lecture 1)

$$\phi(n) = \#\{1 \leq a \leq n : (a, n) = 1\}.$$

$$\frac{\phi(n)}{n} = \text{“probability” that a “random chosen” integer is relatively prime to } n.$$

$\frac{\phi(n)}{n}$ has a (limiting) distribution function for every $\alpha \in (0, 1)$.

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\phi(n)}{n} < \alpha \right\} \text{ exists.}$$

Prime Number Theorem

$$\pi(x) := \#\{p \leq x : p \text{ is prime}\} \sim \frac{x}{\log x}.$$

Riemann Asserted :: If $\zeta(s) = 0$ and $s \neq -2, -4, -6, \ldots$ then $\mathbf{Re}(s) = \frac{1}{2}$. (Also known as the Riemann Hypothesis)

We're trying to understand arithmetic *functions* $f : \mathbb{N} \to \mathbb{C}$.

Notations/Examples ::

$$
\begin{aligned}
0(n) &= 0 \quad \forall n \\
1(n) &= 1 \quad \forall n \\
e(n) &= \begin{cases} 1 \text{ if } n = 1, \\ 0 \text{ if } n > 1. \end{cases} \\
\phi(n) &= \#\{1 \le k \le n : (k, n) = 1\} \\
\sigma(n) &= \text{ sum of positive divisors of } n \\
&= \sum_{d|n} d. \\
\tau(n) &= \text{ number of positive divisors of } n \\
&= \sum_{d|n} 1. \\
\mu(n) &= \text{ (Möbius) :: } \mu(n) = \begin{cases} 0 & \text{if } n \text{ is } \underline{\text{not}} \text{ squarefree,} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct divisors.} \end{cases} \\
|\mu|(n) &= \begin{cases} 1 \text{ if } n \text{ is squarefree,} \\ 0 \text{ if not.} \end{cases} \\
\omega(n) &= \# \text{ of distinct prime divisors of } n \\
&= \sum_{p|n} 1. \\
\Omega(n) &= \# \text{ of prime factors of } n \text{, counted with multiplicity} \\
&= \sum_{p^k \| n} k.
\end{aligned}
$$

Reality Check :: $\mu(n) = (-1)^{\omega(n)} e(\Omega(n) + 1 - \omega(n))$.

We're interested in these sets of questions about arithmetic functions.

- Range of values?
- (Best case scenario) Distribution of values?
- Average value - usually $\frac{1}{x} \sum_{n \le x} f(n)$.

Also correlations between them.

Example :: $\omega(n)$

- Range :: $\{0\} \cup \mathbb{N}$.

  But $\omega(n) \le n$.

  $\omega(n) \le \tau(n) \le 2\sqrt{n}$.

  $\omega(n) \le \Omega(n) \le \frac{\log n}{\log 2}$.

- Average Value ::

$$\sum_{n \le x} \omega(n) = \sum_{n \le x} \left(\sum_{p \mid n} 1 \right) = \sum_{p \le x} 1 \left(\sum_{\substack{n \le x \\ p \mid n}} 1 \right) = \sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor .$$

Use O-notation for errors. $O(f(x))$ denotes an unspecified functions $g(x)$ satisfying $|g(x)| \le C|f(x)|$ for some $C > 0$. Thus,

$$\begin{aligned} \sum_{n \le x} \omega(n) &= \sum_{p \le x} \left(\frac{x}{p} + O(1) \right) \\ &= x \sum_{p \le x} \frac{1}{p} + \sum_{p \le x} O(1) \\ &\quad \text{(Triangle Inequality)} \\ &= x \sum_{p \le x} \frac{1}{p} + O\left(\sum_{p \le x} 1 \right). \end{aligned}$$

Turns out that

$$\sum_{p \le x} \frac{1}{p} = \log \log x + O(1).$$

Therefore,

$$\begin{aligned} \sum_{n \le x} \omega(n) &= x(\log \log x + O(1)) + O(\pi(x)) \\ &= x \log \log x + O(x) + O(\pi(x)) \\ &= x \log \log x + O(x) \end{aligned}$$

We conclude "$\omega(n)$ is $\log \log n$ on average."

$$\left(\text{Since } \sum_{n \le x} \omega(n) \sim \sum_{n \le x} \log \log n \right).$$

Notation :: $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Because

$$\sum_{n\le x} \omega(n) = x \log\log x + O(x),$$

we have

$$\frac{\sum_{n\le x} \omega(n)}{x\log\log x} = 1 + O\left(\frac{1}{\log\log x}\right).$$

Hence,

$$\lim_{x\to\infty} \frac{\sum_{n\le x} \omega(n)}{x\log\log x} = 1.$$

So,

$$\sum_{n\le x} \omega(n) \sim x\log\log x.$$

(Lecture 2)

Question :: How frequent are squarefree numbers?

$$|\mu|(n) = \mu^2(n) = \begin{cases} 1 & \text{if } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $l(n)$ denote the largest d such that $d^2|n$. We'll use

$$\sum_{d|k} \mu(d) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \;=\; e(k).$$

Therefore,

$$\begin{aligned} |\mu|(n) = \left\{\begin{matrix} 1 & \text{if } l(n) = 1 \\ 0 & \text{if } l(n) > 1 \end{matrix}\right\} &= \sum_{d|l(n)} \mu(d) \\ &= \sum_{d\colon d^2|n} \mu(d). \end{aligned}$$

Now we look at the summatory function

$$\sum_{n\le x}|\mu|(n)=\sum_{n\le x}\left(\sum_{d\colon d^2|n}\mu(d)\right)=\sum_{d\le\sqrt{x}}\mu(d)\sum_{\substack{n\le x\\ d^2|n}}1$$
$$=\sum_{d\le\sqrt{x}}\mu(d)\left\lfloor\frac{x}{d^2}\right\rfloor=\sum_{d\le\sqrt{x}}\mu(d)\left(\frac{x}{d^2}+O(1)\right)$$
$$=x\sum_{d\le\sqrt{x}}\frac{\mu(d)}{d^2}+O\left(\sum_{d\le\sqrt{x}}|\mu(d)|\right).$$

The error term (Trivially?) is

$$O\left(\sum_{d\le\sqrt{x}}1\right)=O(\sqrt{x}).$$

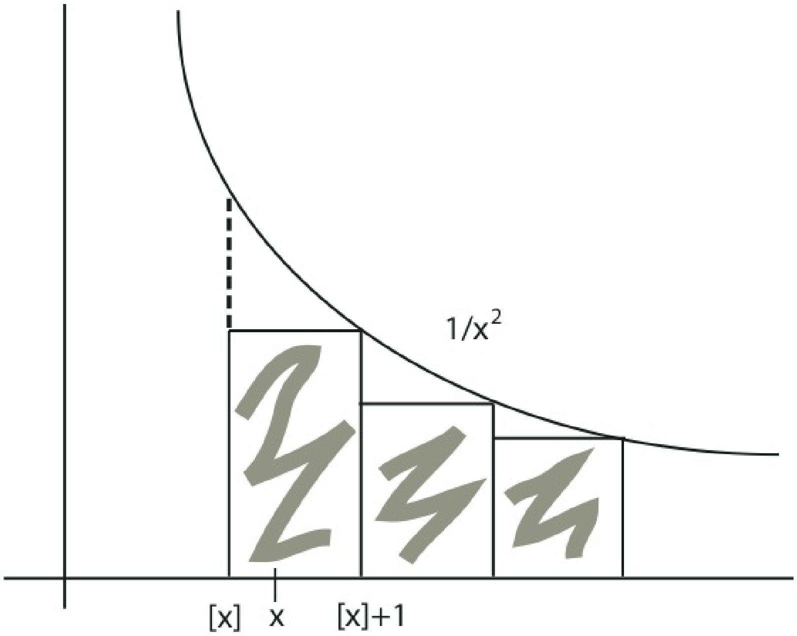
The sum in the main term converges (by comparison with the sum $\sum\frac{1}{d^2}$), and so becomes

$$x\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}-\sum_{d\ge\sqrt{x}}\frac{\mu(d)}{d^2}\right).$$

Moreover,

$$\left|\sum_{d>\sqrt{x}}\frac{\mu(d)}{d^2}\right|\le\sum_{d>\sqrt{x}}\frac{|\mu(d)|}{d^2}\le\sum_{d>\sqrt{x}}\frac{1}{d^2}.$$

Now,



By Integral comparison test, we get

$$\sum_{d>\sqrt{x}} \frac{1}{d^2} < \int_{\lfloor\sqrt{x}\rfloor}^{\infty} \frac{dt}{t^2} = -\frac{1}{t}\bigg|_{\lfloor\sqrt{x}\rfloor}^{\infty} = \frac{1}{\lfloor\sqrt{x}\rfloor}.$$

Notice that $\lfloor t\rfloor \geq \frac{t}{2}$ for $t \geq 1$. So

$$\frac{1}{\lfloor\sqrt{x}\rfloor} \leq \frac{1}{(1/2)\sqrt{x}} = O\left(\frac{1}{\sqrt{x}}\right).$$

We now have

$$\sum_{n\leq x} |\mu|(n) = x\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(\sqrt{x}).$$

(We'll evaluate $\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}$ the next time we see it.)

More generally, if $F(n) = \sum_{d|n} f(d)$, then

$$\sum_{n\leq x} F(n) = x\sum_{d\leq x} \frac{f(d)}{d} + O\left(\sum_{d\leq x} |f(d)|\right).$$

For some sequence $\{a_n\}$, using a power series as a generating function is helpful.

$$\{a_n\} \rightsquigarrow \sum_{n=0}^{\infty} a_n x^n,$$

since

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i b_j\right).$$

This motivates the "convolution" :: $\{a_n\} * \{b_n\} = \left\{\sum_{i+j=n} a_i b_j\right\}$.

In Number Theory, its more common to use <u>Dirichlet series</u>,

$$\{a_n\} \rightsquigarrow \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We see that

$$\left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{b_n}{n^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}\left(\sum_{cd=n} a_c b_d\right).$$

This suggests defining the <u>Dirichlet Convolution</u> of two arithmetic functions.

$$(f * g)(n) = \sum_{cd=n} f(c)g(d) = \sum_{d|n} f\left(\frac{n}{d}\right) g(d).$$

For example, $(f * 1)(n) = \sum_{c|n} f(c) 1\left(\frac{n}{c}\right) = \sum_{c|n} f(c)$.

(Lecture 3)

$$\sum_{d|n} \phi(n) = n \qquad \text{but} \qquad \phi * 1 = ?$$

Notation :: Given $\alpha \in \mathbb{R}$, arithmetic function f, let $T^{\alpha} f$ denote the function

$$T^{\alpha} f(n) = n^{\alpha} f(n),$$

and let $T = T^1$. Also, for $k \in \mathbb{Z}_{\geq 0}$, let $L^k f(n) = (\log n)^k f(n)$, $L = L^1$.

Example ::

$$\begin{aligned} L(f * g)(n) &= (\log n)(f * g)(n)) \\ &= \log n \sum_{cd=n} f(c)g(d) \\ &= \sum_{cd=n} (\log c + \log d) f(c)g(d) \\ &= (Lf) * g + (Lg) * f. \end{aligned}$$

Derivation of Arithmetic Functions

One can also show :: $T^\alpha(f * g) = T^\alpha f * T^\alpha g$.

$$\sum_{d|n} \phi(d) = n,$$

by Möbius Inversion,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

And so

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Using the convolution notation $*$,

$$\begin{aligned} \phi * 1 &= T1 \\ \phi &= T1 * \mu \\ T^{-1}\phi &= 1 * T^{-1}\mu \end{aligned}$$

Facts about $*$ ::

- $(f * g) * h = f * (g * h)$
- If $f * g = 0$, then $f = 0$ or $g = 0$ (Integral Domain)
- f has an inverse ($\exists\ g$ with $f * g = e$) if and only if $f(1) \neq 0$ (Unique Maximal Ideal)

Recall :: If $F(n) = \sum_{d|n} f(d)$, then

$$\begin{aligned} \sum_{n \le x} F(n) &= \sum_{n \le x} \sum_{d|n} f(d) = \sum_{d \le x} f(d) \sum_{\substack{n \le x \\ d|n}} 1 = \sum_{d \le x} f(d) \left\lfloor \frac{x}{d} \right\rfloor \\ &= x \sum_{d \le x} \frac{f(d)}{d} + O\left(\sum_{d \le x} |f(d)| \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n \le x} \frac{\phi(n)}{n} &= x \sum_{d \le x} \frac{\mu(d)/d}{d} + O\left(\sum_{d \le x} \left| \frac{\mu(d)}{d} \right| \right) \\ &= x \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\sum_{d > x} \left| \frac{\mu(d)}{d^2} \right| \right) \right) + O\left(\sum_{d \le x} \left| \frac{\mu(d)}{d} \right| \right). \end{aligned}$$

We had before,

$$\sum_{d>x}\left|\frac{\mu(d)}{d^2}\right| = O\left(\frac{1}{x}\right),$$

also,

$$\sum_{d\le x}\left|\frac{\mu(d)}{d}\right| \le \sum_{d\le x}\frac{1}{d} < 1+\int_1^x \frac{dt}{t} = 1+\log x.$$

Therefore,

$$\sum_{n\le x}\frac{\phi(n)}{n} = x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} + O(\log x).$$

General Fact :: If $f(n)$ is multiplicative, and

$$\sum_{d=1}^{\infty}|f(d)| < \infty,$$

then

$$\sum_{d=1}^{\infty} f(d) = \prod_p (1+f(p)+f(p^2)+\cdots).$$

Proof. Consider

$$\prod_{p\le y}(1+f(p)+f(p^2)+\cdots) = \sum_{\substack{n\in\mathbb{N}\\ p|n\Rightarrow p\le y}} f(n). \qquad \text{(Smooth Numbers)}$$

To prove that the right hand side converges to $\sum_{n=1}^{\infty} f(n)$, we look at

$$\left|\sum_{n=1}^{\infty} f(n) - \sum_{\substack{n\in\mathbb{N}\\ p|n\Rightarrow p\le y}} f(n)\right| = \left|\sum_{\substack{n\in\mathbb{N}\\ \exists\, p|n \text{ with } p>y}} f(n)\right| \le \sum_{n>y}|f(n)| \to 0 \text{ as } y\to\infty.$$

(By hypothesis). Going back...

$$\sum_{n\le x}\frac{\phi(n)}{n} = x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} + O(\log x).$$

Taking $f(d) = \frac{\mu(d)}{d^2}$,

$$\begin{aligned}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} &= \prod_p\left(1+\frac{\mu(p)}{p^2}+\frac{\mu(p^2)}{p^4}+\cdots\right)\\ &= \prod_p\left(1-\frac{1}{p^2}\right).\end{aligned}$$

But also,

$$\begin{aligned}\frac{\pi^2}{6} &= \sum_{d=1}^{\infty}\frac{1}{d^2} = \prod_p\left(1+\frac{1}{p^2}+\frac{1}{p^4}+\cdots\right)\\ &= \prod_p\left(\frac{1}{1-\frac{1}{p^2}}\right) = \prod_p\left(1-\frac{1}{p^2}\right)^{-1}.\end{aligned}$$

We conclude that

$$\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} = \frac{6}{\pi^2}.$$

■

Notation ::

$$\kappa(n) = \begin{cases} \frac{1}{j}, & \text{if } n = p^j \ (p \text{ prime}, j \in \mathbb{N}) \\ 0, & \text{if } \omega(n) \geq 2 \text{ or } n = 1. \end{cases}$$

(“von Mangoldt Λ function”)

$$\Lambda(n) = L\kappa(n) = \begin{cases} \log p & \text{if } n = p^j, \\ 0 & \text{else.} \end{cases}$$

We also define several summatory functions ::

$$\pi(x) = \sum_{p\leq x} 1, \qquad \theta(x) = \sum_{n\leq x}\kappa(n) = \sum_{j=1}^{\infty}\frac{\pi(x^{1/j})}{j}, \qquad \psi(x) = \sum_{n\leq x}\Lambda(n).$$

(Lecture 4)

Notation :: $f(x) \ll g(x)$ means $f(x) = O(g(x))$. Also, $f(x) \gg g(x)$ means $g(x) \ll f(x)$.

Note :: If we write $f \gg g$, then both f and g should be non-negative.

(Pros of $\ll$)

$$f \ll g(x) + h(x) \ll g_1(x) + h_1(x) \ll j(x). \qquad \underline{\text{(Less writing)}}$$

(Pros of O)

$$f(x) = g(x) + O(x^{1/2}).$$

Conjecture on Distribution of Prime Pairs ::

$$\#\{p \le x : p+2k \text{ is also prime}\} \sim 2C_2 \left(\prod_{\substack{p|k \\ p>2}} \frac{p-1}{p-2} \right) \frac{x}{\log^2 x},$$

where $C_2 =$ "Twin primes constant"

$$C_2 = \prod_p \left(1 - \frac{1}{(p-1)^2}\right).$$

So we can say

$$\#\{p \le x : p+2k \text{ is prime}\} \ll_k \frac{x}{\log^2 x}.$$

$\ll_k$ or O_k means the implied constant may depend on k.

The standard interpretation of

$$\#\{p \le x : p+2k \text{ is prime}\} \ll \frac{x}{\log^2 x}$$

is provably <u>false</u>.

Lemma ::

$$1 * \Lambda = L1 \Rightarrow \sum_{d|n} \Lambda(d) = \log n.$$

Note that

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d.$$

But

$$\log n \sum_{d|n} \mu(d) \to 0.$$

So

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

Proof.

$$\begin{aligned} \sum_{d|n} \Lambda(d) &= \sum_{\substack{(p,r) \\ p^r|n}} \log p = \sum_{p^k \| n} k \log p \\ &= \sum_{p^k \| n} \log p^k = \log n. \end{aligned}$$

■

Bounds for multiplicative functions

Example :: For $\phi(n)$, we can show that $\phi(n) \gg_\delta n^{1-\delta}$ for any $\delta > 0$. *Proof.* Lets bound from below.

$$\begin{aligned}\frac{\phi(n)}{n^{1-\delta}} &= \prod_{p^k \| n} \frac{\phi(p^k)}{p^{k(1-\delta)}} = \prod_{p^k \| n} \frac{p^{k-1}(p-1)}{p^{k-k\delta}} \\ &= \prod_{p^k \| n} \left(1 - \frac{1}{p}\right) / p^{-k\delta} = \prod_{p^k \| n} \left(1 - \frac{1}{p}\right) p^{k\delta}.\end{aligned}$$

Thus,

$$\frac{\phi(n)}{n^{1-\delta}} \geq \prod_{p|n} \left(1 - \frac{1}{p}\right) p^{\delta} \geq \prod_{\substack{p|n \\ \left(1-\frac{1}{p}\right)p^{\delta}<1}} \left(1 - \frac{1}{p}\right) p^{\delta}.$$

We conclude that

$$\frac{\phi(n)}{n^{1-\delta}} \geq \prod_{\substack{p \\ \left(1-\frac{1}{p}\right)p^{\delta}<1}} \left(1 - \frac{1}{p}\right) p^{\delta} = C(\delta).$$

This proves $\phi(n) \geq C(\delta) n^{1-\delta}$.

■

This argument can yield explicit constants as well.

Example :: $\delta = \frac{1}{20}$,

$$\begin{aligned} p = 7, &\quad \left(1 - \frac{1}{7}\right) 7^{1/20} \approx 0.945, \\ p = 11, &\quad \left(1 - \frac{1}{11}\right) 11^{1/20} \approx 1.025. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\phi(n)}{n^{19/20}} &\geq \prod_{p \leq 7} \left(1 - \frac{1}{p}\right) p^{1/20} \\ &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)(2 \cdot 3 \cdot 5 \cdot 7)^{1/20} \\ &= 0.298\ldots\end{aligned}$$

We conclude that $\phi(n) > 0.298n^{19/20}$.

(Lecture 5)

Warm-up Question :: Which is bigger,

$$\sum_{n\le x}\phi(n)^{2005} \qquad \text{or} \qquad \sum_{n\le x} n^{2004} \ ?$$

Answer 1 :: We showed that

$$\phi(n) \gg_\delta n^{1-\delta}, \qquad \text{for any } \delta > 0.$$

Take $\delta = \frac{1}{4010}$, then

$$\begin{aligned}
\phi(n) &\gg n^{1-1/4010} \\
\phi(n)^{2005} &\gg n^{2005-\frac{1}{2}} = n^{2004\frac{1}{2}} \\
\sum_{n\le x}\phi(n)^{2005} &\gg \sum_{n\le x} n^{2004\frac{1}{2}} \approx \int_0^x t^{2004\frac{1}{2}}dt \\
&\gg t^{2005\frac{1}{2}}. \\
\sum_{n\le x} n^{2004} &\approx \int_0^x t^{2004}dt \ll x^{2005}.
\end{aligned}$$

Answer 2 ::

$$\begin{aligned}
&\sum_{n\le x}\phi(n)^{2005} \ge \sum_{p\le x}(p-1)^{2005} \\
&\gg \sum_{p\le x} p^{2005} \ge \sum_{x/2<p\le x}\left(\frac{x}{2}\right)^{2005} \\
&\gg x^{2005}\sum_{x/2<p\le x} 1 = x^{2005}\left(\pi(x) - \pi\left(\frac{x}{2}\right)\right).
\end{aligned}$$

Assuming $\pi(x) \sim \frac{x}{\log x}$, we get

$$\sum_{n\le x}\phi(n)^{2005} \gg x^{2005}\left(\frac{x}{\log x} - \frac{x/2}{\log(x/2)}\right).$$

Note that

$$\frac{x/2}{\log(x/2)} = \frac{x/2}{\log x - \log 2} = \frac{x/2}{(\log x)\left(1 + O\left(\frac{1}{\log x}\right)\right)}.$$

Fact :: If $f(x) \to 0$, then

$$\frac{1}{1+O(f(x))} = 1 + O(f(x)).$$

Since

$$\frac{1}{1+g(x)} = 1 - g(x) + g(x)^2 - g(x)^3 + \cdots$$

We know $|g(x)| \leq Cf(x)$, then $|g(x)|^k \leq C^k f(x)^k$. So

$$\begin{aligned} |-g(x) + g(x)^2 - g(x)^3 + \cdots| &\leq |Cf(x)| + |C^2 f(x)^2| + \cdots \\ &= \frac{|Cf(x)|}{1-|Cf(x)|} \leq 2|Cf(x)|, \qquad \text{when } x \gg 1. \end{aligned}$$

Therefore,

$$\frac{x/2}{\log(x/2)} = \frac{x/2}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right) = \frac{x}{2\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Example :: Discuss the convergence of $\sum_{n\leq x} \frac{\mu(n)}{n}$.

Method 1 :: Bound n uniformly. ie., $\frac{1}{n} \leq 1$, so

$$\left|\sum_{n\leq x} \frac{\mu(n)}{n}\right| \leq 1 \cdot \sum_{n \leq x} |\mu(n)| \sim \frac{6}{\pi^2}x.$$

Method 2 :: Split into dyadic blocks,

$$\sum_{U<n\leq 2U} \frac{\mu(n)}{n}.$$

In this range, $\frac{1}{n} \leq \frac{1}{u}$. So

$$\begin{aligned} \left|\sum_{U<n\leq 2U} \frac{\mu(n)}{n}\right| &\leq \frac{1}{U} \sum_{U<n\leq 2U} |\mu(n)| \\ &= \frac{1}{U}(Q(2U) - Q(U)) \leq \frac{Q(2U)}{U} \sim \frac{\frac{6}{\pi^2}2U}{U} \\ &= \frac{12}{\pi^2} \ll 1. \end{aligned}$$

More precisely,

$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

So

$$\begin{aligned}\frac{1}{U}(Q(2U)-Q(U)) &= \frac{1}{U}\left(\frac{6}{\pi^2}2U-\frac{6}{\pi^2}U+O(\sqrt{U})\right)\\ &= \frac{6}{\pi^2}+O\left(\frac{1}{\sqrt{U}}\right).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{n\leq 2^k}\frac{|\mu(n)|}{n} &= 1+\sum_{k=0}^{k-1}\sum_{2^k<n\leq 2^{k+1}}\frac{|\mu(n)|}{n}\\ &= 1+\sum_{k=0}^{k-1}\left(\frac{6}{\pi^2}+O\left(\frac{1}{\sqrt{2^k}}\right)\right)\\ &\qquad \text{(Convergent geometric series)}\\ &= 1+\frac{6}{\pi^2}k+O(1).\end{aligned}$$

In other words,

$$\sum_{n\leq x}\frac{|\mu(n)|}{n}=\frac{6}{\pi^2}\frac{\log x}{\log 2}+O(1).$$

Conclusion :: $\sum_n \frac{\mu(n)}{n}$ does <u>not</u> converge absolutely.

(Lecture 6)

$$Q(x):=\sum_{n\leq x}|\mu(n)| = \text{ \# of square-free integers } \leq x.$$

Warm Up :: If $h=1*j$, then

$$\sum_{h\leq x}h = x\sum_{d\leq x}\frac{j(d)}{d}+O\left(\sum_{d\leq x}|j(d)|\right).$$

Example 1 :: Show that $\sum_{d\leq x}\frac{\mu(d)}{d}$ is a bounded function of x.

Solution. Using that $1 * \mu = e$, we have

$$\begin{aligned} x\sum_{d\le x}\frac{\mu(d)}{d} &= \sum_{n\le x} e(n) + O\left(\sum_{d\le x}|\mu(d)|\right) \\ &= 1 \text{ (if } x\ge 1) \; + O(x) = O(x) \\ \Rightarrow \sum_{d\le x}\frac{\mu(d)}{d} &= O(1). \end{aligned}$$

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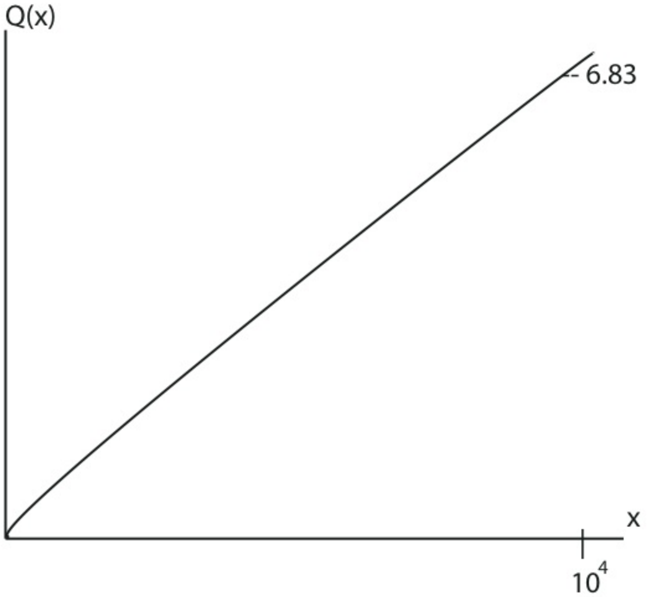
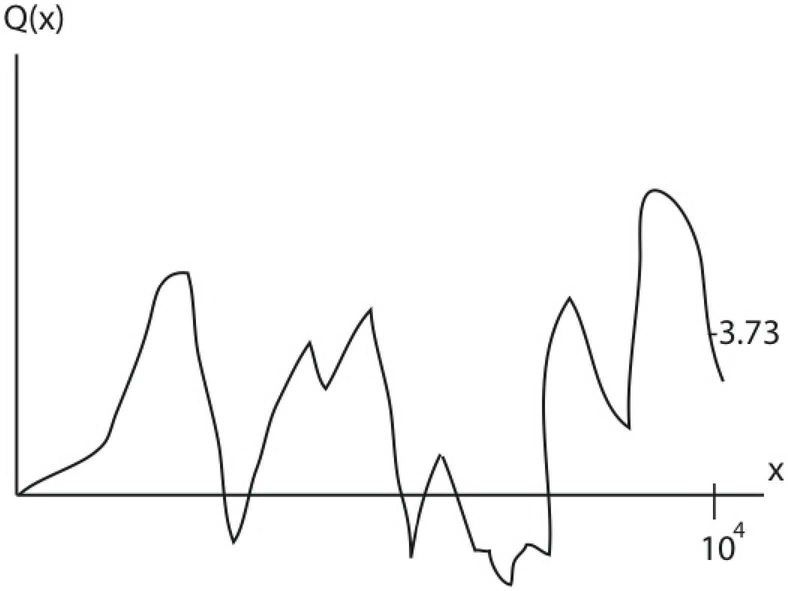
Example 2 :: Let $h=\tau$, $\tau(n)=\sum_{d|n}1=(1*1)(n)$. Then

$$\begin{aligned} \sum \tau(n) &= x\sum_{d\le x}\frac{1}{d} + O\left(\sum_{d\le x}1\cdot 1\right) \\ &= x(\log x + O(1)) + O(x) \\ &= x\log x + O(x). \end{aligned}$$

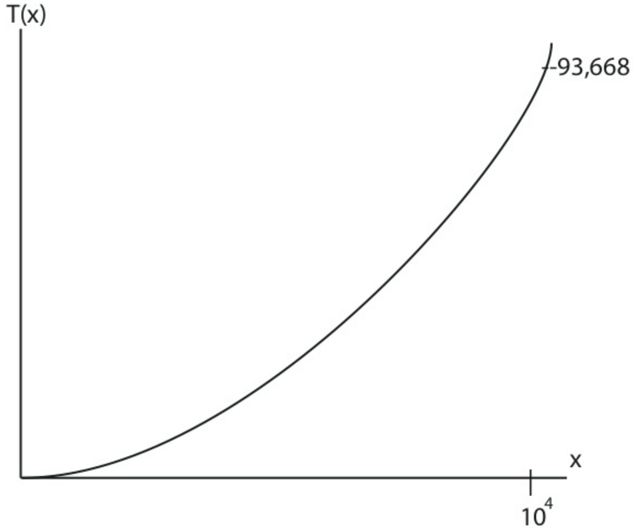
What do graphs of these functions look like?



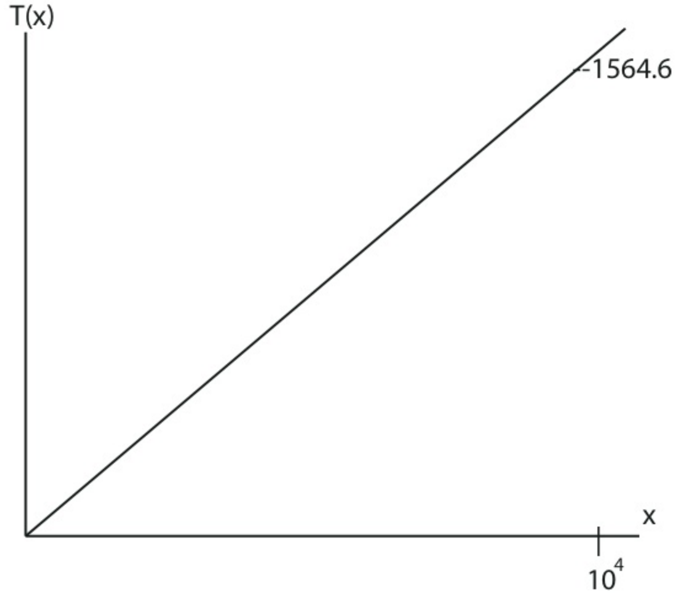
$$Q(x)=\sum_{n\le x}|\mu|(n)$$



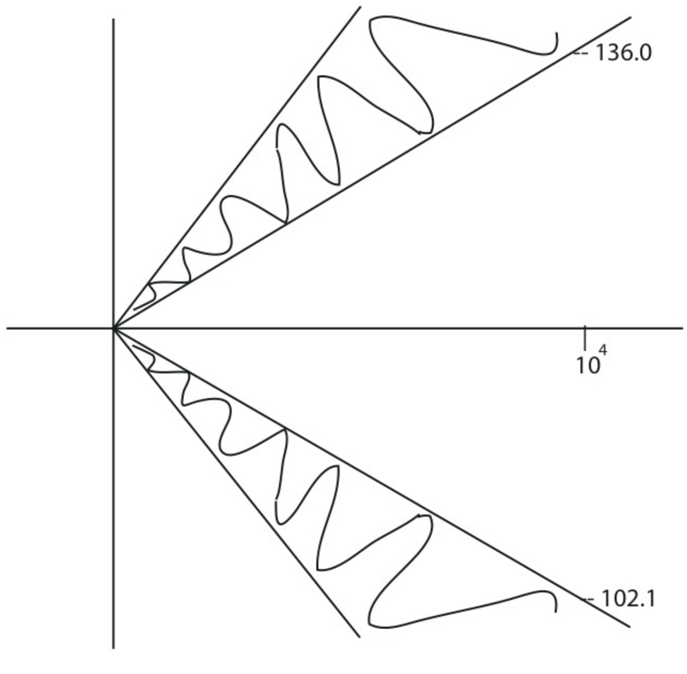
$$Q(x) - \frac{6}{\pi^2}x$$



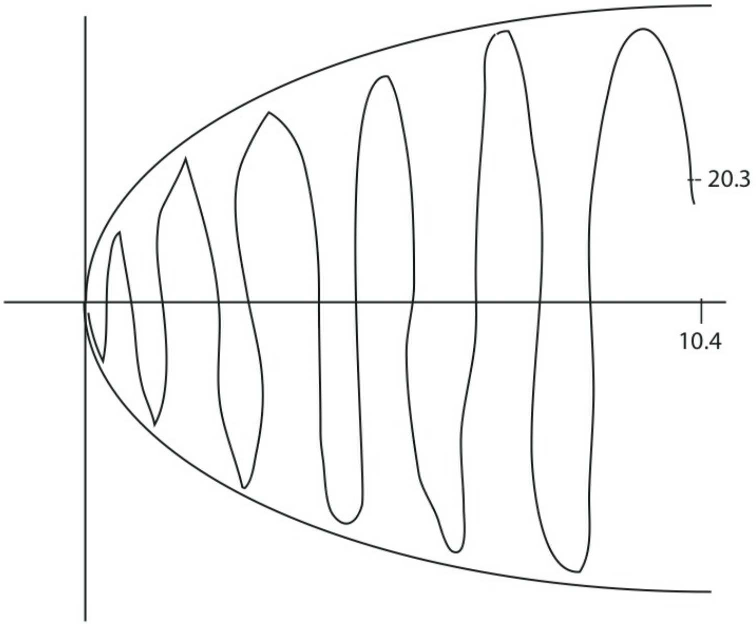
$$T(x) = \sum_{n \le x} \tau(n)$$



$T(x) - x \log x$



$T(x) - x \log x - \frac{1}{7}x$ (Top), $\quad T(x) - x \log x - \frac{1}{6}x$ (Bottom)



$$T(x) = x \log x - cx$$

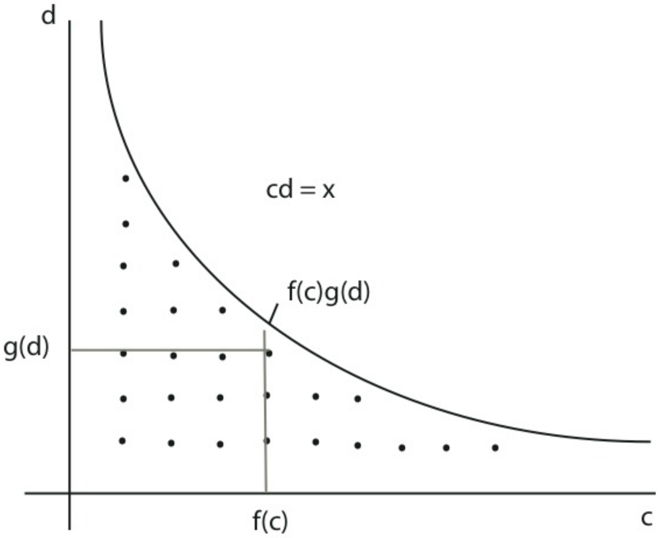
Recall $\sum_{n \le x} \tau(n) = \sum_{d \le x} \left\lfloor \frac{x}{d} \right\rfloor$, then $\left\lfloor \frac{x}{d} \right\rfloor$ can have small error with $\left\lfloor \frac{x}{d} \right\rfloor - 1$, say $x = 5000, d = 2$. but if $x = 5000, d = 2500$, then $\left\lfloor \frac{x}{d} \right\rfloor = 2$, so $\left\lfloor \frac{x}{d} \right\rfloor - 1 = 1$, big error.

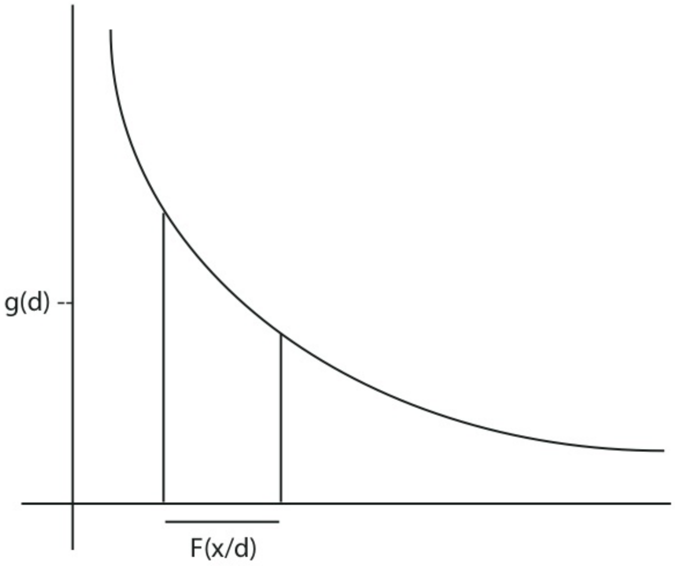
Lets consider

$$\begin{aligned} \sum_{n \le x} (f * g)(n) &= \sum_{n \le x} \sum_{ij = n} f(i) g(j) \\ &= \sum_{c \le x} f(c) \sum_{d \le \frac{x}{c}} g(d) \\ &= \sum_{c \le x} f(c) G\left(\frac{x}{c}\right). \end{aligned}$$

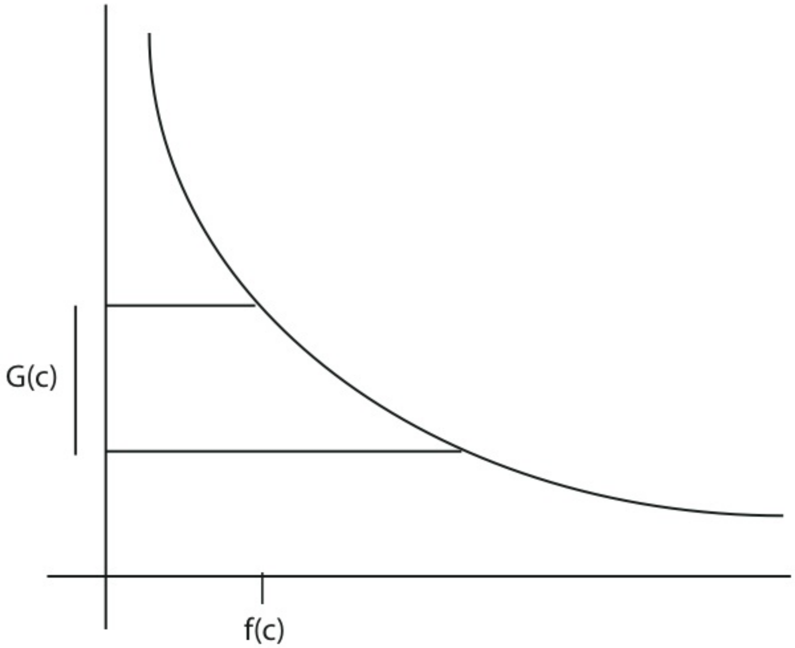
Notation ::

$$F(x) = \sum_{n \le x} f(n), \qquad G(x) = \sum_{n \le x} g(n).$$





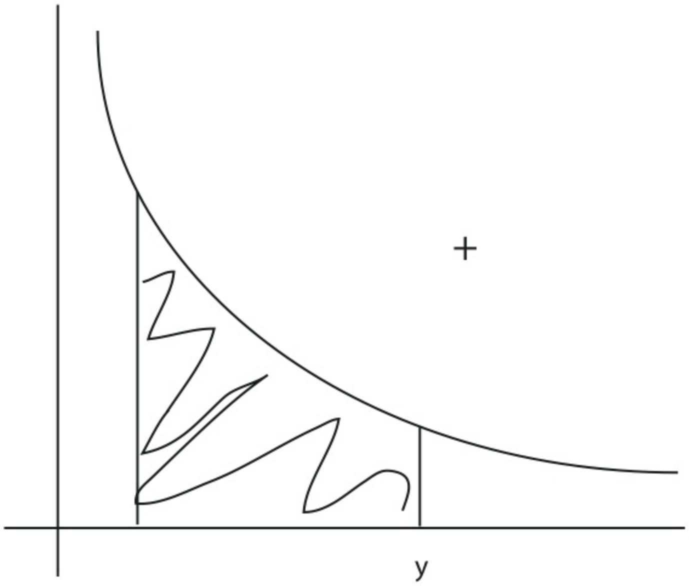
$$\sum_{d \le x} g(d) F\left(\frac{x}{d}\right)$$

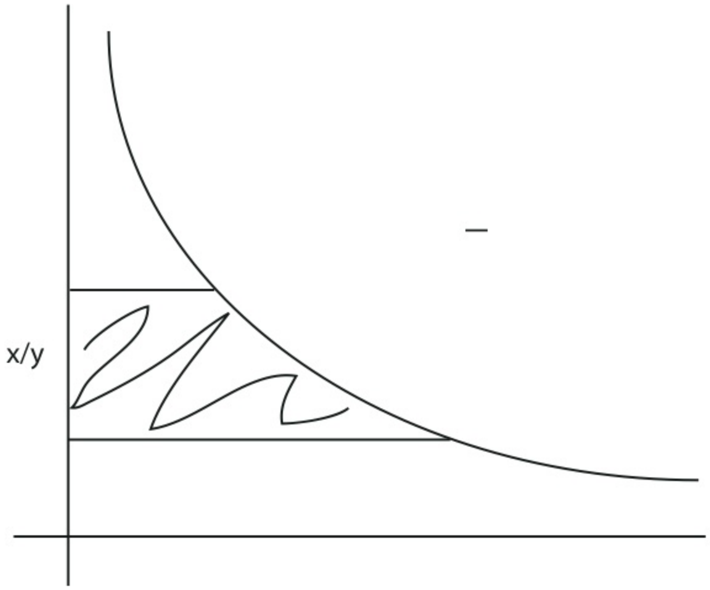


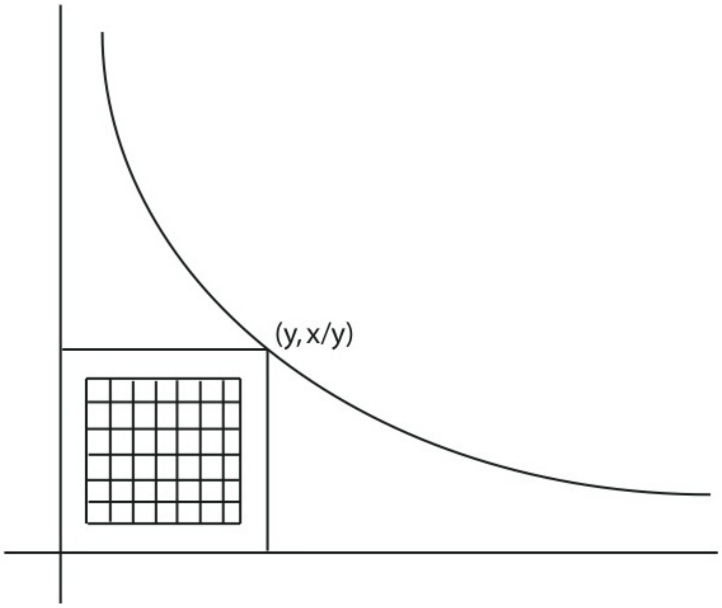
$$\sum_{c \leq x} f(c) G\left(\frac{x}{c}\right)$$

Dirichlet's Hyperbola Method

We can group these lattice points in the following way...







This turns out to be

$$\sum_{n\le x} f*g(n) = \sum_{c\le y} f(c)G\left(\frac{x}{c}\right) + \sum_{d\le x/y} g(d)F\left(\frac{x}{d}\right) - F(y)G\left(\frac{x}{y}\right).$$

Dirichlet's Divisor Problem

$$\sum_{n\le x}\tau(n)=\sum_{n\le x}1*1(n).$$

Using

$$\sum_{n\le x}1(n)=\lfloor x\rfloor,$$

then

$$\begin{aligned}T(x)&=\sum_{c\le y}\left\lfloor\frac{x}{c}\right\rfloor+\sum_{d\le x/y}1(d)\left\lfloor\frac{x}{d}\right\rfloor-\lfloor y\rfloor\left\lfloor\frac{x}{y}\right\rfloor\\&=\sum_{c\le y}\left(\frac{x}{c}-O(1)\right)+\sum_{d\le x/y}\left(\frac{x}{d}+O(1)\right)-(y+O(1))\left(\frac{x}{y}+O(1)\right)\\&=x\sum_{c\le y}\frac{1}{c}+O(y)+x\sum_{d\le x/y}\frac{1}{d}+O\left(\frac{x}{y}\right)-\left(y\frac{x}{y}+O(y)+O\left(\frac{x}{y}\right)\right).\end{aligned}$$

(Lecture 7)

...We will do an aside here....

Lemma 1 :: Fix $m\in\mathbb{N}$. Set

$$F(x)=\int_{m-1}^{x}\frac{1}{t}dt-\left.\frac{\{t\}}{t}\right|_{m-1}^{x}-\int_{m-1}^{x}\frac{\{x\}}{t^2}dt,$$

where $\{t\}=t-\lfloor t\rfloor$. Then

$$F(x)=\begin{cases}0 & \text{if } x\in(m-1,m),\\ \frac{1}{m} & \text{if } x=m.\end{cases}$$

Proof. Start by noting

$$\begin{aligned}\int_{m-1}^{x}\frac{\{t\}}{t^2}dt&=\int_{m-1}^{x}\frac{t-(m-1)}{t^2}dt\qquad\text{when } m-1<x\le m,\ \text{so}\\ \int_{m-1}^{x}\frac{1}{t}dt-\int_{m-1}^{x}\frac{\{t\}}{t^2}dt&=\int_{m-1}^{x}\frac{m-1}{t^2}dt-\left.\frac{-(m-1)}{t}\right|_{m-1}^{x}.\end{aligned}$$

Thus, if $m-1<x<m$, we have

$$F(x)=-\left(\frac{\{t\}}{t}+\frac{m-1}{t}\right)\Bigg|_{m-1}^{x}=-\left.\frac{t}{t}\right|_{m-1}^{x}=0.$$

If $x = m$, then

$$\begin{aligned} F(m) &= -\left(\frac{\{t\}+m-1}{t}\right)\Bigg|_{m-1}^{m} = -\frac{(m-1)}{t}\Bigg|_{m-1}^{m} = -\frac{(m-1)}{m} + \frac{(m-1)}{m-1} \\ &= -\frac{(m-1)}{m} + \frac{m}{m} = \frac{1}{m}. \end{aligned}$$

■

Lemma 2 :: When $x \geq 1$, we have

$$\sum_{1,n \leq x} \frac{1}{u} = \int_1^x \frac{1}{t} dt - \frac{\{t\}}{t}\Bigg|_1^x - \int_1^x \frac{\{t\}}{t^2} dt.$$

Proof. Apply Lemma 1 with $m = x = 2$, $m = x = 3$, …, $m = x = \lfloor x \rfloor$, and the extra piece $m = \lfloor x \rfloor + 1, x = x$. Then

$$\int_1^2 + \int_2^3 + \cdots + \int_{\lfloor x \rfloor - 1}^{\lfloor x \rfloor} + \int_{\lfloor x \rfloor}^{x} = \int_1^x \text{etc...}$$

■

Proposition (Lemma 3.13 in B. & D.) :: For $x \geq 1$,

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where

$$\gamma = \lim_{x \to \infty} \left(\sum_{n \leq x} \frac{1}{n} - \log x \right) = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt \approx 0.577215\ldots \qquad \text{(Euler's Constant)}$$

Proof.

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} = 1 + \sum_{1 < n \leq x} \frac{1}{n} &= 1 + \int_1^x \frac{1}{t} dt - \frac{\{t\}}{t}\Bigg|_1^x - \int_1^x \frac{\{t\}}{t^2} dt \\ &= 1 + \log x - \frac{\{x\}}{x} - \int_1^\infty \frac{\{t\}}{t^2} dt + \int_x^\infty \frac{\{t\}}{t^2} dt \\ &= \log x + \left(1 - \int_1^\infty \frac{\{t\}}{t^2} dt\right) + O\left(\frac{\{x\}}{x} + \int_x^\infty \frac{\{t\}}{t^2} dt\right). \end{aligned}$$

The error term is

$$\ll \frac{1}{x} + \int_x^\infty \frac{1}{t^2} dt \ll \frac{1}{x}.$$

■

We were asymtotically evaluating $T(x) = \sum_{n\leq x}\tau(n)$. We had shown using Dirichlet's hyperbola method that for any $1 \leq y \leq x$,

$$T(x) = x\sum_{c\leq y}\frac{1}{c} + O(y) + x\sum_{c\leq (x/y)}\frac{1}{d} + O\left(\frac{x}{y}\right) - \left(y\frac{x}{y} + O(y) + O\left(\frac{x}{y}\right)\right).$$

By proposition,

$$\begin{aligned} T(x) &= x\left(\log y - \gamma + O\left(\frac{1}{y}\right)\right) + x\left(\log\frac{x}{y} + \gamma + O\left(\frac{x}{y}\right)\right) - x + O\left(y + \frac{x}{y}\right) \\ &= x\log x + (2\gamma - 1)x + O\left(y + \frac{x}{y}\right). \end{aligned}$$

Since $y + \frac{x}{y}$ is minimized at $y = \sqrt{x}$. we conclude that

$$T(x) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$

This leads to the Dirichlet Divisor Problem. ■

Note on Minimizing Error Terms :: We often encounter error terms of the form $O(I(x)) + D(x))$, when I is increasing and D is decreasing. We could use calculus to find the exact minimum but we can also use

$$\max\{I(x), D(x)\} \leq I(x) + D(x) \leq 2\max\{D(x), I(x)\},$$

and $\max\{I(x), D(x)\}$ is minimized when $I(x) = D(x)$ (ie. $y = \frac{x}{y} \Rightarrow y = \sqrt{x}$).

Let's do one more example of the hyperbola method.

Example :: Let $s(n)$ be the indicator function of squares.

$$s(n) = \begin{cases} 1 & \text{if } n \in \mathbb{N}^2 \\ 0 & \text{else.} \end{cases}$$

We have

$$\sum_{n\leq x} s(n) = \lfloor\sqrt{x}\rfloor.$$

So lets try to evaluate

$$\sum_{n\le x} \mu^2 * s(n).$$

First try (Dead End) ::

$$\begin{aligned}\sum_{n\le x} \mu^2 * s(n) &= \sum_{c\le x} \mu^2(c) S\left(\frac{x}{c}\right),\ \text{where}\\ S(x) &= \sum_{n\le x} s(n).\end{aligned}$$

So

$$\begin{aligned}\sum_{c\le x} \mu^2(c) S\left(\frac{x}{c}\right) &= \sum_{c\le x} \mu^2(c)\left(\left\lfloor\sqrt{\frac{x}{c}}\right\rfloor\right)\\ &= \sum_{c\le x} \mu^2(c)\left(\sqrt{\frac{x}{c}} + O(1)\right)\\ &= \sqrt{x}\sum_{c\le x} \frac{\mu^2(c)}{\sqrt{c}} + O(x).\end{aligned}$$

(The error term is x but the main term is $\sqrt{x}$)

Second try (Dead End) ::

$$\begin{aligned}\sum_{n\le x} (\mu^2 * s)(n) &= \sum_{d\le x} s(d) Q\left(\frac{x}{d}\right)\\ &= \sum_{d\le x} s(d)\left(\frac{6}{\pi^2}\frac{x}{d} + O\left(\sqrt{\frac{x}{d}}\right)\right)\\ &= \frac{6}{\pi^2} x \sum_{d\le x} \frac{s(d)}{d} + O\left(\sqrt{x}\sum_{d\le x}\frac{s(d)}{\sqrt{d}}\right).\end{aligned}$$

The error term is (setting $d = l^2$)

$$\ll \sqrt{x}\sum_{l\le\sqrt{x}} \frac{1}{l} \ll \sqrt{x}\log\sqrt{x} \ll \sqrt{x}\log x.$$

(Lecture 8)

Let's recall some facts ::

$\mu^2 * s$, where μ^2 is the indicator function of squarefrees, and s is the indicator function of squares. Hence every number can be represented as such.

$$\sum_{n\le x}(\mu^2 * s)(n),$$

with $(c,d) : cd = n$, $\mu^2(c) = 1$, and $s(d) = 1$. We can draw a chart....

r	0	1	2	3	4	5	$\cdots$
$\mu^2(p^r)$	1	1	0	0	0	0	$\cdots$
$s(p^r)$	1	0	1	0	1	0	$\cdots$
$\mu^2 * s(p^r)$	1	1	1	1	1	1	$\cdots$

$$\mu^2 * s(p^r) = \sum_{cd=p^r} \mu^2(c)s(d).$$

Riemann-Stieltjes Integrals

Goal :: Use our understanding of $\sum_{n\le x} f(n)$ to gain understanding of $\sum_{n\le x} f(n)g(n)$ for various smooth functions g.

Examples ::

$$\sum_{n\le x} f(n)n^\alpha, \qquad \sum_{n\le x} f(n)\log n, \qquad \sum_{n\le x}\frac{f(n)}{\log n}.$$

Definition :: (F, G) is a compatible pair of functions if

- Both F and G are locally of bounded variation (Difference of two increasing functions).
- One of F, G is right-continuous and the other left-continuous.

Think!! Two standard situations ::

- F, G are both smooth.
- One is smooth and the other is a summatory function $\sum_{n \le x} a(n)$.

Definition :: The Riemann-Stieltjes Integral

$$\int_a^b F(t) dG(t),$$

is defined to be the limit of

$$\sum_{i=1}^{N} F(\xi_i)(G(x_i) - G(x_{i-1}))$$

over all partitions $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$ and each $\xi_i \in [x_{i-1}, x_i]$.

Fact :: If (F, G) is a compatible pair of functions, then

$$\int_a^b F(t) dG(t) \qquad \text{exists.}$$

Two most important classes of examples ::

1. If $G(x) = \sum_{n \le x} g(n)$, then
$$\int_a^b F(t) dG(t) = \sum_{a < n \le b} F(n) g(n) \qquad \text{say for } F \text{ smooth.}$$
2. If $G(x) = \int_c^x g(t) dt$, then
$$\int_a^b F(t) dG(t) \quad \text{(A Riemann-Stieltjes Integral)}$$
$$\text{simply equals} \quad \int_a^b F(t) g(t) dt. \quad \text{(A Riemann Integral)}$$

Theorem :: ("Summation by Parts")

If (F,G) are compatible pair of functions, then

$$\int_a^b F(t)dG(t) = F(t)G(t)\Big|_a^b - \int_a^b G(t)dF(t).$$

- Most Important Incarnation.

$$\text{Let} \quad G(x) = \sum_{n\leq x} g(n), \text{ and let } F \text{ be smooth, then}$$

$(*)$

$$\sum_{a<n\leq b} F(n)g(n) = F(t)G(t)\Big|_a^b - \int_a^b G(t)F'(t)dt.$$

Proof of $(*)$. Note that

$$F(n) = F(b) - \int_n^b F'(t)dt.$$

Thus,

$$\begin{aligned}
\sum_{a<n\leq b} F(n)g(n) &= \sum_{a<n\leq b} g(n)\left(F(b) - \int_n^b F'(t)dt\right) \\
&= F(b)(G(b)-G(a)) - \sum_{a<n\leq b} g(n)\int_n^b F'(t)dt \\
&= F(b)(G(b)-G(a)) - \int_n^b F'(t)\left(\sum_{a<n\leq t} g(n)\right)dt \\
&= F(b)(G(b)-G(a)) - \int_a^b F'(t)(G(t)-G(a))dt \\
&= F(b)(G(b)-G(a)) + G(a)\int_a^b F'(t)dt - \int_a^b F'(t)G(t)dt \\
&= F(b)(G(b)-G(a)) + G(a)(F(b)-F(a)) - \int_a^b F'(t)G(t)dt.
\end{aligned}$$

(Lecture 9)

Partial Summation ::

$$\int_a^b F(t)dG(t) = F(t)G(t)\Big|_a^b - \int_a^b G(t)dF(t).$$

If F is differentiable, and $G(x) = \sum_{n\leq x} g(n)$, then

$$\sum_{a<n\leq b} F(n)g(n) = F(t)G(t)\Big|_a^b - \int_a^b G(t)F'(t)dt.$$

Example 1 :: Find an asymptotic formula for

$$H(x) = \sum_{n \le x} \frac{\phi(n)}{n^2 \log n}.$$

Solution. Let

$$G(x) = \sum_{1 < n \le x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + O(\log x).$$

Let

$$F(x) = \frac{1}{x \log x}.$$

Then

$$\begin{aligned}
H(x) &= \int_{3/2}^{x} \frac{1}{t \log t} dG(t) \\
&= \left.\frac{G(t)}{t \log t}\right|_{3/2}^{x} - \int_{3/2}^{x} G(t) d\left(\frac{1}{t \log t}\right) \\
&= \frac{G(x)}{x \log x} - \frac{G(3/2)}{3/2 \log 3/2} - \int_{3/2}^{x} G(t)(-(t \log t)^{-2}(\log t + 1)) dt \\
&= O\left(\frac{1}{\log x}\right) + O(1) + \int_{3/2}^{x} \left(\frac{6}{\pi^2} t + O(\log t)\right)\left(\frac{1}{t^2 \log t} + \frac{1}{t^2 \log^2 t}\right) dt \\
&= O(1) + \frac{6}{\pi^2} \int_{3/2}^{x} \frac{dt}{t \log t} + \int_{3/2}^{x} O\left(\frac{1}{t \log^2 t} + \frac{1}{t^2} + \frac{1}{t^2 \log t}\right) dt.
\end{aligned}$$

The whole error term is

$$O\left(\int_{3/2}^{\infty} \frac{dt}{t \log^2 t}\right) = O(1).$$

So

$$H(x) = \frac{6}{\pi^2} \log \log t \Big|_{3/2}^{x} + O(1) = \frac{6}{\pi^2} \log \log x + O(1).$$

Example 2 :: Define $\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ (valid for $\alpha > 1$). Also, if $\mathbf{Re}(\alpha) > 1$, then

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^\alpha}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{\mathbf{Re}(\alpha)}}.$$

So $\zeta(\alpha)$ converges absolutely if $\mathbf{Re}(\alpha) > 1$.

Remark on Example 1 :: Note that

$$H(x) = \int_{c}^{x} \frac{1}{t \log t} dG(t) \text{ for any } 1 \le c < 2.$$

So we often write

$$\begin{aligned} H(x) &= \lim_{\epsilon \to 0^+} \int_{2-\epsilon}^{x} \frac{1}{t \log t} dG(t) \\ &= \int_{2^-}^{x} \frac{1}{t \log t} dG(t). \end{aligned}$$

Write $F(x) = \frac{1}{x^\alpha}$, $g(n) = 1$, so $G(x) = \lfloor x \rfloor$. Then

$$\begin{aligned} \zeta(\alpha) &= \int_{1^-}^{\infty} \frac{1}{t^\alpha} d\lfloor t \rfloor \\ &= \frac{\lfloor t \rfloor}{t^\alpha}\bigg|_{1^-}^{\infty} - \int_{1^-}^{\infty} \lfloor t \rfloor d\left(\frac{1}{t^\alpha}\right) \\ &= 0 - 0 + \alpha \int_{1^-}^{\infty} \lfloor t \rfloor \frac{dt}{t^{\alpha+1}} \\ &= \alpha \int_{1}^{\infty} (t - \{t\}) \frac{dt}{t^{\alpha+1}}. \quad \text{(Note the change from } 1^- \text{ to } 1\text{)} \end{aligned}$$

Splitting the integral

$$\begin{aligned} \zeta(\alpha) &= \alpha \int_1^\infty \frac{dt}{t^\alpha} - \alpha \int_1^\infty \{t\} \frac{dt}{t^{\alpha+1}} \\ &= \alpha \left(\frac{\infty^{1-\alpha}}{1-\alpha} - \frac{1^{1-\alpha}}{1-\alpha} \right) - \alpha \int_1^\infty \{t\} \frac{dt}{t^{\alpha+1}} \\ &= \frac{\alpha}{\alpha - 1} - \alpha \int_1^\infty \{t\} \frac{dt}{t^{\alpha+1}}. \end{aligned}$$

Notes ::

1. This expression is an analytic function of α in the region $\{\mathbf{Re}(\alpha) > 0\} - \{1\}$, hence it provides an analytic continuation of $\zeta(\alpha)$.

2. Note that

$$\begin{aligned} \zeta(\alpha) - \frac{1}{\alpha - 1} &= \frac{\alpha}{\alpha - 1} - \frac{1}{\alpha - 1} - \alpha \int_1^\infty \frac{\{t\}}{t^{\alpha+1}} dt \\ &= 1 - \alpha \int_1^\infty \{t\} \frac{dt}{t^{\alpha+1}}. \end{aligned}$$

   So $\zeta(\alpha)$ has a simple pole at $\alpha = 1$, with Residue 1. Moreover,

$$\lim_{\alpha \to 1} \left(\zeta(\alpha) - \frac{1}{\alpha - 1} \right) = 1 - 1 \int_1^\infty \{t\} \frac{dt}{t^2} = \gamma,$$

where γ is the Euler's constant. So

$$\zeta(\alpha) = \frac{1}{\alpha - 1} + \gamma + \gamma_1(\alpha - 1) + \gamma_2(\alpha - 1)^2 + \cdots$$

(Lecture 10)

Example 3 :: Using partial summation to get a better upper bound than dyadic blocks.

$$\sum_{d \le x} \frac{\mu(d)\log d}{d}.$$

Recall

$$\sum_{d \le x} \frac{\mu(d)}{d} \ll 1 \quad \text{and} \quad \sum_{d \le x} \frac{|\mu(d)|}{d} \ll \log x.$$

Method 1 ::

$$\sum_{2^{k-1} \le d < 2^k} \frac{\mu(d)\log d}{d} \ll k \sum_{2^{k-1} \le d < 2^k} \frac{|\mu(d)|}{d} \ll k \log 2^k \ll k^2.$$

Thus

$$\begin{aligned} \sum_{d < 2^k} \frac{\mu(d)\log d}{d} &\ll \sum_{j=1}^{k} j^2 \ll k^3 \\ \sum_{d \le x} \frac{\mu(d)\log d}{d} &\ll \log^3 x. \end{aligned}$$

Method 2 :: Let

$$M(x) = \sum_{d \le x} \frac{\mu(d)}{d} = O(1).$$

Then

$$\begin{aligned} \sum_{d \le x} \frac{\mu(d)\log d}{d} &= \int_{1^-}^{x} \log t \, dM(t) \\ &= M(t)\log t \Big|_{1^-}^{x} - \int_{1^-}^{x} M(t)\frac{dt}{t} \\ &\ll \log x + O\left(\int_{1^-}^{x} \frac{dt}{t}\right) \ll \log x. \end{aligned}$$

Define

$$T(x) = \sum_{n \le x} \log n, \qquad \psi(x) = \sum_{n \le x} \Lambda(n).$$

Now,

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d},$$

or

$$\Lambda = \mu * L1.$$

Note that

$$\int_1^x \log t dt < T(x) < \int_1^x \log t dt + \log\lfloor x \rfloor.$$

Therefore, $T(x) = x\log x - x + O(\log x)$. Since $\Lambda = \mu * L1$, we have

$$\begin{aligned}
\psi(x) &= \sum_{d \le x} \mu(d) T\left(\frac{x}{d}\right) \\
&= \sum_{d \le x} \mu(d) \left(\frac{x}{d}\log\frac{x}{d} - \frac{x}{d} + O\left(\log\frac{x}{d}\right)\right) \\
&= (x\log x - x)\sum_{d\le x}\frac{\mu(d)}{d} - x\sum_{d\le x}\frac{\mu(d)\log d}{d} + O\left(\sum_{d\le x}|\mu(d)|\log\frac{x}{d}\right).
\end{aligned}$$

■

Idea :: Replace $\mu(d)$ by some other sequence $\{a_d\}$ with properties we might like.

1. Some finite set D such that $a_d \neq 0 \Rightarrow d \in D$.
2. $\sum_{d\in D} \frac{\mu(d)}{d} = 0$.
3. $\sum_{d\in D} \frac{a_d \log d}{d} \approx -1$.

What can we say if we use a_d?

- $$\begin{aligned}
\sum_{d\in D} a_d T\left(\frac{x}{d}\right) &= \text{(by the same series of manipulation)} \\
&= (x\log x - x)\sum_{d\in D}\frac{a_d}{d} - x\sum_{d\in D}\frac{a_d\log d}{d} + O\left(\sum_{d\in D}|a_d|\log\frac{x}{d}\right) \\
&= 0 - x\sum_{d\in D}\frac{a_d\log d}{d} + O(\log x \cdot \max_{d\in D}\{|a_d|\}),
\end{aligned}$$

  where we can replace the big-O term with $O_{\{a_d\}}(\log x)$ (True when $x > \max(D)$).

-

$$\begin{aligned}\sum_{d\in D} a_d T\left(\frac{x}{d}\right) &= \sum_{d\in D} a_d \sum_{n\le x/d} \log n \\ &= \sum_{d\in D} a_d \sum_{n\le x/d} \sum_{m|n} \Lambda(m).\end{aligned}$$

Using $n = ml$,

$$\begin{aligned} &= \sum_{dlm\le x} a_d\Lambda(m) = \sum_{m\le x} \Lambda(m) \sum_{dl\le x/m} a_d \\ &= \sum_{m\le x} \Lambda(m) E\left(\frac{x}{m}\right), \qquad \text{where} \\ E(y) &= \sum_{dl\le y} a_d = \sum_{d\in D} a_d \left\lfloor \frac{y}{d} \right\rfloor .\end{aligned}$$

The closer $E(y)$ is to 1, the closer this is to $\psi(x)$.

Key Fact ::

$$\begin{aligned}E(y) &= \sum_{d\in D} a_d \left(\frac{y}{d} - \left\{\frac{y}{d}\right\}\right) \\ &= y\sum_{d\in D} \frac{a_d}{d} - \sum_{d\in D} a_d \left\{\frac{y}{d}\right\},\end{aligned}$$

with $\sum_{d\in D} \frac{a_d}{d} \to 0$. So

$$E(y) = -\sum_{d\in D} a_d \left\{\frac{y}{d}\right\}.$$

This is periodic in y, with period LCM(D).

What we do look for is $\{a_d\}$ such that

1. $a_d \neq 0 \Rightarrow d \in D$.
2. $\sum_{d\in D} \frac{a_d}{d} = 0$.
3. $\sum_{d\in D} \frac{a_d \log d}{d} \approx -1$.
4. $E(y) = -\sum_{d\in D} a_d \left\{\frac{y}{d}\right\} \approx 1$.

(Lecture 11)

Summary :: Take any sequence $\{a_d\}$ supported on a finite set D such that

$$\sum_{d\in D} \frac{a_d}{d} = 0.$$

Define

$$E(x) = \sum_{d\in D} a_d \left\lfloor \frac{x}{d} \right\rfloor = -\sum_{d\in D} a_d \left\{ \frac{x}{d} \right\}.$$

We derived

- $$\sum_{d\in D} a_d T\left(\frac{x}{d}\right) = x \sum_{d\in D} \frac{a_d \log d}{d} + O\left(\sum_{d\in D} |a_d| \log \frac{x}{d}\right).$$

- $$\sum_{d\in D} a_d T\left(\frac{x}{d}\right) = \sum_{k\le x} \Lambda(k) E\left(\frac{x}{k}\right).$$

We also know

$$T(x) = x\log x - x + O(\log x).$$

This is Chebyshev's Method. We can choose any $\{a_d\}$ to get numerical bounds.

Lets try $a_1 = 1$, $a_2 = -2$, $a_d = 0$ for $d \ge 3$.

$$E(y) = -\{y\} + 2\left\{\frac{y}{2}\right\}$$

is periodic with period 2.

$$E(y) = \begin{cases} 0 & \text{if } 0 \le y < 1, \\ 1 & \text{if } 1 \le y < 2. \end{cases}$$

So

$$\sum_{d\in D} a_d T\left(\frac{x}{d}\right) = \sum_{\substack{k\le x \\ \lfloor x/k \rfloor \text{ is odd}}} \Lambda(k).$$

In particular, its $\le \psi(x)$, and also $\ge \psi(x) - \psi(x/2)$.

$$\begin{aligned} \sum_{d\in D} a_d T\left(\frac{x}{d}\right) &= -x(-\log 2) + O(\log x) \\ &= x\log 2 + O(\log x). \end{aligned}$$

In other words,

$$x \log 2 + O(\log x) \leq \psi(x).$$

Also,

$$\begin{aligned}
x \log 2 + O(\log x) &\geq \psi(x) - \psi(x/2), \\
(x/2) \log 2 + O(\log x) &\geq \psi(x/2) - \psi(x/4), \\
&\vdots
\end{aligned}$$

Adding $O(\log x)$ of these inequalities together,

$$x \log 2 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{\text{s.t.}}}\right) + O(\log^2 x) \geq \psi(x).$$

$$x \log 2 \left(\sum_{k=0}^{\infty} \frac{1}{2^k} + O\left(\frac{1}{x}\right)\right) + O(\log^2 x) \geq \psi(x).$$

$$x(2 \log 2) + O(\log^2 x) \geq \psi(x).$$

The functions

$$\psi(x) = \sum_{n \leq x} \Lambda(n), \quad \theta(x) = \sum_{p \leq x} \log p, \quad \text{and} \quad \pi(x) = \sum_{p \leq x} 1,$$

are closely related.

$$\begin{aligned}
\psi(x) &= \sum_{p \leq x} \log p + \sum_{p^2 \leq x} \log p + \sum_{p^3 \leq x} \log p + \cdots \\
&= \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \cdots \\
&= \sum_{k=1}^{K} \theta(x^{1/k}), \quad \text{where} \quad K = \left\lfloor \frac{\log x}{\log 2} \right\rfloor.
\end{aligned}$$

Note that $\theta(y) \leq y \log y$ trivially, and so

$$\begin{aligned}
\psi(x) &= \theta(x) + O\left(\sum_{k=2}^{K} x^{1/k} \log x^{1/k}\right) \\
&= \theta(x) + O\left(K\sqrt{x} \log \sqrt{x}\right) \\
&= \theta(x) + O(x^{1/K} \log^2 x).
\end{aligned}$$

Note that

$$\pi(x) = \int_{2^-}^{x} \frac{1}{\log t} d\theta(t).$$

Summing by parts,

$$\begin{aligned}\pi(x) &= \frac{\theta(t)}{\log t}\Big|_{2^-}^{x} - \int_{2^-}^{x} \theta(t) d\left(\frac{1}{\log t}\right)\\ &= \frac{\theta(x)}{\log x} + \int_{2}^{x} \frac{\theta(t)}{t\log^2 t} dt.\end{aligned}$$

If we have, for example, $\theta(x) \leq Cx$, then

$$\begin{aligned}\pi(x) &\leq \frac{Cx}{\log x} + \int_{2}^{x} \frac{Ct}{t\log^2 t} dt\\ &\leq \frac{Cx}{\log x} + \left(\int_{2}^{\sqrt{x}} + \int_{\sqrt{x}}^{x}\right)\left(\frac{C}{\log^2 t} dt\right)\\ &\leq \frac{Cx}{\log x} + \int_{2}^{\sqrt{x}} \frac{C}{\log^2 2} dt + \int_{\sqrt{x}}^{x} \frac{C}{\log^2 \sqrt{x}} dt\\ &\leq \frac{Cx}{\log x} + \frac{C}{\log^2 2}\sqrt{x} + \frac{4C}{\log^2 x} x\\ &\leq \frac{Cx}{\log x} + O\left(\frac{x}{\log^2 x}\right).\end{aligned}$$

In particular, Chebyshev's bounds for $\psi(x)$ imply that

$$\frac{x\log 2}{\log x} + O\left(\frac{x}{\log^2 x}\right) \leq \pi(x) \leq \frac{2x\log 2}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

(Lecture 12)

We already know

$$\psi(x) \asymp x. \quad (\text{“is of order of magnitude } x\text{”})$$

That is, $\psi(x) \ll x$ and $\psi(x) \gg x$. This implies

$$\pi(x) \asymp \frac{x}{\log x}.$$

We also recall

$$T(x) = \sum_{n\leq x} \log n = x\log x + O(x).$$

Merten's Formulas

1.
$$\sum_{n\leq x}\frac{\Lambda(n)}{n}=\log x+O(1).$$

2.
$$\sum_{p\leq x}\frac{\log p}{p}=\log x+O(1).$$

Proof. We write, using $L1=1*\Lambda$,

$$T(x)=\sum_{n\leq x}\log n=x\sum_{d\leq x}\frac{\Lambda(d)}{d}+O\left(\sum_{d\leq x}\Lambda(d)\right).$$

By Chebyshev's Bounds, this is

$$x\sum_{d\leq x}\frac{\Lambda(d)}{d}+O(x).$$

Comparing to $T(x)=x\log x+O(x)$, we have

$$x\sum_{d\leq x}\frac{\Lambda(d)}{d}=x\log x+O(x),$$

so divide through by x.

To derive 2, note that

$$\sum_{n\leq x}\frac{\Lambda(n)}{n}-\sum_{p\leq x}\frac{\log p}{p}=\sum_{\substack{p,r\\ r\geq 2\\ p^r\leq x}}\frac{\log p}{p^r}.$$

The sum over r is a geometric series, so this is bounded by

$$\sum_{p}\log p\sum_{r=2}^{\infty}\frac{1}{p^r}=\sum_{p}\frac{\log p}{p(p-1)}\ll\sum_{n}\frac{\log n}{n^2}\ll 1.$$

■

Note that from 1,

$$\begin{aligned}\sum_{n}\frac{\Lambda(n)}{n}&=\int_{1^-}^{x}\frac{1}{t}d\psi(t)\\&=\left.\frac{\psi(t)}{t}\right|_{1^-}^{x}-\int_{1^-}^{x}\psi(t)d\left(\frac{1}{t}\right)\\(*)\qquad&=O(1)+\int_{1}^{x}\psi(x)\frac{dt}{t^2}.\end{aligned}$$

The partial summation argument we saw Lecture 11 shows that

$$\pi(x) \sim \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Therefore, the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x} \quad \text{is equivalent to} \quad \psi(x) \sim x.$$

We can now prove ::

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \geq 1 \quad \text{and} \quad \liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1.$$

Suppose, for the sake of contradiction, that

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} > 1.$$

Choose c so that

$$1 < c < \liminf_{x\to\infty} \frac{\psi(x)}{x}.$$

By the definition of $\liminf$, there is some x_0 such that

$$\frac{\psi(x)}{x} > c \qquad \text{for all } x > x_0.$$

Therefore,

$$\begin{aligned} \int_1^x \psi(x)\frac{dt}{t^2} &> \int_1^{x_0} \frac{\psi(t)}{t^2}dt + \int_{x_0}^x c\frac{dt}{t} \\ &= O(1) + c(\log x - \log x_0) \\ &= c\log x + O(1). \end{aligned}$$

But this is impossible by $(*)$,

$$\begin{aligned} \int_1^x \psi(x)\frac{dt}{t^2} &= \sum_{n\leq x} \frac{\Lambda(n)}{n} + O(1) \\ &= \log x + O(1). \end{aligned}$$

Therefore, $\liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1$. A similar argument establishes the $\limsup_{x\to\infty}$ bound. ∎

Remark :: This implies that if $\frac{\pi(x)}{x/\log x}$ has a limit, then the limit is 1.

Merten's Formulas (Con't) ::

3.

$$\sum_{p\le x}\frac{1}{p}\sim\log\log x+b+O\left(\frac{1}{\log x}\right),\qquad\text{for some } b\in\mathbb{R}.$$

4.

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=e^{c}\log x+O(1),\qquad\text{for some constant } c.$$

Proof. Write

$$L(x)=\sum_{p\le x}\frac{\log p}{p},$$

and define

$$R(x)=L(x)-\log x.$$

So $R(x)\ll 1$ from 2. Then

$$\begin{aligned}\sum_{p\le x}\frac{1}{p}&=\int_{2^-}^{x}\frac{1}{\log t}dL(t)\\&=\frac{L(t)}{\log t}\bigg|_{2^-}^{x}-\int_{2^-}^{x}L(t)\left(-\frac{1}{t\log^2 t}\right)dt\\&=\frac{L(x)}{\log x}+\int_{2}^{x}\frac{(\log t+R(t))}{t\log^2 t}dt\\&=1+\frac{R(x)}{\log x}+\int_2^x\frac{dt}{t\log t}+\int_2^x\frac{R(t)}{t\log^2 t}dt.\end{aligned}$$

Since the last integral converges as $x\to\infty$ (since $R(t)\ll 1$), we have

$$\sum_{p\le x}\frac{1}{p}=1+O\left(\frac{1}{\log t}\right)+\log\log t-\log\log 2+\int_2^\infty\frac{R(t)}{t\log^2 t}dt-\int_x^\infty\frac{R(t)}{t\log^2 t}dt.$$

Define

$$b=1-\log\log 2+\int_2^\infty\frac{R(t)}{t\log^2 t}dt,$$

and noting that

$$\begin{aligned}\int_x^\infty\frac{R(t)}{\log^2 t}dt&\ll\int_x^\infty\frac{dt}{t\log^2 t}\\&=\left(-\frac{1}{\log t}\right)\bigg|_x^\infty=\frac{1}{\log x}.\end{aligned}$$

We conclude

$$\sum_{p\le x}\frac{1}{p}=\log\log x+b+O\left(\frac{1}{\log x}\right).$$

(Lecture 13)

So now we know

- $$\sum_{n\le x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right).$$
- $$\sum_{p\le x}\frac{1}{p}=\log\log x+b+O\left(\frac{1}{\log x}\right).$$

We wish to prove

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=e^{\gamma}\log x+O(1). \qquad \text{(Merten's Formula)}$$

1. Prove $e^c\log x+O(1)$ for some constant c (easy).
2. Prove $c=\gamma$ (annoying).

Proof of 1.

$$\log\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}.$$

Note that

$$\log(1-t)^{-1}=\sum_{k=1}^{\infty}\frac{t^k}{k}=t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots$$

In particular,

$$\log(1-t)^{-1}-1=\sum_{k=2}^{\infty}\frac{t^k}{k}=t^2\cdot\sum_{k=1}^{\infty}\frac{t^k}{k+2}.$$

This power series converges (to an analytic function) for $|t|<1$, hence is bounded uniformly on $|t|\le 1/2$. Therefore,

$$\log\left(1-\frac{1}{p}\right)^{-1}-\frac{1}{p}\ll\left(\frac{1}{p}\right)^2 \quad \text{uniformly in } p.$$

So

$$\begin{aligned}\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1} &= \sum_{p\le x}\frac{1}{p}+\sum_{p\le x}\left(\log\left(1-\frac{1}{p}\right)^{-1}-\frac{1}{p}\right)\\ &= \sum_{p\le x}\frac{1}{p}+\sum_{p}\left(\log\left(1-\frac{1}{p}\right)^{-1}-\frac{1}{p}\right)+O\left(\sum_{p>x}\frac{1}{p^2}\right).\end{aligned}$$

Thus

$$\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1} = \left(\log\log x+b+O\left(\frac{1}{\log x}\right)\right)+d+O\left(\frac{1}{x}\right),$$

where

$$d=\sum_{p}\left(\log\left(1-\frac{1}{p}\right)^{-1}-\frac{1}{p}\right)=\sum_{p}\sum_{k\ge 2}\frac{1}{kp^k}.$$

Setting $c=b+d$,

$$=\log\log x+c+O\left(\frac{1}{\log x}\right).$$

Exponentiating,

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=e^{\log\log x}e^{c}e^{O\left(\frac{1}{\log x}\right)}=e^{c}\log x\left(1+O\left(\frac{1}{\log x}\right)\right),$$

since $e^t=1+O(t)$ for $|t|\le 1/2$. ■

Proof of 2. We have

$$\begin{aligned}\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1} &= \log\log x+c+O\left(\frac{1}{\log x}\right)\\ &= \sum_{n\le \log x}\frac{1}{n}+(c-\gamma)+O\left(\frac{1}{\log x}\right).\end{aligned}$$

Unmotivated Step :: For all $\delta\in(0,1/2)$, define

$$\begin{aligned}A_\delta &= \delta\int_1^\infty x^{-1-\delta}\left(\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}\right)dx,\\ B_\delta &= \delta\int_1^\infty x^{-1-\delta}\left(\sum_{n\le\log x}\frac{1}{n}\right)dx,\\ C_\delta &= \delta\int_1^\infty x^{-1-\delta}(c-\gamma)dx,\\ D_\delta &= \delta\int_1^\infty x^{-1-\delta}O\left(\min\left\{1,\frac{1}{\log x}\right\}\right)dx.\end{aligned}$$

Then $A_\delta=B_\delta+C_\delta+D_\delta$ for all $\delta\in(0,1/2)$.

Lemma 1 :: $A_\delta = \log\zeta(1+\delta) + O(\delta)$ as $\delta \to 0^+$.

Lemma 2 :: $\log\zeta(1+\delta) = \log\delta^{-1} + O(\delta)$.

Lemma 3 :: $B_\delta = \log\delta^{-1} + O(\delta)$.

Lemma 4 :: $C_\delta = c - \delta$. (Proved!)

Lemma 5 :: $D_\delta \ll \delta\log\delta^{-1}$.

Given all these lemmas,

$\log\delta^{-1}+O(\delta) = \log\delta^{-1}+O(\delta)+c-\gamma+O(\delta\log\delta^{-1})$ or $c-\gamma = O(\delta+\delta\log\delta^{-1}) = O(\delta\log\delta^{-1})$.

Taking $\lim_{\delta\to 0^+}$ yields $c-\gamma = 0$.

Helpful observation ::

$$\delta\int_w^\infty x^{-1-\delta}dx = \delta\left(\frac{x^{-\delta}}{-\delta}\right)\bigg|_w^\infty = w^{-\delta} \quad \text{for all } \delta, w > 0.$$

Proof of Lemma 1.

$$\begin{aligned} A_\delta &= \delta\int_1^\infty x^{-1-\delta}\left(\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}\right)dx \\ &= \sum_p \log\left(1-\frac{1}{p}\right)^{-1}\delta\int_p^\infty x^{-1-\delta}dx, \end{aligned}$$

by Tonelli's Theorem. So

$$\begin{aligned} A_\delta &= \sum_p \log\left(1-\frac{1}{p}\right)^{-1}p^{-\delta} \\ &= \sum_p\sum_{k\ge 1}\frac{1}{kp^k}p^{-\delta}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\zeta(1+\delta)=\sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}}&=\prod_p\left(1+\frac{1}{p^{1+\delta}}+\frac{1}{p^{2(1+\delta)}}+\cdots\right)\\&=\prod_p\left(1-\frac{1}{p^{1+\delta}}\right)^{-1}.\end{aligned}$$

So

$$\begin{aligned}\log\zeta(1+\delta)&=\sum_p\log\left(1-\frac{1}{p^{1+\delta}}\right)^{-1}\\&=\sum_p\sum_{k\geq 1}\frac{1}{k(p^{1+\delta})^k}.\end{aligned}$$

Thus

$$\frac{1}{\delta}(A_\delta-\log\zeta(1+\delta))=\sum_p\sum_{k\geq 1}\frac{1}{kp^k}\frac{p^{-\delta}-p^{-k\delta}}{\delta}.$$

We need to show this is $O(1)$. We do this by showing $\lim_{\delta\to 0^+}$ exists. This limit is

$$\begin{aligned}\lim_{\delta\to 0^+}\sum_p\sum_{k\geq 2}\frac{1}{kp^k}\frac{p^{-\delta}-p^{-k\delta}}{\delta}&=\sum_p\sum_{k\geq 2}\frac{1}{kp^k}\lim_{\delta\to 0^+}\frac{p^{-\delta}-p^{-k\delta}}{\delta}\\ \text{(L'Hôspital)}&=\sum_p\sum_{k\geq 2}\frac{1}{kp^k}\lim_{\delta\to 0^+}\frac{(-\log p)p^{-\delta}+(k\log p)p^{-k\delta}}{1}\\&=\sum_p\sum_{k\geq 2}\frac{1}{kp^k}(k-1)\log p.\end{aligned}$$

This is a finite number. It's

$$\begin{aligned}&\leq\sum_p\sum_{k\geq 2}\frac{1}{p^k}\log p\\&=\sum_p\frac{\log p}{p(p-1)}\ll\sum_n\frac{\log n}{n^2}\ll 1.\end{aligned}$$

∎

(Lecture 14)

Proof of Lemma 2. We saw before that

$$\lim_{\alpha\to 1}\left(\zeta(\alpha)-\frac{1}{\alpha-1}\right)=\gamma.$$

In particular, for δ near 0,

$$\begin{aligned}\zeta(1+\delta) &= \frac{1}{\delta} + O(1) \\ &= \frac{1}{\delta}(1+O(\delta)).\end{aligned}$$

Therefore,

$$\begin{aligned}\log\zeta(1+\delta) &= \log\frac{1}{\delta} + \log(1+O(\delta)) \\ &= \log\delta^{-1} + O(\delta).\end{aligned}$$

$\blacksquare$

...We were in the middle of proving

$$\prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1} = e^{\gamma}\log y + O(1).$$

Proof of Lemma 3.

$$\begin{aligned}B_\delta &= \delta\int_1^\infty x^{-1-\delta}\left(\sum_{n\le\log x}\frac{1}{n}\right)dx \\ &= \sum_{n=1}^{\infty}\frac{1}{n}\delta\int_{e^n}^\infty x^{-1-\delta}dx \\ &= \sum_{n=1}^{\infty}\frac{1}{n}(e^n)^{-\delta} \\ &= \log(1-e^{-\delta})^{-1}.\end{aligned}$$

Since $e^t = 1+t+O(t^2)$ for t near 0, this gives

$$\begin{aligned}B_\delta &= \log(1-(1-\delta+O(\delta^2))^{-1} \\ &= \log(\delta+O(\delta^2))^{-1} = -\log\delta(1+O(\delta)) \\ &= -\log\delta - \log(1+O(\delta)) \\ &= \log\delta^{-1} + O(\delta).\end{aligned}$$

$\blacksquare$

Proof of Lemma 5.

$$D_\delta \ll \delta\int_1^\infty x^{-1-\delta}\min\left\{1, \frac{1}{\log x}\right\}dx.$$

We need $D_\delta = o(1)$ as $\delta \to 0$. We estimate this over 3 ranges separately.

(a)

$$\begin{aligned}\delta \int_1^e x^{-1-\delta} \min\left\{1, \frac{1}{\log x}\right\} dx &= \delta \int_1^e x^{-1-\delta} dx \\ &= \delta \left(\frac{x^{-\delta}}{-\delta}\right)\Bigg|_1^e \\ &= 1 - e^{-\delta} \\ &= 1 - (1 + O(\delta)) = O(\delta).\end{aligned}$$

(b)

$$\begin{aligned}&\delta \int_e^{e^{1/\delta}} x^{-1-\delta} \min\left\{1, \frac{1}{\log x}\right\} dx \\ &\le \delta \int_e^{e^{1/\delta}} \frac{1}{x \log x} dx = \delta \log\log x \Bigg|_e^{e^{1/\delta}} \\ &= \delta(\log \delta^{-1} - 0) = \delta \log \delta^{-1}.\end{aligned}$$

(c)

$$\begin{aligned}\delta \int_{e^{1/\delta}}^\infty x^{-1-\delta} \min\left\{1, \frac{1}{\log x}\right\} dx &< \delta \int_{e^{1/\delta}}^\infty x^{-1-\delta} dx \\ &= \delta (e^{1/\delta})^{-\delta} = \delta / e.\end{aligned}$$

Thus $D_\delta = O(\delta \log \delta^{-1}) = o(1)$ as $\delta \to 0$. ■

Reminder :: The notation $f(x) = o(g(x))$ means

$$\lim \frac{f(x)}{g(x)} = 0 \quad \text{as } x \to \text{whatever}.$$

Example ::

$$\begin{aligned}\pi(x) \sim \frac{x}{\log x} \Leftrightarrow \pi(x) &= \frac{x}{\log x}(1 + o(1)) \\ &= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).\end{aligned}$$

I prefer explicit error terms like

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

It turns out that we get better error terms by writing

$$\pi(x) = \mathrm{li}(x) + O(\cdots),$$

where

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}.$$

Proposition :: For all $n \geq 3$,

$$\phi(n) \geq \frac{e^{-\gamma} n}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right),$$

and this is best possible.

Proof. Suppose for a (optimistic) moment that n is of the form

$$n = \prod_{p \leq y} p.$$

Then

$$\begin{aligned}
\frac{\phi(n)}{n} &= \prod_{p|n}\left(1 - \frac{1}{p}\right) \\
&= \prod_{p \leq y}\left(1 - \frac{1}{p}\right) = \frac{1}{e^{\gamma}\log y + O(1)} \\
&= \frac{1}{e^{\gamma}\log y\left(1 + O\left(\frac{1}{\log y}\right)\right)} \\
&= \frac{1}{e^{\gamma}\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right).
\end{aligned}$$

Note that

$$\log n = \sum_{p \leq y} \log p = \theta(y) \asymp y$$

by Chebyshev's estimate. (ie. $cy < \theta(y) < Cy$ for constants c, C). So $\log\log n = \log\theta(y) = \log y + O(1)$, or $\log y = \log\log n + O(1)$. This yields (check!)

$$= \frac{1}{e^{\gamma}\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Now we consider general n. Let y be the $\omega(n)$-th prime, and set

$$m = \prod_{p \le y} p.$$

We see ::

- $$n \ge \prod_{p|n} p \ge \prod_{p \le y} p = m.$$
- $$\frac{\phi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right) \ge \prod_{p|m} \left(1 - \frac{1}{p}\right) = \frac{\phi(m)}{m}.$$

Therefore,

$$\begin{aligned} \frac{\phi(n)}{n} \ge \frac{\phi(m)}{m} &= \frac{1}{e^\gamma \log\log m} \left(1 + O\left(\frac{1}{\log\log m}\right)\right) \\ &\ge \frac{1}{e^\gamma \log\log n} \left(1 + O\left(\frac{1}{\log\log n}\right)\right). \end{aligned}$$

■

A similar proof shows ::

$$\omega(n) \le \frac{\log n}{\log\log n} \left(1 + O\left(\frac{1}{\log\log n}\right)\right),$$

which is also best possible.

We saw that the minimal order of $\phi(n)$ is

$$\frac{n}{e^\gamma \log\log n}.$$

This means

$$\phi(n) \ge (1 + o(1)) \frac{n}{e^\gamma \log\log n},$$

or

$$\phi(n) \gtrsim \frac{n}{e^\gamma \log\log n.}$$

Much earlier I claimed that

$$\omega(n) \leq \frac{\log n}{\log\log n} + O\left(\frac{\log n}{\log\log^2 n}\right),$$

(Here, $\log\log^2 n = (\log\log n)^2$) follows by a similar argument... We'll use the fact that

$$\begin{aligned} \pi(x) &= \frac{\omega(x)}{\log x} + O\left(\frac{\omega(x)}{\log^2 x}\right) \\ &\text{or} \\ \pi(x) &- \frac{\omega(x)}{\log x} \ll \frac{x}{\log^2 x}. \end{aligned}$$

Proof. Recall the argument that

$$\begin{aligned} 2^{\omega(n)} &\ll_\epsilon n^\epsilon. \\ \frac{2^{\omega(n)}}{n^\epsilon} &= \prod_{p^k \| n} \frac{2}{p^{k\epsilon}} \leq \prod_{p \leq 2^{1/\epsilon}} \frac{2}{p^\epsilon}. \end{aligned}$$

Equivalently,

$$\begin{aligned} \omega(n)\log 2 &\leq \epsilon \log n + \sum_{p \leq 2^{1/\epsilon}} (\log 2 - \epsilon \log p) \\ &= \epsilon \log n + (\log 2)\pi(2^{1/\epsilon}) - \epsilon \cdot \omega(2^{1/\epsilon}) \\ &= \epsilon \log n + (\log 2)\left(\pi(2^{1/\epsilon}) - \frac{\omega(2^{1/\epsilon})}{\log 2^{1/\epsilon}}\right) \\ &= \epsilon \log n + O\left(\frac{2^{1/\epsilon}}{\log^2 2^{1/\epsilon}}\right) \\ &= \epsilon \log n + O(\epsilon^2 2^{1/\epsilon}). \end{aligned}$$

We now choose

$$\epsilon = \frac{\log 2}{\log\log n}.$$

$$\begin{aligned} \omega(n)\log 2 &\leq \frac{\log 2}{\log\log n}\log n + O\left(\frac{1}{\log\log^2 n} 2^{\log\log n/\log 2}\right) \\ &= \frac{\log 2 \log n}{\log\log n} + O\left(\frac{1}{\log\log^2 n}\log n\right). \end{aligned}$$

Dividing by $\log 2$ finishes the proof. ∎

A similar argument will show

$$\log \tau(n) \leq \log 2 \frac{\log n}{\log\log n} + O\left(\frac{\log n}{\log\log^2 n}\right),$$

or

$$\tau(n) \leq 2^{(1+o(1))\log n/\log\log n} \ll_\epsilon n^\epsilon.$$

Yet,

$$2^{(1+o(1))\log n/\log\log n} \gg \exp(\log^b n), \quad \text{for } 0 < b < 1.$$

Lemma ::

$$\sum_{n\leq x}(\omega(n) - \log\log x)^2 \ll x\log\log x. \qquad \text{(Variance)}$$

Proof. We expand the left-hand side to

$$\sum_{n\leq x}\omega(n)^2 - 2\log\log x\sum_{n\leq x}\omega(n) + \lfloor x\rfloor \log\log^2 x.$$

We've already seen that

$$\begin{aligned}\sum_{n\leq x}\omega(n) &= x\log\log x + O(x).\\ \sum_{n\leq x}\omega^2(n) &= x\log\log^2 x + O(x\log\log x).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n\leq x}(\omega(n)-\log\log x)^2 &= (x\log\log^2 x + O(x\log\log x))\\ &- 2\log\log x(x\log\log x + O(x))\\ &\quad + x\log\log^2 x + O(\log\log^2 x)\\ &= 0\cdot x\log\log^2 x + O(x\log\log x).\end{aligned}$$

■

It turns out the same estimate holds for

$$\sum_{n\leq x}(\omega(n) - \log\log n)^2.$$

Now let $S = \{n \in \mathbb{N} : |\omega(n) - \log\log n| > (\log\log n)^{3/4}\}$. Then

$$\begin{aligned}
\sum_{\substack{n \le x \\ n \in S}} 1 &\le \sum_{n \le x} \frac{(\omega(n) - \log\log n)^2}{(\log\log n)^{3/2}} \\
&\lesssim (\log\log x)^{-3/2} \sum_{n \le x} (\omega(n) - \log\log n)^2 \\
&\ll (\log\log x)^{-3/2} x \log\log x \\
&= \frac{x}{(\log\log x)^{1/2}}.
\end{aligned}$$

We've shown

$$\omega(n) = \log\log n + O((\log\log n)^{3/4}), \qquad \text{for } \underline{\text{almost all}}\ n.$$

("almost all" means "on a set of density 1.")

$$\text{for density } \delta \Rightarrow \sum_{n \le x} f(n) \sim \delta.$$

Comment :: Averages of $\Omega(n)$ are the same as for $\omega(n)$.

$$\begin{aligned}
\sum_{n \le x} \Omega(n) &= \sum_{p \le x} \left(\left\lfloor \frac{x}{p} \right\rfloor + \left\lfloor \frac{x}{p^2} \right\rfloor + \left\lfloor \frac{x}{p^3} \right\rfloor + \cdots \right) \\
&\sim x \sum_{p \le x} \frac{1}{p-1} \\
&= x \sum_{p \le x} \left(\frac{1}{p} + O\left(\frac{1}{p^2} \right) \right).
\end{aligned}$$

(Lecture 15)

We saw that

$$\sum_{n \le x} (\omega(n) - \log\log n)^2 \ll x \log\log x.$$

This implies that if $E(n)$ goes to infinity faster than $\sqrt{\log\log n}$, then <u>almost all</u> integers n will satisfy

$$|\omega(n) - \log\log n| < E(n).$$

(In fact it's true that $\sum_{n \le x} (\omega(n) - \log\log n)^2 \sim x \log\log x$.)

Thinking in terms of

$$\text{average}(\omega(n)) \sim \log\log n.$$

$$\text{variance}(\omega(n)) \sim \log\log n.$$

We define

$$S(a,b) = \left\{ n \in \mathbb{N} : a < \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} < b \right\}.$$

We have the Erdös-Kac Theorem ::

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{x}\#\{n \le x : n \in S(a,b)\} \\ &= \text{density of } S(a,b) \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt = \text{Normal Distribution!} \end{aligned}$$

Analogy :: $\mathbb{X}_p$ a random variable, with

$$\begin{aligned} \mathbb{P}_r(\mathbb{X}_p = 1) &= \frac{1}{p}, \\ \mathbb{P}_r(\mathbb{X}_p = 0) &= 1 - \frac{1}{p}. \end{aligned}$$

Then

$$\mathbb{E}\left(\sum_{p\le x} \mathbb{X}_p\right) \sim \log\log x \sim \sigma^2 \left(\sum_{p \le x} \mathbb{X}_p\right).$$

This analogy is imperfect, because

$$\frac{1}{x}\#\{n \le x : p|n\} = \frac{1}{x}\left\lfloor \frac{x}{p} \right\rfloor = \frac{1}{p} - \frac{1}{x}\left\{\frac{x}{p}\right\}.$$

In the random variable analogy, $\mathbb{X}_p$ and $\mathbb{X}_q$ are independent. However,

$$\begin{aligned} &\frac{1}{x}\{n \le x : p|n \text{ and } q|n\} \\ &= \frac{1}{pq} - \frac{1}{x}\left\{\frac{x}{pq}\right\} \\ &\ne \left(1 - \frac{1}{p} - \frac{1}{x}\left\{\frac{x}{p}\right\}\right)\left(\frac{1}{q} - \frac{1}{x}\left\{\frac{x}{q}\right\}\right). \end{aligned}$$

Probabilistic Heuristics

1. Squarefree Numbers

   n is a squarefree $\Leftrightarrow$ for all primes p, $p^2 \nmid n$.

   Heuristics ::

   - "Probability" that $p^2 \nmid n$ is $\left(1 - \frac{1}{p^2}\right)$.
   - These events are "independent."

   Therefore, the "probability" that n is squarefree should be

   $$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

   (Got the right answer!)

2. Prime numbers

   n is prime $\Leftrightarrow$ for all primes $p \le \sqrt{n}$, $p \nmid n$. The "probability" that $p \nmid n$ is $\left(1 - \frac{1}{p}\right)$, so we get the prediction that n is prime with probability

   $$\prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p}\right) \sim \frac{1}{e^{\gamma} \log \sqrt{n}}$$
   $$= 2e^{-\gamma} \frac{1}{\log n}.$$

   This would imply the prediction

   $$\pi(x) \sim 2e^{-\gamma} \frac{x}{\log x},$$

   but this is provably false from Chebyshev (Note that $2e^{-\gamma} \approx 1.12$).

3. Prime values of Polynomials

   Example :: $f(x) = x^2 + 1$.

Question :: Given a prime p, what's the "probability" that $p | f(n) = n^2 + 1$?.

The number of solutions to $n^2 + 1 \equiv 0 \pmod{p}$ is

$$1 + \left(\frac{-1}{p}\right) = \begin{cases} 1 \text{ if } p = 2, \\ 2 \text{ if } p \equiv 1 \pmod{4}, \\ 0 \text{ if } p \equiv 3 \pmod{4}. \end{cases}$$

Thus the probability that $p|f(n)$ is

$$\frac{1}{p}\left(1+\left(\frac{-1}{p}\right)\right).$$

So we might predict something like

$$\prod_{p\leq\sqrt{f(n)}}\left(1-\frac{1}{p}\left(1+\left(\frac{-1}{p}\right)\right)\right)$$

as the probability that $f(n)$ is prime.

From example 2, we're suspicious...

So instead, lets predict

$$\begin{aligned}&\#\{n\leq x : n^2+1 \text{ is prime}\}\\ \sim\ &\frac{x}{2\log x}\prod_p\left(1-\frac{1}{p}\left(1+\left(\frac{-1}{p}\right)\right)\right)\left(1-\frac{1}{p}\right)^{-1}\\ =\ &\frac{x}{2\log x}\prod_{p\equiv 1 \pmod 4}\frac{p-2}{p-1}\prod_{p\equiv 3 \pmod 4}\frac{p}{p-1}.\end{aligned}$$

In general, if f is irreducible, and

$$\sigma(p)=\#\{a \pmod p : f(a)\equiv 0 \pmod p\},$$

then

$$\#\{n\leq x : f(n)\text{is prime}\}\sim\frac{x}{\deg(f)\log x}\prod_p\left(1-\frac{1}{p}\right)^{-1}\left(1-\frac{\sigma(p)}{p}\right).$$

(Lecture 16)

Definition :: A Dirichlet Series is an expression of the form

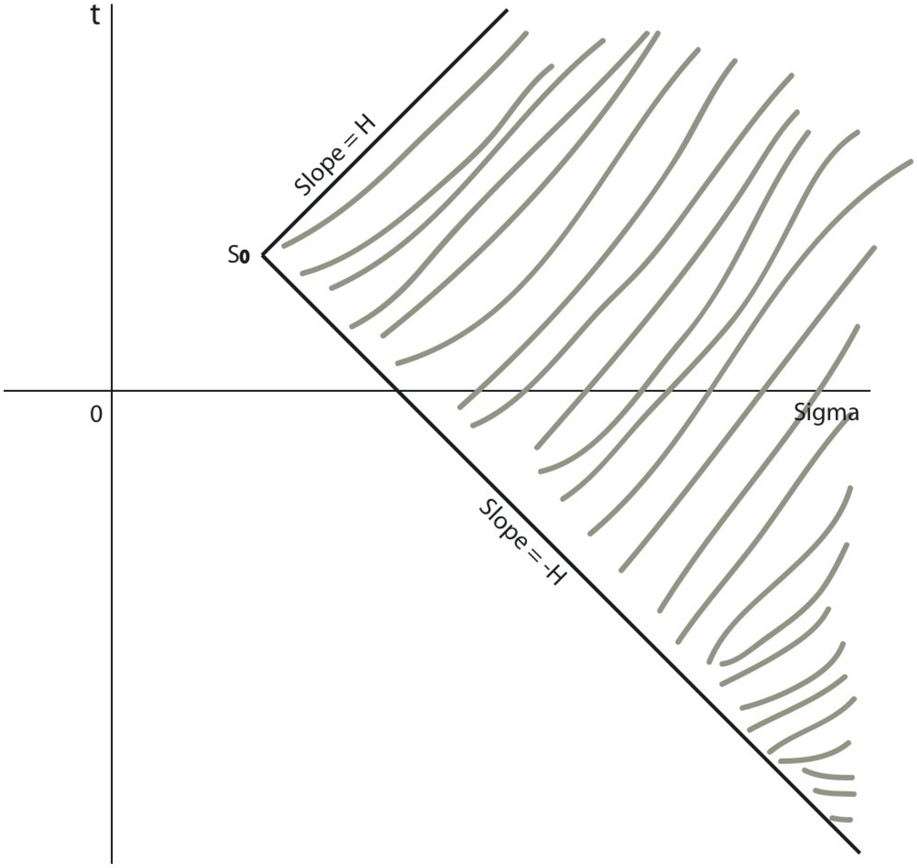
$$A(s)=\sum_{n=1}^{\infty}a_n n^{-s}\quad\text{for some sequence }\{a_n\}.\ \text{(Think arithmetic function)}$$

Convention :: In Analytic Number Theory, we tend to use s (rather than z) as the complex variable, writing $s=\sigma+it$, where $\sigma,t\in\mathbb{R}$.

Goal #1 :: To understand the region of convergence of a Dirichlet series $A(s)$.

Lemma :: Suppose that $A(s_0)$ converges, where $s_0 = \sigma_0 + it_0$. Then for any $H > 0$, the series $A(s)$ converges uniformly in the sector

$$S = \{s = \sigma + it : \sigma \geq \sigma_0, |t - t_0| \leq H(\sigma - \sigma_0)\}.$$



Proof. Define

$$R(u) = \sum_{n>u} a_n n^{-s_0}.$$

Then given $\epsilon > 0$, there exists M such that $|R(u)| < \epsilon$ for all $u \geq M$. (Sequences of 1 and $M + 1$ and $1 + N$ are Cauchy sequences). Then

$$\begin{aligned} \sum_{n=M+1}^{N} a_n n^{-s} &= \int_M^N u^{s_0 - s} d(-R(u)) \\ &= -R(u)u^{s_0-s}\Big|_M^N + \int_M^N R(u)(s_0 - s)u^{s_0-s-1} du. \end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\sum_{n=M+1}^{N} a_n n^{-s}\right| &\leq |R(M)|M^{\sigma_0-\sigma} + |R(N)|N^{\sigma_0-\sigma} + |s_0 - s| \int_M^N |R(u)|u^{\sigma_0-\sigma-1} du \\
&< \epsilon \cdot 1 + \epsilon \cdot 1 + |s_0 - s|\epsilon \int_M^\infty u^{\sigma_0-\sigma-1} du \\
&= \epsilon \left(2 + M^{\sigma_0-\sigma} \frac{|s_0 - s|}{\sigma - \sigma_0}\right) \\
&\ll \epsilon \left(1 + \frac{|s_0 - s|}{\sigma - \sigma_0}\right).
\end{aligned}$$

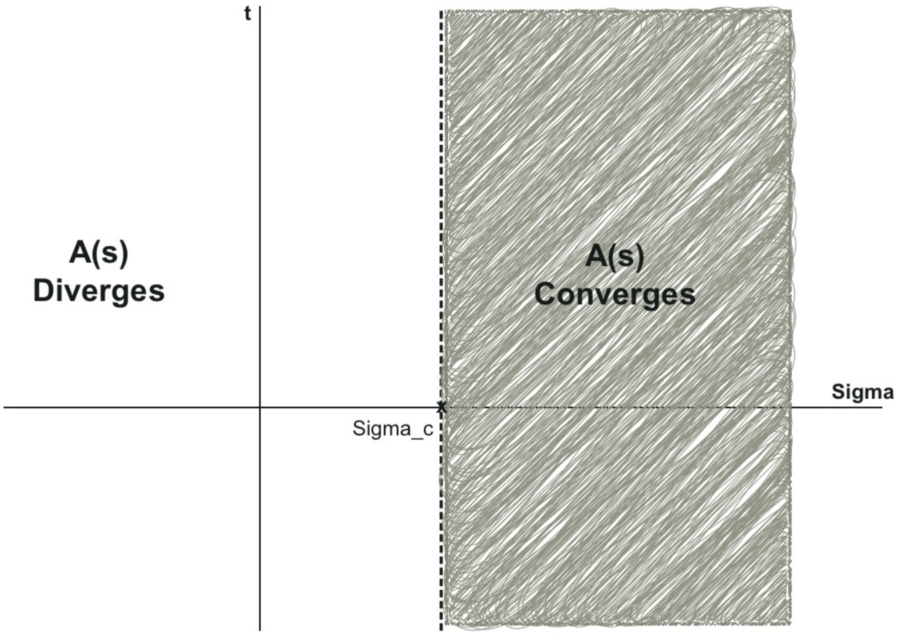
This shows that $\{\sum_{n=1}^N a_n n^{-s}\}$ is a Cauchy sequence. Hence $A(s)$ converges. Moreover, for $s \in S$,

$$\begin{aligned}
|s_0 - s| &\leq |t_0 - t| + (\sigma - \sigma_0) \\
&\leq (H+1)(\sigma - \sigma_0),
\end{aligned}$$

and so the upper bound is $\ll_S \epsilon$. Hence $A(s)$ converges uniformly on S. ■

Theorem :: Any Dirichlet series $A(s)$ has an abscissa of convergence $\sigma_c \in \mathbb{R} \cup \{-\infty, \infty\}$ with the properties ::

- $A(s)$ converges if $\sigma > \sigma_c$,
- $A(s)$ doesn't converge if $\sigma < \sigma_c$.

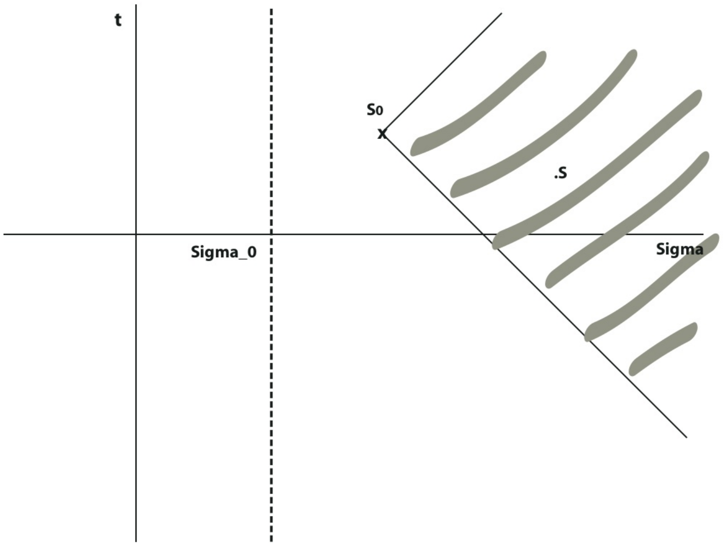


Moreover, $A(s)$ is locally uniformly convergent in $\{s : \sigma > \sigma_c\}$. Consequently, $A(s)$ is analytic on $\{s : \sigma > \sigma_c\}$.

Proof. Define $\sigma_c = \inf\{\mathbf{Re}(s) : A(s) \text{ converges}\}$. Obviously, $A(s)$ diverges if $\sigma < \sigma_c$. If $\sigma > \sigma_c$, then there exists s_0 with $\sigma > \sigma_0 > \sigma_c$ such that $A(s_0)$ converges. Then take

$$H > \frac{t - t_0}{\sigma - \sigma_0},$$

and apply Lemma. ∎



Example :: Define

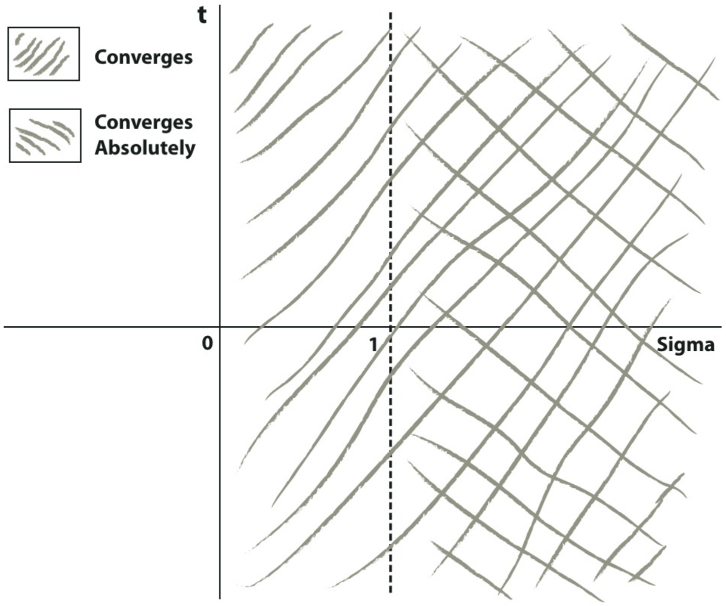
$$A(s) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$$
$$= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

First of all, $A(s)$ converges absolutely precisely where

$$\sum_{n=1}^{\infty}|(-1)^{n-1}n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma} = \zeta(\sigma) \quad \text{converges,}$$

that is, on $\{s : \sigma > 1\}$.

However, $A(\sigma)$ ($\sigma \in \mathbb{R}$) is an alternating series of terms decreasing in absolute value when $\sigma > 0$. Thus $A(\sigma)$ converges for $\sigma > 0$; by the Theorem we concludes that $A(s)$ converges on $\{s : \sigma > 0\}$. (Note :: $A(s)$ diverges for $\sigma \leq 0$ simply because the summand doesn't tend to zero).



Definition :: The abscissa of absolute convergence, σ_a, of $A(s)$ is the abscissa of the convergence of

$$|A|(s) = \sum_{n=1}^{\infty} |a_n| n^{-s}.$$

Example Above ::

$$A(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \qquad \sigma_c = 0, \qquad \sigma_a = 1.$$

Theorem :: We have

$$\sigma_a - 1 \leq \sigma_c \leq \sigma_a.$$

Proof. $\sigma_c \leq \sigma_a$ is obvious. Suppose $\epsilon > 0$, then

$$\sum_{n=1}^{\infty} a_n n^{-(\sigma_c+\epsilon)} \quad \text{converges.}$$

Hence $a_n n^{-(\sigma_c+\epsilon)} \to 0$ as $n \to \infty$. Then

$$\sum_{n=1}^{\infty} a_n n^{-(\sigma_c+1+2\epsilon)}$$

converges absolutely by comparison with

$$\sum_{n=1}^{\infty} n^{-(1+\epsilon)}.$$

This shows $\sigma_a - 1 \le \sigma_c$. ■

(Lecture 17)

Remark ::

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} &= \sum_{n \text{ odd}} n^{-s} - \sum_{n \text{ even}} n^{-s} \\ &= \sum_{n=1}^{\infty} n^{-s} - 2\sum_{n \text{ even}} n^{-s} \\ &= \zeta(s) - 2\sum_{m=1}^{\infty}(2m)^{-s} \\ &= \zeta(s) - 2^{1-s}\sum_{m=1}^{\infty} m^{-s} = \zeta(s)(1-2^{1-s}).\end{aligned}$$

(Manipulations valid for $\sigma > 1$.)

Therefore,

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{m=1}^{\infty}(-1)^{n-1}n^{-s} \qquad \text{for } \sigma > 1.$$

Note ::

$$\begin{aligned}1 - 2^{1-s} = 0 &\Leftrightarrow 2^{1-s} = 1 \\ &\Leftrightarrow \exp((1-s)(\log 2)) = 1 \\ &\Leftrightarrow (1-s)\log 2 = 2\pi i k,\ k \in \mathbb{Z} \\ &\Leftrightarrow s = 1 - \frac{2\pi i k}{\log 2},\ k \in \mathbb{Z}.\end{aligned}$$

So this provides another analytic continuation of $\zeta(s)$ to

$$\{\sigma > 0\} \backslash \left\{1 + \frac{2\pi i k}{\log 2},\ k \in \mathbb{Z}\right\}.$$

Remark on General Dirichlet Series

Given $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, let F be the first integer with $a_F \neq 0$. Then $A(s) \sim a_F F^{-s}$ as $\sigma \to \infty$. In particular, if $\sum_{n=1}^{\infty} a_n n^{-s} = 0$ for all $\sigma > K$, say. Then $a_n = 0$ for all $n \in \mathbb{N}$. It follows that

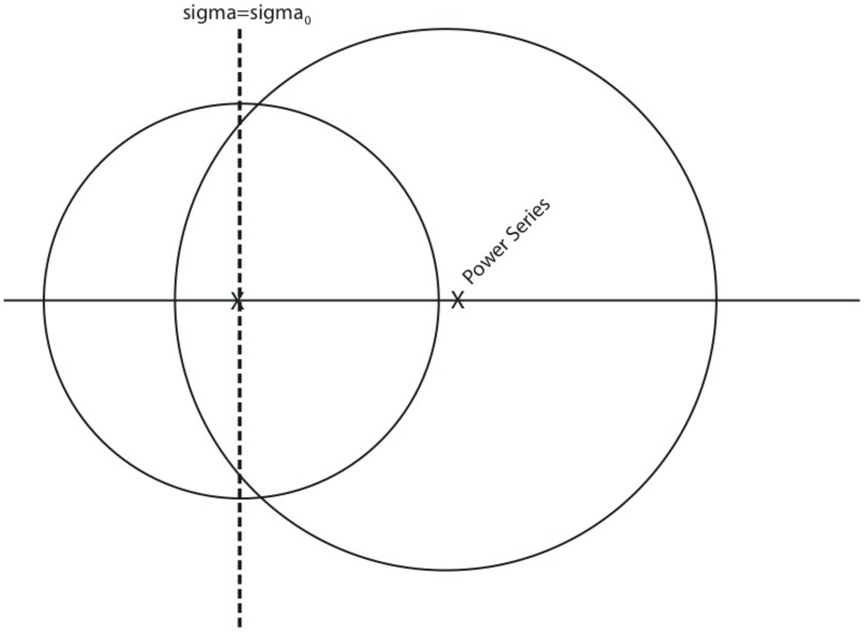
$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s} \quad \text{for } \sigma > K \;\Rightarrow\; \{a_n\} = \{b_n\}.$$

Theorem :: (Landau, 1905)

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Suppose σ_c is finite and $a_n \geq 0$ for all $n \in \mathbb{N}$. Then $A(s)$ has a singularity at σ_c.

The Idea ::



Proof. By replacing a_n with $a_n n^{-\sigma_c}$, we may assume $\sigma_c = 0$. Suppose $A(s)$ is analytic at 0, that is, analytic on some $\{|s| < \delta\}$. Expand $A(s)$ in a power series around $s = 1$.

$$A(s) = \sum_{k=0}^{\infty} \frac{1}{k!} A^{(k)}(1)(s-1)^k. \quad \text{(Power Series Expansion)}$$

Sideshow :: If $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, for $\sigma > \sigma_c$, then since $A(s)$ is locally uniform convergent, then we can differentiate term-by-term. ie., $A'(s) = \sum_{n=1}^{\infty} (-\log n) a_n n^{-s}$.

Thus,

$$A^{(k)}(1) = (-1)^k \sum_{n=1}^{\infty} (\log n)^k a_n n^{-1}.$$

Therefore,

$$A(s) = \sum_{k=0}^{\infty} \frac{1}{k!} (s-1)^k (-1)^k \sum_{n=1}^{\infty} \frac{a_n}{n} \log^k n.$$

Now the power series $A(s)$ converges in a disk (about $s = 1$) of radius at least $\sqrt{1+\delta^2}$ so it converges at $s = -\delta'$, say, for $0 < \delta' < \sqrt{1+\delta^2} - 1$.

$$A(-\delta') = \sum_{k=0}^{\infty} \frac{(1+\delta')^k}{k!} \sum_{n=1}^{\infty} \frac{a_n}{n} (\log n)^k.$$

Now, everything in sight is non-negative, hence

$$\begin{aligned} A(-\delta') &= \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{1}{k!} ((1+\delta') \log n)^k \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n} e^{(1+\delta') \log n} \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n} n^{1+\delta'} = \sum_{n=1}^{\infty} a_n n^{-(-\delta')}. \end{aligned}$$

But this asserts that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges at $s = -\delta'$, contradicting $\sigma_c = 0$. Thus $A(s)$ must have a singularity at $s = 0$. ■

(Lecture 18)

Recall :: If g is a multiplicative function, and

$$\text{either} \quad \sum_{n=1}^{\infty} |g(n)| < \infty \quad \text{or} \quad \prod_p (1 + |g(p)| + |g(p^2)| + \cdots) < \infty,$$

then

$$\sum_{n=1}^{\infty} g(n) = \prod_p (1 + g(p) + g(p^2) + \cdots).$$

Let $F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}$, where f is multiplicative, and $\sigma>\sigma_a$. Then

$$\sum_{n=1}^{\infty} |f(n)n^{-s}| < \infty$$

by definition of σ_a. But $f(n)n^{-s}=(T^{-s}f)(n)$ is multiplicative, hence by the "recall,"

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \left(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\right), \quad \text{for } \sigma>\sigma_a.$$

This is called the Euler product for $F(s)$.

Note :: If f is totally multiplicative, then the factors in the Euler product are geometric series, and so

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \left(1-\frac{f(p)}{p^s}\right)^{-1}.$$

Examples ::

(a)

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}, \quad \sigma>1.$$

(b)

$$\begin{aligned}\sum_{n=1}^{\infty} \mu(n)n^{-s} &= \prod_p \left(1-\frac{1}{p^s}+0+0+\cdots\right)\\ &= \prod_p (1-p^{-s}) = \frac{1}{\zeta(s)}, \quad \sigma>1.\end{aligned}$$

Personal Notes ::

- In the function with μ, σ_a is exactly 1 because there are enough squarefrees so that it cannot get anymore left (justified by a partial summation argument)
- Infinite products converges implies the product being non-zero - think in terms of log-land (product goes to zero $\Rightarrow$ log $\Rightarrow -\infty$.)

(c)

$$\begin{aligned}\sum_{n=1}^{\infty}\mu^2(n)n^{-s} &= \prod_p\left(1+\frac{1}{p^s}\right)\\ &= \prod_p\frac{1-p^{-2s}}{1-p^{-s}}\\ &= \prod_p(1-p^{-s})^{-1}\prod_p(1-p^{-2s})\\ &= \frac{\zeta(s)}{\zeta(2s)},\quad \sigma>1.\end{aligned}$$

(d) Define the Liouville lambda-function

$$\lambda(n) = (-1)^{\Omega(n)}.$$

(Recall $\Omega(n)$ counts the number of prime factors of n with multiplicity)

Then

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda(n)n^{-s} &= \prod_p(1-p^{-s}+p^{-2s}-p^{-3s}+\cdots)\\ &= \prod_p(1+p^{-s})^{-1}\\ &= \frac{\zeta(2s)}{\zeta(s)},\quad \sigma>1.\end{aligned}$$

Example ::

$$\eta(n) = \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}. \quad \text{(Dirichlet Eta function)}$$

Recall that $\sigma_a = 1$, $\sigma_c = 0$ for $\eta(s)$ and that $\eta(s) = (1-2^{1-s})\zeta(s)$.

Note that

$$(-1)^{n-1} = \begin{cases} -1 \text{ if } 2|n,\\ \;\;1 \text{ if } 2\nmid n.\end{cases}$$

That is, it's the multiplicative function f defined by

$$f(p^k) = \begin{cases} -1 \text{ if } p=2,\\ \;\;1 \text{ if } p>2.\end{cases}$$

Therefore, for $\sigma > 1$,

$$\begin{aligned} \eta(s) &= (1 - 2^{-s} - 2^{-2s} - \cdots) \prod_{p>2} (1 + p^{-s} + p^{-2s} + \cdots) \\ (*) \qquad &= \left(1 - \frac{2^{-s}}{1-2^{-s}}\right) \prod_{p>2} (1-p^{-s})^{-1}. \end{aligned}$$

■

Notes ::

- This actually equals to
$$\zeta(s)\left(1 - \frac{2^{-s}}{1-2^{-s}}\right)(1-2^{-s}) = \zeta(s)(1-2^{-s}-2^{-s}),$$
but we know this already...

- The infinite product in $(*)$ converges exactly where $\sum_p p^{-s}$ converges, which is $\sigma > 1$. In particular, it is <u>false</u> to say $\eta(s) =$ (Euler product) for $\sigma > \sigma_c$.

We've use the fact that

$$\prod_n (1 + x_n) \quad \text{converges} \quad \Leftrightarrow \quad \sum_n \log(1 + x_n) \quad \text{converges.}$$

(This is from the definition of convergence of infinite product)

$$\Leftrightarrow \sum_n x_n \quad \text{converges.}$$

(Since $\log(1 + x_n) = x_n + O(x_n^2)$)

Personal Note :: This power series expansion is only good when x_n is small, that said, if x_n isn't small, then it won't converge anyways...

More Examples ::

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-s} &= \sum_{n=1}^{\infty} n^{-s} \left\{ \begin{array}{l} 0 \text{ if } \exists\, p|q,\, p|n, \\ 1 \text{ if not.} \end{array} \right\} \\ &= \prod_{p\nmid q} (1-p^{-s})^{-1} \prod_{p|q} (1 + 0 + 0 + \cdots) \\ &= \zeta(s) \prod_{p|q} (1 - p^{-s}), \quad \sigma > 1. \end{aligned}$$

This is easy to generalize to $\sum_{(n,q)=1} f(n)n^{-s}$.

Non-Euler product stuff

Suppose

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \quad \text{for } \sigma > \sigma_c.$$

Then

- $$\begin{aligned}\sum_{n=1}^{\infty}(T^{\alpha}f)(n)n^{-s} &= \sum_{n=1}^{\infty}(f(n)n^{\alpha})n^{-s} \\ &= F(s-\alpha), \qquad \text{for } \sigma > \sigma_c + \mathbf{Re}(\alpha).\end{aligned}$$

- $$\begin{aligned}\sum_{n=1}^{\infty}(L^{k}f)(n)n^{-s} &= \sum_{n=1}^{\infty}(f(n)\log^{k} n)n^{-s} \\ &= (-1)^{k}F^{(k)}(s), \qquad \text{for } \sigma > \sigma_c.\end{aligned}$$

Example ::

$$\begin{aligned}-\zeta'(s) &= -\left(\sum_{n=1}^{\infty} 1(n)n^{-s}\right)' \\ &= \sum_{n=1}^{\infty}(L1)(n)n^{-s} \\ &= \sum_{n=1}^{\infty}(\log n)n^{-s}, \quad \text{valid for } \sigma > 1.\end{aligned}$$

$$\begin{aligned}\log\zeta(s) &= \log\left(\prod_{p}(1-p^{-s})^{-1}\right) \\ &= \sum_{p}\log(1-p^{-s})^{-1} \\ &= \sum_{p}\left(\sum_{k=1}^{\infty}\frac{(p^{-s})^{k}}{k}\right) \\ &= \sum_{n=1}^{\infty}\kappa(n)n^{-s}, \quad \text{for } \sigma > 1.\end{aligned}$$

$$\begin{aligned}-\frac{\zeta'(s)}{\zeta(s)} &= -(\log \zeta(s))' \\ &= \sum_{n=1}^{\infty} (\log n)\kappa(n) n^{-s} \\ &= \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.\end{aligned}$$

(Lecture 19)

Theorem :: Suppose

$$A(s) = \sum_{n=1}^{\infty} a(n) n^{-s} \quad \text{and} \quad B(s) = \sum_{n=1}^{\infty} b(n) n^{-s},$$

both converge absolutely at s. Then the Dirichlet series

$$\sum_{n=1}^{\infty} (a * b)(n) n^{-s}$$

converge absolutely at s to $A(s)B(s)$.

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty} (a * b)(n) n^{-s} &= \sum_{n=1}^{\infty} \left(\sum_{cd=n} a(c) b(d) (cd)^{-s} \right) \\ &= \sum_{c=1}^{\infty} a(c) c^{-s} \sum_{d=1}^{\infty} b(d) d^{-s} \cdot 1,\end{aligned}$$

and this is valid (in reverse?) by absolutely convergence. ■

Note :: If A, B are just convergent. ie., $A(s) = B(s) = \eta(s)$. η converges for $\sigma > 0$, but $\eta * \eta(s) = \eta^2(s) =$ converges for $\sigma > 1/4$. (Information is lost when taken the convolution).

Example 1 :: The convolution identity $1 * \mu = e$ corresponds to

$$\zeta(s) \cdot \frac{1}{\zeta(s)} = 1. \qquad (\sigma > 1)$$

Example 2 ::

$$A(s) = \sum_{n=1}^{\infty} \tau(n) n^{-s}.$$

We know $\tau(n) \ll_\epsilon n^\epsilon$, and so $\sigma_c = \sigma_a = 1$. Also by the Theorem,

$$\tau = 1 * 1, \qquad \text{so } A(s) = \zeta^2(s). \qquad (\sigma > 1)$$

This gives an analytic continuation of $A(s)$ to $\{\sigma > 0\}\backslash\{1\}$. We have

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \quad \text{near } s = 1.$$

Then

$$\begin{aligned} A(s) &= \left(\frac{1}{s-1} + \gamma + O(s-1)\right)^2 \\ &= \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + O(1). \end{aligned}$$

So $A(s)$ has a double pole at $s = 1$. Also,

$$\mathrm{Res}_{s=1} A(s) = 2\gamma. \qquad (\underline{\text{Not}}\ 1!)$$

Note :: Notice that for simple poles,

$$\mathrm{Res}_{s=s_0} f(s) = \lim_{s \to s_0} (s - s_0) f(s).$$

But for higher order poles, the residue exists but the limit does not.

Example 3 ::

$$B(s) = \sum_{n=1}^{\infty} \sigma(n) n^{-s}.$$

We have

$$\sigma(n) = \sum_{d|n} d, \quad \text{or} \quad \sigma = 1 * T1.$$

Therefore, $B(s) = \zeta(s)\zeta(s-1)$. $\qquad (\sigma > 2)$

Since

$$\sum_{n=1}^{\infty} T1(n) n^{-s} = \sum_{n=1}^{\infty} n \cdot n^{-s} = \sum_{n=1}^{\infty} n^{-(s-1)}.$$

Remark :: $\mathrm{Res}_{s=2} B(s) = \zeta(2)$.

Example 4 :: The Dirichlet series Identity

$$\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\zeta(s) = -\zeta'(s),$$

corresponds to

$$\left(\sum_{n=1}^{\infty}\Lambda(n)n^{-s}\right)\left(\sum_{n=1}^{\infty}n^{-s}\right) = \left(\sum_{n=1}^{\infty}L1(n)n^{-s}\right),$$

$$\text{or}\quad \Lambda * 1 = L1, \quad \text{or} \quad \sum_{d|n}\Lambda(d) = \log n.$$

Example 5 ::

$$C(s) = \sum_{n=1}^{\infty}\phi(n)n^{-s}.$$

Solution 1. Note

$$\sum_{d|n}\phi(n) = n, \qquad \text{or} \quad \phi * 1 = T1.$$

So $C(s)\zeta(s) = \zeta(s-1)$, or

$$C(s) = \frac{\zeta(s-1)}{\zeta(s)}. \quad (\sigma > 2)$$

Solution 2. Note

$$\frac{\phi(n)}{n} = \sum_{d|n}\frac{\mu(d)}{d},$$

(depends on primes dividing n, not the prime powers)

so $T^{-1}\phi = 1 * T^{-1}\mu$. This gives

$$C(s+1) = \zeta(s)\frac{1}{\zeta(s+1)}.$$

(Lecture 20)

Remarks From The Past :: If

$$F(s) = \sum_{n=1}^{\infty}f(n)n^{-s}, \quad G(s) = \sum_{n=1}^{\infty}g(n)n^{-s}, \quad \text{and}$$

$$H(s) = F(s)G(s) = \sum_{n=1}^{\infty}(f*g)(n)n^{-s},$$

for $\sigma > \max\{\sigma_a(F), \sigma_a(G)\}$. In other words,

$$\sigma_a(H) \leq \max\{\sigma_a(F), \sigma_a(G)\}.$$

Note :: This might not be an equality!

Example ::

$$F(s) = \zeta(s), \quad G(s) = \frac{1}{\zeta(s)}. \quad \text{Then } H(s) = 1.$$

Then $\sigma_a(F) = 1$, $\sigma_a(G) = 1$, but $\sigma_a(H) = -\infty$. (Verify)

Analogy :: Let $\mathrm{ord}_p(n) = k$ if and only if $p^k \| n$. Then $\mathrm{ord}_p(a+b) \geq \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\}$.

Note that this might not be an equality, but the smallest two values among $\{\mathrm{ord}_p(a), \mathrm{ord}_p(b), \mathrm{ord}_p(a+b)\}$ are the same (this is somewhat related to the p-adic metric...).

Example 1 :: Define

$$D_k(s) = \sum_{n=1}^{\infty} \left(\frac{n}{\phi(n)}\right)^k n^{-s}. \quad (k \in \mathbb{N})$$

Show that $\sigma_a = \sigma_b = 1$. But that D_k can be meromorphically continued to $\{\sigma > 0\}$, with its only pole a simple pole at $s = 1$. Calculate $\mathrm{Res}_{s=1} D_k(s)$.

Solution. Notice that

$$\begin{aligned} \left(\frac{n}{\phi(n)}\right)^k &\leq ((e^\gamma + o(1)) \log\log n)^k \\ &\ll_{k,\epsilon} n^\epsilon. \end{aligned}$$

Therefore, $\sigma_a = \sigma_c = 1$.

The function $(n/\phi(n))^k$ is multiplicative.

Lets experiment ::

r	0	1	2	3	$\cdots$
$\left(\frac{p^r}{\phi(p^r)}\right)^k$	1	$\left(\frac{p}{p-1}\right)^k$	$\left(\frac{p}{p-1}\right)^k$	$\left(\frac{p}{p-1}\right)^k$	$\cdots$
$\mu(p^r)$	1	-1	0	0	$\cdots$
$*$	1	$\left(\frac{p}{p-1}\right)^k - 1$	0	0	$\cdots$

So we define $f_k(n)$ to be the multiplicative function satisfying

$$f(p^r) = \begin{cases} \left(\frac{p}{p-1}\right)^k - 1 & \text{if } r = 1, \\ 0 & \text{if } r > 1. \end{cases}$$

Let

$$F_k(s) = \sum_{n=1}^{\infty} f_k(n) n^{-s}.$$

Then

$$\left(\frac{n}{\phi(n)}\right)^k * \mu(n) = f_k(n), \quad \text{so}$$

$$D_k(s)\frac{1}{\zeta(s)} = F_k(s), \quad \sigma > 1.$$

Since F_k converges absolutely for $\sigma > 1$, we have the Euler product

$$\begin{aligned} F_k(s) &= \prod_p \left(1 + \frac{f_k(p)}{p^s} + \frac{f_k(p^2)}{p^{2s}} + \cdots\right) \\ &= \prod_p (1 + f(p)p^{-s}), \quad \sigma > 1. \end{aligned}$$

Where does F_k converge?

We have

$$\begin{aligned}(1-x)^k &= 1+kx+O_k(x^2)\\ &= 1+O_k(x) \quad \text{uniformly for } |x|\le 1/2.\end{aligned}$$

Thus

$$f_k(p)=\left(\frac{p}{p-1}\right)^k-1=\left(1-\frac{1}{p}\right)^{-k}-1=O_k\left(\frac{1}{p}\right).$$

So

$$\prod_p(1+f_k(p)p^{-s}) \quad \text{converges} \quad \Leftrightarrow \quad \sum_p f_k(p)p^{-s} \quad \text{converges.}$$

But

$$\sum_p |f_k(p)p^{-s}| \ll_k \sum_p \frac{1}{p}p^{-\sigma} \quad \text{converges for } \sigma>0.$$

Therefore, $D_k(s)=\zeta(s)F_k(s)$ is an analytic continuation of D_k to $\{\sigma>0\}$, except for a simple pole at $s=1$.

The residue is

$$\begin{aligned}\mathrm{Res}_{s=1}\zeta(s)F_k(s) &= \lim_{s\to1}(s-1)\zeta(s)F_k(s)\\ &= \left(\lim_{s\to1}(s-1)\zeta(s)\right)\left(\lim_{s\to1}F(s)\right)\\ &= \mathrm{Res}_{s=1}\zeta(s)F_k(1)\\ &= 1\cdot F_k(1)=\sum_{n=1}^{\infty}\frac{f_k(n)}{n}\\ &= \prod_p\left(1+\frac{1}{p}\left(\left(\frac{p}{p-1}\right)^k-1\right)\right).\end{aligned}$$

■

For example, take $k = 1$.

$$
\begin{aligned}
&= \operatorname{Res}_{s=1}\left(\sum_{n=1}^{\infty} \frac{n}{\phi(n)} n^{-s}\right) = F_1(1) \\
&= \prod_p \left(1 + \frac{1}{p}\left(\frac{p}{p-1} - 1\right)\right) \\
&= \prod_p \left(1 + \frac{1}{p-1} - \frac{1}{p}\right) \\
&= \prod_p \left(\frac{p^2 - p + 1}{p(p-1)}\right) \\
&= \prod_p \frac{p^3+1}{p(p^2-1)} = \prod_p \frac{p^6-1}{p(p^2-1)(p^3-1)} \\
&= \prod_p \frac{1-p^{-6}}{(1-p^{-2})(1-p^{-3})} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}.
\end{aligned}
$$

Hold on!

$$\frac{\zeta(2)\zeta(3)}{\zeta(6)} \quad \text{was the } \underline{\text{average value}} \text{ of } \quad \frac{n}{\phi(n)}!$$

Lets look at

$$\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}. \quad \text{(Not really a connection to the previous example...)}$$

General Remark :: If

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

then for $k \in \mathbb{N}$,

$$
\begin{aligned}
A(ks) &= \sum_{n=1}^{\infty} a_n n^{-ks} \\
&= \sum_{n=1}^{\infty} b_m m^{-s}, \quad \text{where} \quad b_m = \begin{cases} a_n \text{ if } m = n^k, \\ 0 \text{ otherwise.} \end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)} &= \sum_{n=1}^{\infty} n^{-s} \begin{pmatrix} 1 \text{ if } n = \text{square,} \\ 0 \text{ else.} \end{pmatrix} * \begin{pmatrix} 1 \text{ if } n = \text{cube,} \\ 0 \text{ else.} \end{pmatrix} * \begin{pmatrix} \mu(m) \text{ if } n = m^6, \\ 0 \quad \text{else.} \end{pmatrix} \\
&= \text{(From Homework)} \quad \sum_{n \text{ power-full}} n^{-s}.
\end{aligned}
$$

Where are the poles of

$$\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}?$$

- Whenever $\zeta(\rho) = 0$, this function has a pole at $\rho/6$, however, there are none with $\mathbf{Re}(\rho/6) > 1/6$.
- Pole at $s = 1/2$, $\mathrm{Res}_{s=1/2} = \zeta(3/2)/\zeta(3)$.
- Pole at $s = 1/3$, $\mathrm{Res}_{s=1/3} = \zeta(2/3)/\zeta(2)$.

(Lecture 21)

Goal :: Write down some contour integral involving

$$A(s) := \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{that relates to} \quad \sum_{n\le x} a_n.$$

Convention :: When we write

$$\int_{c-iT_1}^{c+iT_1} \quad \text{or} \quad \int_{c-i\infty}^{c+i\infty}$$

we mean a contour integral over a vertical line (segment).

Lemma :: (Sound Bite Version) Let $c > 0$. Then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds = \begin{cases} 1 \text{ if } y > 1, \\ 0 \text{ if } 0 < y < 1, \\ \frac{1}{2} \text{ if } y = 1. \end{cases}$$

Note ::

$$\int_{c-i\infty}^{c+i\infty} \quad \text{means} \quad \lim_{T_1,T_2\to\infty}\int_{c-iT_1}^{c+iT_2}.$$

If the integral converges absolutely, then we can let $T_1, T_2 \to \infty$ however we like. Given this remark, the $y = 1$ case of the lemma is technically a lie ::

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1}{s} ds = \frac{1}{2}.$$

Lemma :: (Explicit Version) Let $c, y, T > 0$. Define

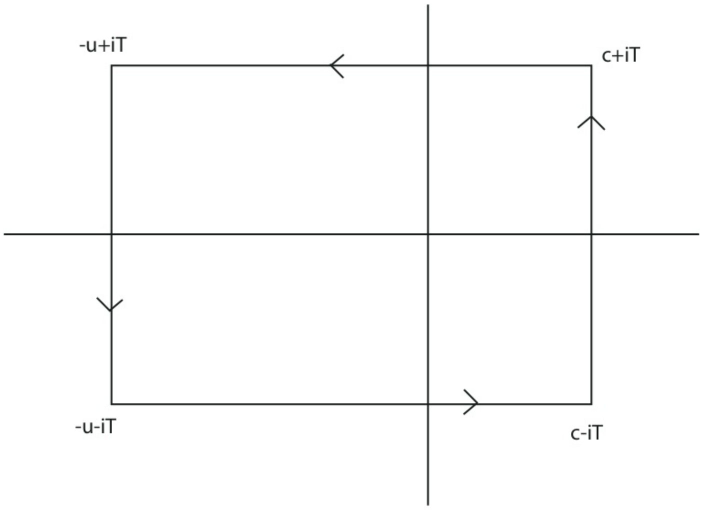
$$I_c(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s} ds.$$

Then

- If $y > 1$, then
$$|I_c(y,T) - 1| < y^c \min\left\{1, \frac{1}{\pi T \log y}\right\}.$$
- If $0 < y < 1$, then
$$|I_c(y,T)| < y^c \min\left\{1, \frac{1}{\pi T \log y^{-1}}\right\}.$$
- If $y = 1$, then
$$\left|I_c(y,T) - \frac{1}{2}\right| < \frac{c}{\pi T}.$$

Proof. ($y > 1$) Consider

$$\frac{1}{2\pi i}\oint \frac{y^s}{s} ds \quad \text{around the following contour ::}$$



By the residue theorem,

$$\frac{1}{2\pi i}\oint \frac{y^s}{s} ds = \text{Res}_{s=0}\frac{y^s}{s} = 1.$$

Also, the right-hand segment gives exactly $I_c(y, T)$. On the bottom segment,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{-u-iT}^{c-iT}\frac{y^s}{s}ds\right| &\leq \int_{-u}^{c}\frac{y^\delta}{|\delta - iT|}d\delta \\ &\quad \text{(change of variable } s = \delta - iT\text{)} \\ &< \frac{1}{2\pi T}\int_{-\infty}^{c} y^\delta d\delta = \frac{1}{2\pi T}\frac{y^\delta}{\log y}\bigg|_{-\infty}^{c} \\ &= \frac{y^c}{2\pi \log y}.\end{aligned}$$

We get the same estimate for the top segment. On the left-hand segment:

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{-u-iT}^{-u+iT}\frac{y^s}{s}ds\right| &\leq \frac{1}{2\pi}\int_{-T}^{T}\frac{y^{-u}}{|-u+it|}dt \\ &< \frac{y^{-u}}{2\pi u}2T.\end{aligned}$$

Therefore,

$$|I_c(y, T) - 1| < 2\frac{y^c}{2\pi T \log y} + \frac{y^{-u}}{2\pi u}2T.$$

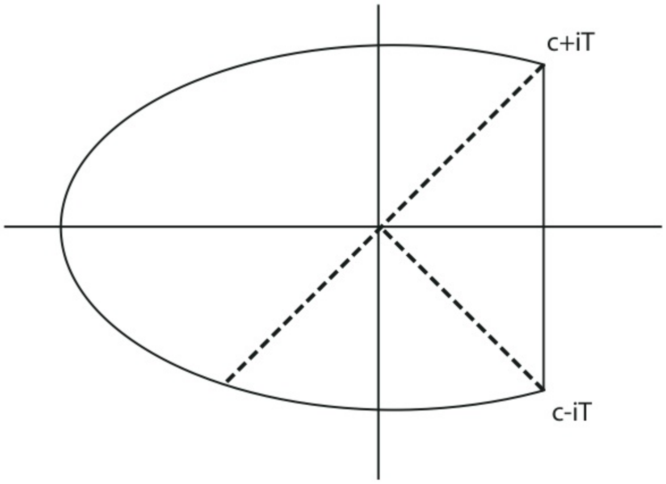
Taking $u \to \infty$ yields

$$|I_c(y, T) - 1| \leq \frac{y^c}{\pi T \log y}.$$

Consider instead

$$\frac{1}{2\pi i}\oint \frac{y^s}{s}ds.$$

On this contour ::

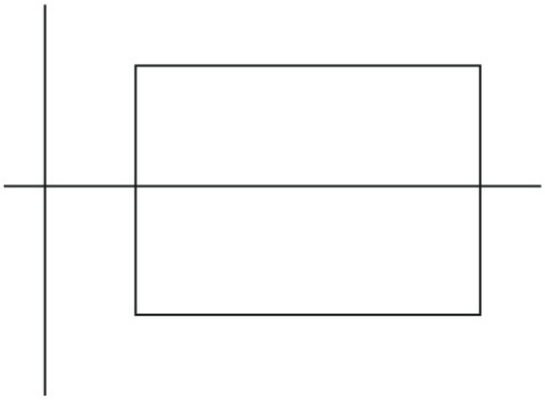


The length of the arc is $< 2\pi R$. On this arc, $|y^s| \le y^c$, and $1/|s| = 1/R$. Therefore,

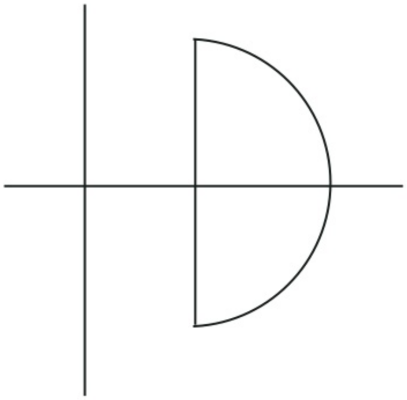
$$\left|\frac{1}{2\pi i}\int_{\text{arc}} \frac{y^s}{s} ds\right| < \frac{1}{2\pi} 2\pi \cdot R \cdot y^c \cdot \frac{1}{R} = y^c.$$

$\blacksquare$

Remark :: The case $0 < y < 1$ is very similar, only using the contours



and



(The left diagram :: No pole inside, we get 0 instead of 1)

The case $y = 1$ actually 1st year calculus:

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1^s}{s} ds = \frac{1}{2\pi i}\int_{-T}^{T} \frac{i\,dt}{c+it} = \frac{1}{2\pi}\int_{0}^{T}\left(\frac{1}{c+i} + \frac{1}{c-it}\right) dt = \cdots$$

In the sound Bite version, put $y = x/n$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \begin{cases} 1 \text{ if } 0 < n < x, \\ 0 \text{ if } n > x > 0, \\ \frac{1}{2} \text{ if } n = x > 0. \end{cases}$$

Therefore,

$$\sum_{n=1}^{\infty} a_n \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s}\right) = \sum_{n \le x} a_n \left(+\frac{1}{2} a_x \text{ if } x \in \mathbb{N}\right).$$

This leads to:

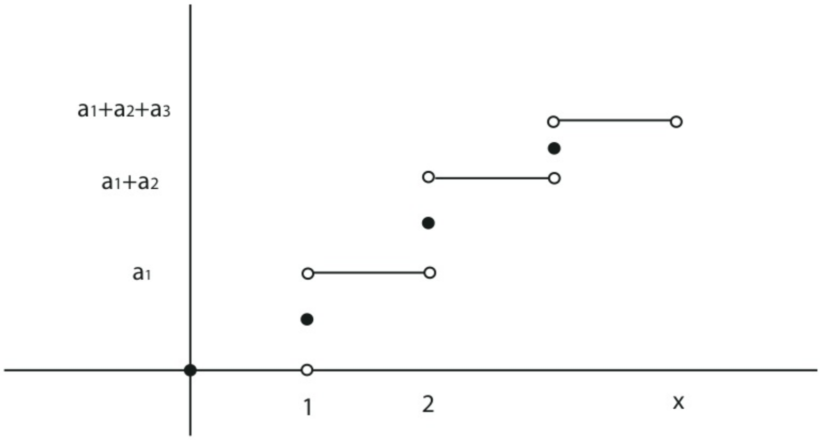
Perron's Formula :: (Sound Bite version) Let

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

have abscissa of absolute convergence σ_a. Let $x > 0$ and $c > \max\{0, \sigma_a\}$. Then

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A(s) \frac{x^s}{s} ds = \sum_{n \le x}{}' a_n,$$

where $\sum_{n \le x}'$ means, if $n \in \mathbb{N}$, include $\frac{1}{2} a_x$ in the sum instead of a_x.



Perron's Formula :: (Explicit Version) Let

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Let $x > 0$ and $C > \max\{0, \sigma_a\}$, and $T > 0$. Then

$$\left| \sum_{n \le x}{}' a_n - \frac{1}{2\pi i} \int_{c-iT}^{c+iT} A(s) \frac{x^s}{s} ds \right|$$
$$\ll V \frac{x^c}{T} + \sum_{x/2 < n < 3x/2} |a_n| \min\left\{1, \frac{c}{T|x-n|}\right\} + \left(\frac{a_x}{T} \text{ if } n \in \mathbb{N}.\right).$$

Here,

$$V = \sum_{n=1}^{\infty} |a_n| n^{-c}.$$

(Lecture 22)

Proof of Perron's Formula (Explicit Version). Let

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

let $c > \max\{0, \sigma_a\}$, and let $x, T > 0$. Then

$$\begin{aligned}
{\sum_{n\le x}}' a_n - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} A(s)\frac{x^s}{s}ds &= \sum_{n=1}^{\infty} a_n \left\{\begin{matrix} 1 \text{ if } \frac{x}{n} > 1, \\ 0 \text{ if } 0 < \frac{x}{n} < 1, \\ \frac{1}{2} \text{ if } \frac{x}{n} = 1. \end{matrix}\right\} \\
&\quad - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\frac{x^s}{s}ds \\
&= \sum_{n=1}^{\infty} a_n \left(\left\{\begin{matrix} 1 \\ 0 \\ \frac{1}{2} \end{matrix}\right\} - \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^s \frac{ds}{s}\right),
\end{aligned}$$

valid by absolute converges of $A(s)$.

By the lemma from Lecture 21, the left-hand side

$$\ll \sum_{n=1}^{\infty} |a_n| \left(\frac{x}{n}\right)^c \min\left\{1, \frac{1}{\left|T\log\left(\frac{x}{n}\right)\right|}\right\} + \left(|a_x|\frac{c}{T} \text{ if } n \in \mathbb{N}\right).$$

The terms in the sum with $n \le \frac{x}{2}$ or $n \ge \frac{3x}{2}$ have $\left|\log\left(\frac{x}{n}\right)\right| \gg 1$, and hence their contribution is

$$\begin{aligned}
&\ll \sum_{|x-n|\ge x/2} |a_n|\left(\frac{x}{n}\right)^c \cdot \frac{1}{T} \\
&\ll \frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^c} \\
&= \frac{x^c}{T}V.
\end{aligned}$$

For the terms $x < n < 3x/2$, use the "1" for the smallest n, and for the others (by a power series expansion),

$$\log\left(\frac{x}{n}\right) = \log\left(1 - \frac{x-n}{n}\right)^{-1} \asymp \frac{n-x}{x}.$$

Therefore,

$$\sum_{\substack{x/2<n<3x/2 \\ n\neq x}} |a_n| \left(\frac{x}{n}\right)^c \min\left\{1, \frac{1}{T\left|\log\left(\frac{x}{n}\right)\right|}\right\} \ll 2^c \sum_{\substack{x/2<n<3x/2 \\ n\neq x}} \left\{1, \frac{x}{T|n-x|}\right\}.$$

Final Answer :: Left-hand side

$$\ll \frac{x^c}{T}V + 2^c \sum_{\substack{x/2<n<3x/2 \\ n\neq x}} |a_n| \min\left\{1, \frac{x}{T|n-x|}\right\} + \left(|a_x|\frac{c}{T} \text{ if } n \in \mathbb{N}\right).$$

■

Example 1 :: Express $\sum_{n\leq x}' \tau(n)^2$ as a contour integral.

Solution. Let

$$B(s) = \sum_{n=1}^{\infty} \tau^2(n) n^{-s}.$$

Lets try to write $B(s)$ in terms of ζ, for example. Since τ^2 is multiplicative, we look at values on prime powers. ($*\mu$ here would be "like" a "difference operator"...)

r	0	1	2	3	4	5	$\cdots$
$\tau^2(p^r)$	1	4	9	16	25	36	$\cdots$
$\tau^2 * \mu(p^r)$	1	3	5	7	9	11	$\cdots$
$\tau^2 * \mu * \mu(p^r)$	1	2	2	2	2	2	$\cdots$
$\tau^2 * \mu * \mu * \mu(p^r)$	1	1	0	0	0	0	$\cdots$
$\tau^2 * \mu * \mu * \mu * \mu(p^r)$	1	0	-1	0	0	0	$\cdots$

Hence for $\sigma > 1$,

$$B(s)\left(\frac{1}{\zeta(s)}\right)^4 = \sum_{n=1}^{\infty}(\tau^2 * \mu * \mu * \mu * \mu)(n)n^{-s}$$
$$= \prod_p \left(1 + \frac{0}{p^s} - \frac{1}{p^{2s}} + 0 + \cdots\right) = \frac{1}{\zeta(2s)}.$$

Hence

$$B(s) = \frac{\zeta^4(s)}{\zeta(2s)},$$

and so

$$\sum_{n\le x}{}' \tau(n)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds. \quad (c > 1)$$

■

Note ::

$$\sum_{n=1}^{\infty} \tau^2(n)n^{-s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

The other rows give ::

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \tau(n^2)n^{-s}.$$
$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} 2^{\omega(n)}n^{-s}.$$
$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \mu^2(n)n^{-s}.$$

Example 2 :: Express

$$\psi_0(x) = \sum_{n\le x}{}' \Lambda(n)$$

using explicit Perron Formula.

Solution. We know that

$$\sum_{n=1}^{\infty} \Lambda(n)n^{-s} = -\frac{\zeta'}{\zeta}(s).$$

Therefore, for $c > 1$,

$$\psi_0(x) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}ds$$
$$\ll \frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|\Lambda(n)|}{n^c} + 2^c \sum_{\substack{x/2<n<3x/2 \\ n\neq x}} |\Lambda(n)| \min\left\{1, \frac{x}{T|x-n|}\right\} + \left(\frac{\Lambda(x)c}{T} \text{ if } x \in \mathbb{N}\right).$$

Using $\Lambda(n) \leq \log n$,

$$\ll \frac{x^c}{T}\left(-\frac{\zeta'}{\zeta}(c)\right) + 2^c \log x \sum_{\substack{x/2<n<3x/2 \\ n\neq x}} \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{c\log x}{T}.$$

The sum is

$$\ll 1 + 2\sum_{k=1}^{x/2} \frac{x}{Tk} \ll 1 + \frac{x\log x}{T},$$

and so

$$\begin{aligned} &\psi_0(x) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right)ds \\ &\ll \frac{x^c}{T}\left(-\frac{\zeta'}{\zeta}(c)\right) + 2^c\left(\log x + \frac{x\log^2 x}{T}\right) + \frac{c\log x}{T}. \end{aligned}$$

■

(Lecture 23)

In Lecture 22, we showed

$$\begin{aligned} \psi_0(x) = {\sum_{n\leq x}}' \Lambda(n) = &\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}ds \\ &+O\left(\frac{x^c}{T}\left(-\frac{\zeta'}{\zeta}(c)\right) + 2^c\log x\left(1+\frac{x\log x}{T}\right) + \frac{c}{T}\log x\right), \end{aligned}$$

for $c > 1$, $T > 0$. So in particular, lets stipulate $1 < T \leq x$ and choose

$$c = 1 + \frac{1}{\log x},$$

so $x^c = ex$. Then

$$\psi_0(x) = \frac{1}{2\pi i}\int_{c+1/\log x - iT}^{c+1/\log x+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)\frac{x^s}{s}ds + O\left(\frac{x\log^2 x}{T}\right),$$

since

$$-\frac{\zeta'}{\zeta}(1+\delta) \sim \frac{1}{\delta}.$$

Reason :: $-\zeta'/\zeta$ has a simple pole of residue 1 at $s = 1$, so

$$-\frac{\zeta'}{\zeta} \sim \frac{1}{s-1} \quad \text{near } s = 1.$$

At a minimum, we'd like to know that $\zeta(s) \neq 0$ for $\sigma = 1$. (In fact, this statement is equivalent to the prime number theorem $\pi(x) \sim x/\log x$). To see this, we'll use

$$\log \zeta(s) = \sum_{n=1}^{\infty} \kappa(n) n^{-s},$$

where $\kappa(p^r) = r^{-1}$ and $\kappa(n) = 0$ otherwise ($\sigma > 1$).

Consider ::

$$\mathbf{Re}(3\log\zeta(\sigma) + 4\log\zeta(\sigma+it) + \log\zeta(\sigma+2it)).$$

Since $\log\zeta$ converges absolutely for $\sigma > 1$,

$$\begin{aligned} &= \mathbf{Re}\left(\sum_{n=1}^{\infty} \kappa(n) n^{-\sigma}\left(3 + 4n^{it} + n^{-2it}\right)\right) \\ &= \sum_{n=1}^{\infty} \kappa(n) n^{-\sigma}\left(3 + 4\cos(-t\log n) + \cos(-2t\log n)\right). \end{aligned}$$

Note that

$$\begin{aligned} 3 + 4\cos\theta + \cos 2\theta &= 3 + 4\cos\theta + (2\cos^2\theta - 1) \\ &= 2(1+\cos\theta)^2 \geq 0. \end{aligned}$$

We conclude that

$$\mathbf{Re}(3\log\zeta(\sigma) + 4\log\zeta(\sigma+it) + \log\zeta(\sigma+2it)) \geq 0.$$

And so

$$\zeta(\sigma)^3|\zeta(\sigma+it)^4||\zeta(\sigma+2it)| \geq 1. \quad (\sigma > 1)$$

Suppose $\zeta(1+it) = 0$, then $\zeta(\sigma+it) \ll \sigma - 1$ for $\sigma > 1$, σ near 1. We know that

$$\zeta(\sigma) \sim \frac{1}{\sigma - 1} \quad \text{and} \quad \zeta(\sigma+2it) \ll 1.$$

Then

$$\zeta^3(\sigma)|\zeta^4(\sigma+it)||\zeta(\sigma+2it)| \ll \frac{1}{(\sigma-1)^3}(\sigma-1)^4\cdot 1 = \sigma - 1.$$

This is a contradiction as $\sigma \to 1^+$. Hence $\zeta(1+it) \neq 0$. ■

To extend this argument, we need results from classical complex analysis: Section 8.4 of Bateman and Diamond.

Borel-Carathéodory Lemma :: Suppose that $f(z)$ is analytic in $\{|z| < R\}$, $f(0) = 0$, and satisfies $\mathbf{Re}(f(z)) \leq U$ there. Then for $\{|z| < r\}$, where $0 < r < R$, we have

$$|f(z)| \leq \frac{2r}{R-r}U.$$

In fact,

$$\frac{|f^{(k)}(z)|}{k!} \leq \frac{2R}{(R-r)^{k+1}}U.$$

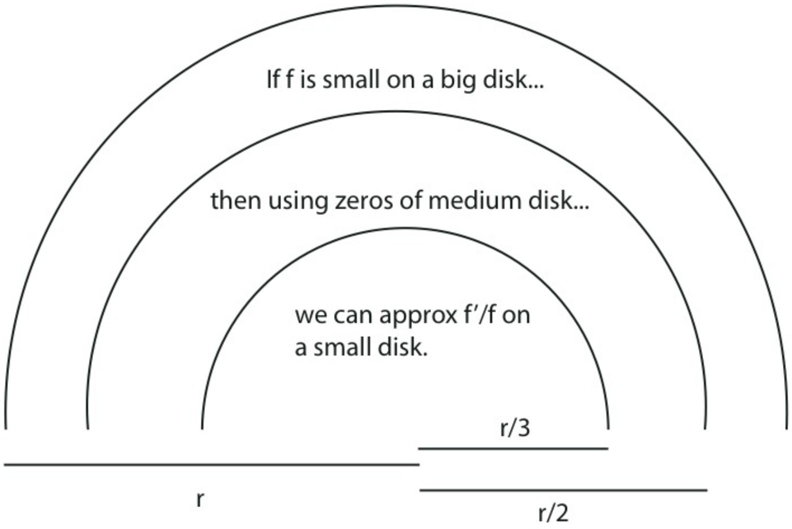
Lemma 8.10 :: Let f be analytic on $\{|z-z_0| \leq r\}$, with $f(z_0) \neq 0$. Suppose that

$$\left|\frac{f(z)}{f(z_0)}\right| < e^M,$$

for $\{|z-z_0| \leq r\}$. Then for $\{|z-z_0| \leq r/3\}$, we have

$$\left|\frac{f'}{f}(z) - \sum_{\rho}\frac{1}{z-\rho}\right| < \frac{36M}{r},$$

where ρ runs over the zeros of f in $\{|z-z_0| < r/2\}$.



(Lecture 24)

Consequence of Lemma 8.10 (and the Borel-Carathéodory Lemma)

Proposition :: Let $|t| \geq 6/7$, $6/7 \leq \delta \leq 10/7$. Then

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\log T),$$

where the sum is over

$$\left\{ \rho : \zeta(\rho) = 0, \left| \frac{8}{7} + it - \rho \right| \leq \frac{3}{7} \right\}.$$

Notation :: We let $T = |t| + 4$.

This can be proved by applying Lemma 8.10 to

$$f(z) = \zeta\left(z + \frac{8}{7} + it\right), \quad r = \frac{6}{7}.$$

The error term $O(\log T)$ uses ::

- $\zeta(s) \ll T$ (See Lemma 8.4 in B. & D.).
- $\zeta(8/7 + it) \gg 1$ (Euler Product).

Remark :: If $\delta > 1$, then each $\frac{1}{s-\rho}$ has positive real part.

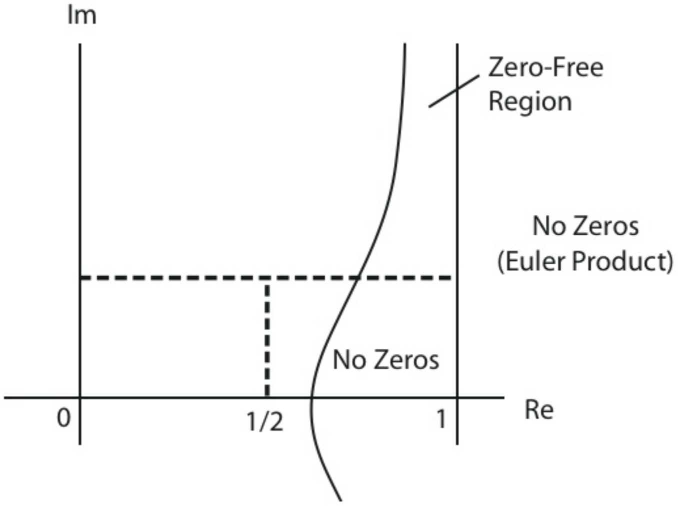
Zero-free Region for $\zeta(s)$

Theorem :: Suppose $\rho_1 = 1 - \delta + it$ satisfies

$$0 < \delta \leq \frac{1}{10}, \quad |t| \geq \frac{6}{7}, \quad \text{and} \quad \zeta(\rho_1) = 0.$$

Then there exists an absolute constant c such that

$$\delta > \frac{c}{\log T}.$$



Proof. Let $s_0 = 1 + 4\delta$, $s_1 = 1 + 4\delta + it$, $s_2 = 1 + 4\delta + 2it$. Then

$$\begin{aligned}&\mathbf{Re}\left(-3\frac{\zeta'}{\zeta}(s_0) - 4\frac{\zeta'}{\zeta}(s_1) - \frac{\zeta'}{\zeta}(s_2)\right)\\ &= \sum_{n=1}^{\infty} \Lambda(n) n^{-1-4\delta}(3 + 4\cos(t\log n) + \cos(2t\log n)) \geq 0.\end{aligned}$$

On the other hand,

- $-\frac{\zeta'}{\zeta}(s_0)$ is real, and

$$-\frac{\zeta'}{\zeta}(s_0) = \frac{1}{s_0 - 1} + O(1) = \frac{1}{4\delta} + O(1).$$

- $$\begin{aligned}\mathbf{Re}\left(-\frac{\zeta'}{\zeta}(s_1)\right) &= \mathbf{Re}\left(-\sum_\rho \frac{1}{s_1-\rho}+O(\log T)\right)\\ &\le -\mathbf{Re}\left(\frac{1}{s-\rho_1}\right)+O(\log T)\\ &= -\frac{1}{5\delta}+O(\log T).\end{aligned}$$

- $$\mathbf{Re}\left(-\frac{\zeta'}{\zeta}(s_2)\right) = \mathbf{Re}\left(-\sum_\rho \frac{1}{s_2-\rho}+O(\log T)\right) \le 0+O(\log T).$$

Therefore,

$$\begin{aligned}0 &\le \mathbf{Re}\left(-3\frac{\zeta'}{\zeta}(s_0)-4\frac{\zeta'}{\zeta}(s_1)-\frac{\zeta'}{\zeta}(s_2)\right)\\ &\le 3\left(\frac{1}{4\delta}+O(1)\right)+4\left(-\frac{1}{5\delta}+O(\log T)\right)+O\left(\log T\right).\end{aligned}$$

This gives

$$\frac{1}{2\delta} \le O(\log T), \quad \text{or} \quad \delta \gg \frac{1}{\log T}.$$

■

We'll need the following estimate ::

Lemma 8.11 in B. & D. :: If $c > 0$ is such that $\zeta(s) \neq 0$ in the region

$$1-\delta < \frac{c}{\log T}. \qquad \text{Then}$$

$$\frac{\zeta'}{\zeta}(s) \ll \log T$$

in the region

$$\left\{ s : 1-\delta < \frac{2c}{3\log T},\ |s-1| > \frac{c}{3\log T}\right\}.$$

Recall ::

$$\psi_0(x) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds + O\left(\frac{x\log^2 x}{T}\right).$$

Consider the rectangle with vertices

$$1+\frac{1}{\log x}\pm iT, \quad 1-\frac{c}{2\log T}\pm iT.$$

Inside the rectangle, $\zeta(s)$ has <u>no</u> zeros, and so

$$\frac{1}{2\pi i}\oint -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds = \mathrm{Res}_{s=1}\left(-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds\right) = \frac{x^1}{1} = x.$$

On the top and bottom segments ::

$$\begin{aligned}\frac{1}{2\pi i}\int_{1+\frac{c}{2\log T}\pm iT}^{1+\frac{1}{\log x}\pm iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds \ll (\log T)\frac{1}{T}\int_{-\infty}^{1+\frac{1}{\log x}} x^\sigma d\sigma \\ = \frac{\log T}{T}\frac{x^\sigma}{\log x}\bigg|_{-\infty}^{1+\frac{1}{\log T}} \\ = \frac{x\log T}{T\log x}.\end{aligned}$$

On the left side ::

$$\frac{1}{2\pi i}\int_{1-\frac{c}{2\log T}-iT}^{1-\frac{c}{2\log T}+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds \ll (\log T)x^{1-\frac{c}{2\log T}}\int_{1-\frac{c}{2\log T}-iT}^{1-\frac{c}{2\log T}+iT}\left|\frac{ds}{s}\right|.$$

Furthermore,

$$\int\left|\frac{ds}{s}\right| < 2\int_0^T \frac{dt}{\frac{1}{2}+t} \ll \log T.$$

We conclude that

$$\begin{aligned}\psi_0(x) &= O\left(\frac{x\log^2 x}{T}\right)+x+O\left(\frac{x\log T}{T\log x}\right)+O\left((\log T)^2 x^{1-\frac{c}{2\log T}}\right) \\ &= x+O\left(x\log^2 x\left(\frac{1}{T}+x^{\frac{-c}{2\log T}}\right)\right).\end{aligned}$$

We choose T so that

$$\frac{1}{T} = x^{\frac{-c}{2\log T}}.$$

$$\begin{aligned}\log T &= \left(\frac{c}{2\log T}\right)\log x \\ \Rightarrow \log T &= \sqrt{\frac{c\log x}{2}}.\end{aligned}$$

Final Formula ::

$$\psi_0(x) = x + O\left(x \log^2 x \cdot \exp\left(-\sqrt{\frac{c\log x}{2}}\right)\right).$$

(Lecture 26)

Two main Achievements from Lecture 25

1. There exists $c > 0$ such that if $\zeta(\sigma + it) = 0$, then

$$1 - \sigma > \frac{c}{\log \tau}, \quad \text{where } \tau = |t| + 4.$$

2.

$$\psi_0(x) = x + O\left(x \log^2 x \cdot \exp\left(-\sqrt{\frac{c\log x}{2}}\right)\right).$$

Road from $\psi_0(x)$ to $\pi(x)$

1. Let $c < \sqrt{c/2}$ (note that it's from the old c to a new c) and note that

$$\psi(x) - \psi_0(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{N}, \\ \frac{1}{2}\Lambda(x) \ll \log x & \text{if } x \in \mathbb{N}. \end{cases}$$

   So

$$\psi(x) = x + O(x \cdot \exp(-c\sqrt{\log x})).$$

2. From

$$\begin{aligned} \psi(x) &\geq \theta(x) \geq \psi(x) - 2\psi(\sqrt{x}) \\ \theta(x) &= \psi(x) + O(\psi(\sqrt{x})) = \psi(x) + O(\sqrt{x}). \end{aligned}$$

   Thus $\theta(x) = x + O(x \cdot \exp(-c\sqrt{\log x}))$.

3. Going from

$$\theta(x) = \sum_{p \leq x} \log p \quad \text{to} \quad \pi(x) = \sum_{p \leq x} 1.$$

   is partial summation.

$$\pi(x) = \int_{2^-}^{x} \frac{1}{\log t} d\theta(t) = \int_{2^-}^{x} \frac{dt}{\log t} + \int_{2^-}^{x} \frac{1}{\log t} d(\theta(t) - t).$$

We define

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Note that

$$\begin{aligned}\int_{2^-}^x \frac{1}{\log t} d(\theta(t) - t) &= \left.\frac{\theta(t) - t}{\log t}\right|_2^x + \int_2^x \frac{(\theta(t) - t)}{t \log^2 t} dt \\ &\ll O(x \cdot \exp(-c\sqrt{\log x})) + \int_2^x \frac{\exp(-c\sqrt{\log t})}{\log^2 t} dt.\end{aligned}$$

If we split this integral into

$$\int_2^y + \int_y^x,$$

where $y = x\exp(-c\sqrt{\log x})$, we find that

$$\pi(x) = \mathrm{Li}(x) + O(x \cdot \exp(-c\sqrt{\log x})).$$

4. Weaker but simpler statements

$$\begin{aligned}\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} &= \left.\frac{t}{\log t}\right|_2^x - \int_2^x t d\left(\frac{1}{\log t}\right) \\ &= \frac{x}{\log x} + O(1) + \int_2^x \frac{dt}{\log^2 t}.\end{aligned}$$

We can deduce that

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Integrating by parts again gives

$$\int_2^x \frac{dt}{\log^2 t} + 2\int_2^x \frac{dt}{\log^3 t} \quad \cdots \quad \text{and so on} \quad \cdots$$

In fact, for any $K \in \mathbb{N}$,

$$\mathrm{Li}(x) = \sum_{k=1}^K \frac{(k-1)!x}{\log^k x} + O_K\left(\frac{x}{\log^{K+1} x}\right).$$

Note that

$$x \cdot \exp(-c\sqrt{\log x}) \ll_{k,c} \frac{x}{\log^{k+1} x} \quad \text{for any } k \in \mathbb{N}.$$

Riemann's Functional Equation for $\zeta(s)$ (1859)

Define

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Riemann Proved ::

- $\xi(s)$ is entire.
- $\xi(1-s) = \xi(s)$.

(Lecture 27)

Lemma :: Define

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-n^2\pi x}, \quad (x > 0)$$

then

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\theta(x).$$

(Modular Form of Weight 1/2, whatever this means...)

Proof. Using Poisson Summation Formula ::

If $f(x)$ is smooth, f, f' integrable, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m),$$

where

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}dx.$$

So take $f(u) = e^{-u^2\pi/x}$.

$$\begin{aligned}\theta\left(\frac{1}{x}\right) &= \sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)\\ &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-u^2\pi/x}e^{-2\pi ium}du.\end{aligned}$$

Set $u = vx$,

$$\begin{aligned}\theta\left(\frac{1}{x}\right) &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-v^2\pi x} e^{-2\pi i v x m} x dv \\ &= x \sum_{m\in\mathbb{Z}} e^{-\pi^2 m x} \int_{-\infty}^{\infty} e^{-\pi x(v+m)^2} dv,\end{aligned}$$

since $-\pi^2 m x - \pi x(v+im)^2 = -v^2\pi x - 2\pi i v x m$.

Note ::

- $$\int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\pi}.$$

- $$\int_{-\infty}^{\infty} e^{-\lambda v^2} dv = \sqrt{\frac{\pi}{\lambda}} \quad \text{for } \lambda > 0.$$

It turns out that

$$\int_{-\infty}^{\infty} e^{-\lambda(v+im)^2} dv = \sqrt{\frac{\pi}{\lambda}}$$

So

$$\begin{aligned}\theta\left(\frac{1}{x}\right) &= x \sum_{m\in\mathbb{Z}} e^{-\pi^2 m x} \sqrt{\frac{\pi}{\pi x}} \\ &= \sqrt{x} \cdot \theta(x).\end{aligned}$$

∎

Proof of... We start from

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-t} t^{s/2} \frac{dt}{t} \quad (\mathbf{Re}(s) > 0).$$

Changing $t = n^2\pi x$, and note that

$$\frac{dt}{t} = \frac{n^2\pi dx}{n^2\pi x} = \frac{dx}{x}.$$

So

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-n^2\pi x} n^s \pi^{s/2} x^{s/2} \frac{dx}{x},$$

or

$$\pi^{-s/2}n^{-s}\Gamma\left(\frac{s}{2}\right)=\int_0^\infty x^{s/'2}\omega(x)\frac{dx}{x},$$

where

$$\omega(x)-\sum_{n=1}^\infty e^{-n^2\pi x}=\frac{\theta(x)-1}{2}.$$

Splitting the integral,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_1^\infty x^{s/2}\omega(x)\frac{dx}{x}+\int_0^1 x^{s/2}\omega(x)\frac{dx}{x}.$$

In the second integral, set

$$u=\frac{1}{x},\quad du=-\frac{1}{x^2}dx,\quad \text{so}\quad \frac{du}{u}=-\frac{dx}{x}.$$

Hence the second integral is

$$\int_1^\infty u^{-s/2}\omega\left(\frac{1}{u}\right)\frac{du}{u}.$$

The Lemma gives

$$\omega\left(\frac{1}{u}\right)=\sqrt{u}\cdot\omega(u)+\frac{1}{2}\sqrt{u}-\frac{1}{2}.$$

Hence

$$\begin{aligned}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)&=\int_1^\infty x^{s/2}\omega(x)\frac{dx}{x}+\int_1^\infty u^{-s/2}(\sqrt{u}\cdot\omega(u)+\frac{1}{2}\sqrt{u}+\frac{1}{2})\frac{du}{u}\\&=\int_1^\infty x^{s/2}\omega(x)\frac{dx}{x}+\int_1^\infty u^{\frac{1-s}{2}}\omega(u)\frac{du}{u}+\frac{1}{2}\int_1^\infty u^{-\frac{s}{2}-\frac{1}{2}}du\\&\quad-\frac{1}{2}\int_1^\infty u^{-\frac{s}{2}-1}du\end{aligned}$$

The last two integrals are $\frac{1}{s-1}$ and $-\frac{1}{s}$. Hence

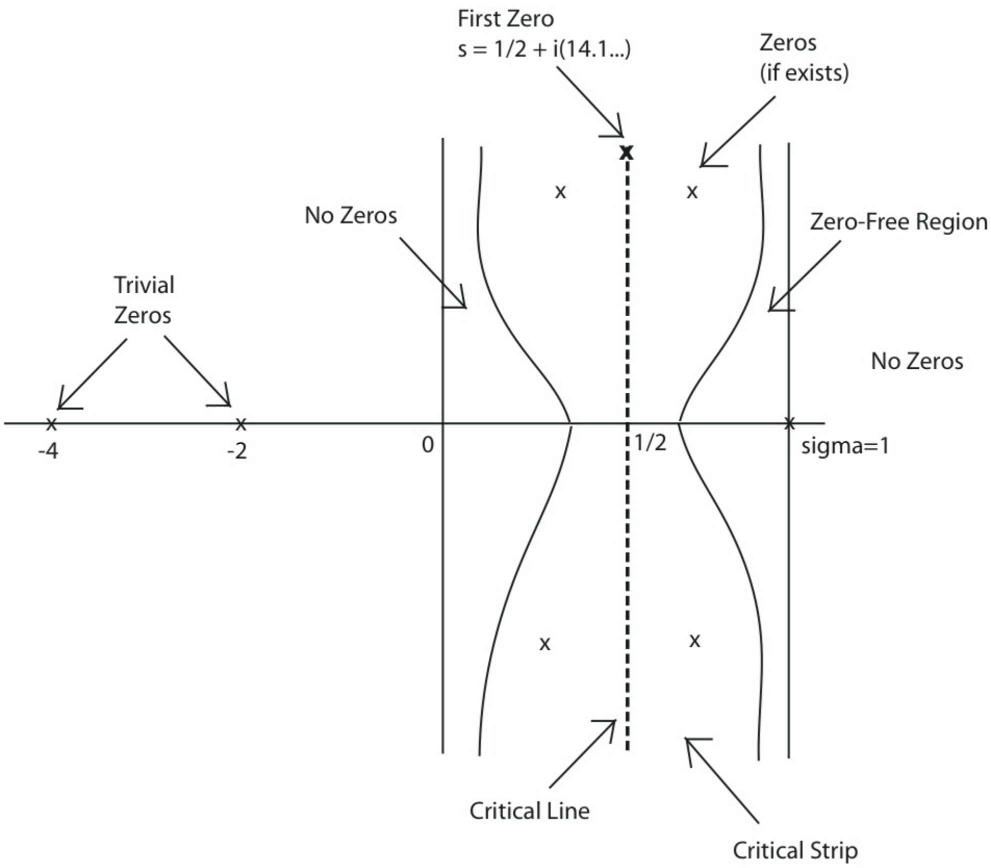
$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_1^\infty(x^{\frac{s}{2}}+x^{\frac{1-s}{s}})\,\omega(x)\frac{dx}{x}+\frac{1}{s-1}-\frac{1}{s},$$

or

$$\xi(s)=\frac{1}{2}s(s-1)\int_1^\infty(x^{\frac{s}{2}}+x^{\frac{1-s}{2}})\,\omega(x)\frac{dx}{x}+\frac{1}{2}.$$

1. The right-hand side converges for all $s\in\mathbb{C}$.

2. $\xi(1-s)=\xi(s)$.



$$\begin{aligned} N(T) &= \#\{\rho \text{ in critical strip} : 0 < \mathbf{Im}(\rho) < T, \zeta(\rho) = 0\} \\ &= \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + O(\log T). \end{aligned}$$

(Lecture 28)

Conjectures from Riemann's Memoir :: (Exerpt from Davenport)

1. Asymptotic Formula

$$\begin{aligned} N(T) &= \#\{s : 0 \le \sigma \le 1, |t| \le T; \zeta(s) = 0\} \\ &\sim \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \end{aligned}$$

2. Hadamard Factorization

$$\xi(s) = e^{A+Bs} \prod_{\rho:\, \zeta(\rho)=0} \left(1 - \frac{s}{\rho}\right) e^{s/e}.$$

Similar to ::

$$\begin{aligned}\sin(\pi z) &= \pi z \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)\\ &= \pi z \prod_{n\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{n}\right)e^{z/n}.\end{aligned}$$

We have ::

$$\xi(s) = \frac{1}{2}s(s-1)\int \text{blah } + \frac{1}{2},$$

so $e^A = \xi(0) = 1/2$.

It turns out that

$$\begin{aligned}B &= -\frac{\gamma}{2} - 1 + \frac{1}{2}\log(4\pi)\\ &= -\sum_{\substack{\rho\text{: non-trivial}\\ \text{zeros of }\zeta}} \frac{1}{\rho},\end{aligned}$$

where we interpret $\sum_\rho$ as

$$\lim_{T\to\infty} \sum_{\rho\text{: }|\text{Im }\rho|\,\le\, T}.$$

3. Explicit Formula :: (analogous to)

$$\psi_0(x) = x - \sum_{\substack{\rho\text{: }\zeta(\rho)=0\\ 0\le\text{Re}(\rho)\le 1}} \frac{x^\rho}{\rho} - \underbrace{\log 2\pi}_{\zeta'(0)/\zeta(0)} - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right),$$

for $x > 0$.

4. Riemann Hypothesis . . .

Example of using Perron's Formula ::

Determine $\sum_{n\le x}\tau(n)^2$ asymptotically.

We saw that

$$\sum_{n=1}^{\infty}\tau(n)^2 n^{-s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

Hence, by Perron's formula :: for $1 < c < 2$,

$$\sum_{n\le x}{}' \tau(n)^2 = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds + O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}|\tau(n)^2|n^{-c}\right.$$
$$\left. + \sum_{\substack{x/2<n<3x/2\\ n\neq x}} |\tau(n)^2| \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{\tau(x)^2}{T} \text{ if } x\in\mathbb{N}\right)$$

In this example :: Use $\tau(n) \ll n^\epsilon$, and we'll allow all $O-$ or $\ll-$ constants to depend on ϵ. Taking $c = 1 + 1/\log x$ as before, we get ::

The error term is

$$\ll \frac{x}{T}\frac{\zeta^4\left(1+\frac{1}{\log x}\right)}{\zeta\left(2+\frac{2}{\log x}\right)} + x^{2\epsilon}\sum_{x/2<n<3x/2} \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{x^\epsilon}{T}.$$

The sum is $\ll \frac{x\log x}{T}$ as before; also,

$$\frac{\zeta^4(1+\delta)}{\zeta(2(1+\delta))} \sim \frac{\delta^{-4}}{\zeta(2)} + O(\delta^{-3})$$

from Laurent expansion. Thus the error is

$$\ll \frac{x^{1+\epsilon}}{T}$$

for $T \le x$.

(Lecture 29)

In the example started on Lecture 28, we have shown ::

$$\sum_{n\le x}\tau(n)^2 = \frac{1}{2\pi i}\int_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT} \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds + O\left(\frac{x^{1+\epsilon}}{T}\right),$$

valid for $1 \le T \le x$ and any $\epsilon > 0$.

(Recall that in this example, all $O-$ and $\ll-$ constants can depend on ϵ)

Plan from here ::

- Look at
$$\frac{1}{2\pi i}\oint \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds$$
over the rectangle with vertices
$$1+\frac{1}{\log x}\pm iT \quad \text{and} \quad \frac{1}{2}+\epsilon\pm iT.$$

- Calculate
$$\mathrm{Res}_{s=1}\left(\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}\right).$$

To do the first part, we need the following estimates ::

- $$\left|\frac{1}{\zeta(2s)}\right| = \left|\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2s}}\right| \leq \sum_{n=1}^{\infty}|n^{-2s}| = \zeta(2\sigma), \quad \text{for } \sigma > \frac{1}{2}.$$

  In particular, for $\sigma \geq 1/2+\epsilon$, we have
  $$\left|\frac{1}{\zeta(2s)}\right| \leq \zeta\left(2\left(\frac{1}{2}+\epsilon\right)\right) \ll 1.$$

- We'll use the estimate :: for $1/2 \leq \sigma \leq 1$,
  $$\zeta(s) \ll \begin{cases} |t|^{1-\sigma}\log|t| & \text{if } |t| \geq 2, \\ \frac{1}{1-\sigma} & \text{if } |t| \leq 2. \end{cases}$$

  (Also $\zeta(s) \ll \log|t|$ for $\sigma \geq 1$ (Lemma 8.4))

Horizontal sides
$$\frac{1}{2\pi i}\int_{1/2+\epsilon+iT}^{1+1/\log x+iT}\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds \ll \int_{1/2+\epsilon}^{1+1/\log x}\frac{|\zeta^4(s)|\cdot 1\cdot x^\sigma}{T}d\sigma$$
$$\ll \frac{1}{T}\int_{1/2+\epsilon}^{1}\left(T^{1-\sigma}\log T\right)^4 x^\sigma d\sigma + \frac{1}{T}\int_{1}^{1+1/\log x}(\log T)^4 x^\sigma d\sigma. \quad (T \geq 2)$$

These integrals are
$$T^3\log^4 T\int_{1/2+\epsilon}^{1}\left(\frac{x}{T^4}\right)^\sigma d\sigma = \left.\frac{T^3\log^4 T(x/T^4)^\sigma}{\log(x/T^4)}\right|_{1/2+\epsilon}^{1} \quad (T^4 \leq x).$$

If $T < x^{1/5}$ say, then

$$\frac{1}{\log(x/T^4)} < \frac{1}{\log x^{1/5}} \ll \frac{1}{\log x}.$$

So this is

$$\begin{aligned}
&\ll \frac{T^3 \log^4 T}{\log x}\left(\frac{x}{T^4} - \left(\frac{x}{T^4}\right)^{1/2+\epsilon}\right) \\
&< \frac{T^3 \log^4 T}{\log x}\frac{x}{T^4} \ll \frac{x^{1+\epsilon}}{T}.
\end{aligned}$$

The second integral is

$$\ll \frac{\log^4 T}{T}\frac{1}{\log x} x^{1+1/\log x} \ll \frac{x^{1+\epsilon}}{T}.$$

For the left-hand side ::

$$\begin{aligned}
\frac{1}{2\pi i}\int_{1/2+\epsilon-iT}^{1/2+\epsilon+iT} \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds &\ll x^{1/2+\epsilon}\int_{-T}^{T}\frac{|\zeta(1/2+\epsilon+it)|^4}{|1/2+\epsilon+it|}dt \\
&\ll \int_0^2 \left(\frac{1}{1-(1/2+\epsilon)}\right)^4 \frac{dt}{1/2+\epsilon} + \int_2^T \left(t^{1-(1/2+\epsilon)}\log t\right)^4 \frac{dt}{t} \\
&\ll 1 + (\log x)^4 \int_2^T t^{1-4\epsilon}dt \\
&\ll 1 + \left.\frac{x^\epsilon t^{2-4\epsilon}}{2-4\epsilon}\right|_2^T \\
&\ll x^\epsilon T^{2-4\epsilon} \ll x^\epsilon T^2.
\end{aligned}$$

Now we know

$$\sum_{n\le x}\tau(n)^2 = \frac{1}{2\pi i}\oint \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds + O\left(\frac{x^{1+\epsilon}}{T} + x^\epsilon T^2\right).$$

From Complex Analysis we conclude

$$\sum_{n\le x}\tau(n)^2 = \operatorname{Res}_{s=1}\left(\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}\right) + O\left(x^\epsilon\left(\frac{x}{T}+T^2\right)\right).$$

The Laurent Expansions of

- $$\begin{aligned}
\zeta^4(s) &= \left(\frac{1}{s-1} + \gamma + \gamma_1(s-1) + \cdots\right)^4 \\
&= \frac{1}{(s-1)^4} + \frac{4\gamma}{(s-1)^3} + \frac{?}{(s-1)^2} + \frac{?}{s-1} + O(1).
\end{aligned}$$

- $$\frac{1}{s\zeta(2s)} = \frac{1}{\zeta(2)} + ?(s-1) + ?(s-1)^2 + \cdots$$

- $$\begin{aligned} x^s &= x(x^{s-1}) = xe^{(s-1)\log x} \\ &= x\sum_{k=0}^{\infty} \frac{(\log x)^k}{k!}(s-1)^k \\ &= x + x\log x(s-1) + \frac{x(\log x)^2}{2}(s-1)^2 + \frac{x(\log x)^3}{6}(s-1)^3 + O((s-1)^4). \end{aligned}$$

Multiplying these together yields

$$\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s} = \frac{x}{\zeta(2)}\frac{1}{(s-1)^4} + \cdots + \frac{1}{s-1}\left(\frac{x\log^3 x}{6\zeta(2)} + ?x\log^2 x + ?x\log x + ?\right) + O(1).$$

So

$$\mathrm{Res}_{s=1}\frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s} = xP(\log x),$$

where $P(y)$ is a cubic polynomial with leading coefficient

$$\frac{1}{6\zeta(2)} = \frac{1}{\pi^2}.$$

(Lecture 30)

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+1/3} = \lim_{T\to\infty} \sum_{\substack{n\in\mathbb{Z}\\ |n|\le T}} \frac{1}{n+1/3} = \frac{\pi}{\sqrt{3}}.$$

We saw

1. $$\sum_{n\le x}\tau(n)^2 = \frac{1}{2}\int_{1+1/\log x-iT}^{1+1/\log x+iT} \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds + O\left(\frac{x^{1+\epsilon}}{T}\right).$$

2. $$\sum_{n\le x}\tau(n)^2 = \frac{1}{2\pi i}\oint \frac{\zeta^4(s)}{\zeta(2s)}\frac{x^s}{s}ds + O\left(x^{\epsilon}\left(\frac{x}{T} + x^{1/2}T^2\right)\right),$$

   where the $\oint$ is over the rectangle with vertices $1+1/\log x + iT$ and $1/2+\epsilon \pm iT$.

3.
$$\sum_{n\le x}\tau(n)^2 = xP(\log x) + O\left(x^\epsilon\left(\frac{x}{T}+x^{1/2}T^2\right)\right),$$
where $P(u)$ is a cubic polynomial with leading coefficient
$$\frac{1}{3!}\frac{1}{\zeta(s)} = \frac{1}{\pi^2}. \quad (2\le T\le x^{1/2})$$

We choose T so that
$$\frac{x}{T} = x^{1/2}T^2,$$
that is, $T\le x^{1/6}$. So
$$\sum_{n\le x}\tau(n)^2 = xP(\log x) + O(x^{5/6+\epsilon}),$$
or
$$\sum_{n\le x}\tau(n)^2 = \frac{1}{\pi^2}x\log^2 x + O(x\log^2 x).$$

Notation ::

- G is a finite abelian group
- $\text{Hom}(G,\mathbb{C}^\times)$ is the set of (group) homomorphim from G to $\mathbb{C}$ - itself a group.
- $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ for $q\in\mathbb{N}$.
- $\mathbb{Z}_q^\times = (\mathbb{Z}/q\mathbb{Z})^\times$ the multiplicative group of unites in $\mathbb{Z}/q\mathbb{Z}$ / reduced residue classes (mod q).

Given $h:\mathbb{Z}_q^\times\to\mathbb{C}^\times$, define an arithmetic function
$$\chi(n) = \begin{cases} h\ (n \bmod q) & \text{if } (n,q)=1,\\ 0 & \text{if } (n,q)>1.\end{cases}$$

Such χ have the property ::

- χ is periodic with period q.
- $\chi(n) \neq 0 \iff (n, q) = 1$.
- $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$.

Such an arithmetic function χ is called a Dirichlet Character (mod q).

Example :: $q = 12$.

	1	2	3	4	5	6	7	8	9	10	11	12
χ_0	1	0	0	0	1	0	1	0	0	0	1	0
χ_1	1	0	0	0	-1	0	-1	0	0	0	1	0
χ_2	1	0	0	0	-1	0	1	0	0	0	-1	0
$\chi_1\chi_2$	1	0	0	0	1	0	-1	0	0	0	-1	0

In general, we let χ_0 denote the principal character (mod q) ::

$$\chi_0(n) = \begin{cases} 1 \text{ if } (n, q) = 1, \\ 0 \text{ if } (n, q) > 1. \end{cases}$$

Examples ::

- $\chi(1) = 1$, and $\chi(n)$ is a root of unity for $(n, q) = 1$.
- The set of { Dirichlet Characters (mod q) } is a group for all $q \in \mathbb{N}$ (for $z \in \{|z| = 1\}$, $z^{-1} = \overline{z}$).

In fact, this group is isomorphic to $\mathbb{Z}_q^\times$ itself.

Fundamental Theorem for Finitely Generated Abelian Groups :: Any G is "isomorphic" to exactly one group of the form

$$\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k},$$

where $r_1 | r_2 | \cdots | r_k$.

Alternatively, of the form

$$\mathbb{Z}_{p_1}^{b_1} \times \cdots \times \mathbb{Z}_{p_l}^{b_l},$$

where each $p_i^{b_i}$ is a prime power, not necessarily distinct.

Example :: $q = 7$.

	1	2	3	4	5	6	7
χ_0	1	1	1	1	1	1	0
χ	1	α^2	α	$\overline{\alpha}^2$	$\overline{\alpha}$	-1	0
χ^2	1	$\alpha^4 = \overline{\alpha}^2$	α^2	α^2	$\overline{\alpha}^2$	1	0
χ^3	1	1	$\alpha^3 = -1$	1	-1	-1	0
χ^4	1	α^2	$\overline{\alpha}^2$	$\overline{\alpha}^2$	α^2	1	0
$\chi^5 = \overline{\chi} = \chi^{-1}$	1	$\overline{\alpha}^2$	$\overline{\alpha}$	α^2	α	-1	0

3 is a primitive root (mod 7), its powers are $3, 2, 6, 4, 5, 1$ (mod 7). So we let $\alpha = e^{2\pi i/6}$...

Generality ::

- $\chi(-1)$ is always -1, +1, we call χ odd or even, respectively.
- If $\chi^2 = \chi_0$ (that is, $\chi(n) \in \{-1, 0, 1\}$ for all n), then χ is called real or quadratic.

(Greg :: I guess we don't call χ_0 quadratic)

- Let q be prime, then

$$\chi(n) = \left(\frac{n}{q}\right) \quad \text{(Legendre Symbol)}$$

is a quadratic character (mod q).

(Lecture 31)

Warm-up Question :: Describe the structure of $\mathbb{Z}_{108}^{\times}$.

- $1008 = 2^4 \times 3^2 \times 7$.

So

$$\begin{aligned}\mathbb{Z}_{1008}^{\times} &\cong \mathbb{Z}_{16}^{\times} \times \mathbb{Z}_{9}^{\times} \times \mathbb{Z}_{7}^{\times} \\ &\cong (\mathbb{Z}_2 \times \mathbb{Z}_4) \times \mathbb{Z}_6 \times \mathbb{Z}_6.\end{aligned}$$

Standard Form 1 ::

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_3).$$

Standard Form 2 ::

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{12}.$$

Orthogonality Relations for Characters

For any finite Abelian group G, any $\chi \in \mathrm{Hom}(G, \mathbb{C}^{\times})$,

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| \text{ if } \chi \text{ is trivial,} \\ 0 \quad \text{otherwise.} \end{cases}$$

Proof. If χ is nontrivial, choose $h \in G$ with $\chi(h) \neq 1$. Then

$$\chi(h) \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(hg) = \sum_{g \in G} \chi(g).$$

So since $\chi(h) \neq 1$, we get

$$\sum_{g \in G} \chi(g) = 0.$$

More generaly, for any $\chi, \psi \in \mathrm{Hom}(G, \mathbb{C}^\times)$,

$$\sum_{g \in G} \chi(g) \overline{\chi}(g) = \begin{cases} |G| \text{ if } \chi = \psi, \\ 0 \quad \text{if } \chi \neq \psi. \end{cases}$$

For any $h, g \in G$,

$$\sum_{\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)} \chi(g) \overline{\chi(h)} = \begin{cases} |G| \text{ if } g = h, \\ 0 \quad \text{otherwise.} \end{cases}$$

"Proof." Choose $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ such that

$$\chi(g) \neq \chi(h) \cdots$$

(Reality Check Problem...why does such a ψ exist?)

Given $g \in \mathbb{N}$, this means ::

- For any $a, b, \in \mathbb{N}$,

$$\sum_{\chi \pmod q} \chi(b) \overline{\chi}(a) = \begin{cases} \phi(q) \text{ if } a \equiv b \pmod q \text{ and } (a, q) = 1, \\ 0 \qquad \text{otherwise.} \end{cases}$$

- For any $\chi \pmod q$,

$$\sum_{n \pmod q} \chi(n) = \begin{cases} \phi(q) \text{ if } \chi = \chi_0, \\ 0 \qquad \text{if } \chi \neq \chi_0. \end{cases}$$

Note :: The last fact can be restated if $\chi \neq \chi_0$. Then for any $k \in \mathbb{N}$,

$$\sum_{n=k}^{k+q-1} \chi(n) = 0.$$

So what can we say about

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^\beta}?$$

(Homework #2 implies converges for $\beta > 0$)

Therefore, if we define the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

then $L(s,\chi)$ converges for $\mathbf{Re}(s) > 0$ ($\chi \neq \chi_0$). In fact, $\sigma_c = 0$ (for $\mathbf{Re}(s) < 0$, $\left|\frac{\chi(n)}{n^s}\right| \nrightarrow 0$), we still have $\sigma_a = 1$.

Remark :: If χ_0 is the principal character (mod q), then

$$L(s,\chi_0) = \sum_{n=1}^{\infty} \chi_0(n) n^{-s} = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-s} = \zeta(s) \prod_{p|q} (1 - p^{-s}).$$

Note that, for any $(a,q) = 1$,

$$\begin{aligned}
\sum_{\chi \,(\mathrm{mod}\ q)} \overline{\chi}(a) L(s,\chi) &= \sum_{\chi\ (\mathrm{mod}\ q)} \overline{\chi}(a) \sum_{n=1}^{\infty} \chi(n) n^{-s} \\
&= \sum_{n=1}^{\infty} n^{-s} \sum_{\chi\ (\mathrm{mod}\ q)} \overline{\chi}(a)\chi(n) \\
&= \phi(q) \sum_{\substack{n=1 \\ n\equiv a\ (\mathrm{mod}\ q)}} n^{-s}. \quad (\mathbf{Re}(s) > 1)
\end{aligned}$$

In fact, for $\mathbf{Re}(s) > 1$,

$$L(s,\chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}.$$

Thus, taking logarithmic derivatives,

$$\frac{L'}{L}(s,\chi) = -\sum_{n=1}^{\infty} \Lambda(n)\chi(n) n^{-s}.$$

Therefore,

$$\sum_{\chi\ (\mathrm{mod}\ q)} \overline{\chi}(a) \left(-\frac{L'}{L}(s,\chi) \right) = \phi(q) \sum_{\substack{n=1 \\ n\equiv a\ (\mathrm{mod}\ q)}}^{\infty} \Lambda(n) n^{-s}.$$

(Lecture 32)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
χ	1	-1	0	1	-1	0	0	-1	0	1	-1	0	1	0	0	1	-1	0	1	-1	0
χ^*	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	1	-1	0

General Fact :: If χ_1 and χ_2 are characters modulo q_1 and q_2, $\chi_1\chi_2$ is a character modulo **LCM**(q_1, q_2).

Special Case :: $q^* = q$, any χ^* (mod q), χ_0 (mod q). Then $\chi^*\chi_0$ is a character modulo **LCM**$(q^*, q) = q$. This new character $\chi^*\chi_0$ has the same values as the old character χ^*, except its zero when it has to be.

Example :: $q = 12$.

	1	2	3	4	5	6	7	8	9	10	11	12
χ_0	1	0	0	0	1	0	1	0	0	0	1	0
χ_1	1	0	0	0	-1	0	-1	0	0	0	1	0
χ_2	1	0	0	0	-1	0	1	0	0	0	-1	0
$\chi_1\chi_2$	1	0	0	0	1	0	-1	0	0	0	-1	0

χ_0 is induced by χ (mod 1),

χ_2 is induced by χ (mod 4),

$\chi_1\chi_2$ is induced by $\chi \neq \chi_0$ (mod 3).

Definition :: Given a character χ (mod q), let q^* denote the smallest modulus for which there exists a character χ^* satisfying

$$\underbrace{\chi^*}_{q^*}\underbrace{\chi_0}_{q} = \underbrace{\chi}_{q}.$$

We say ::

- χ^* <u>induces</u> χ.
- χ^* is a <u>primitive</u> character modulo q^*.

In particular, χ is primitive if and only if $q^* = q$, q^* is called the conductor of χ^*.

Important Fact ::

$$\{\text{character modulo } q\} \leftrightarrow \dot{\bigcup_{q^*|q}} \{\text{primitive characters modulo } q^*\}.$$

Plan :: For

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} ::$$

- Analytic Continuation / Functional Equation.
- Number of Zeros in Critical Strip.
- Zero-free Region.
- Asymptotic / Explicit formula for ::

$$\begin{aligned}
\psi(\chi; q, a) &= \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n), \\
\theta(\chi; q, a) &= \sum_{\substack{p \le x \\ p \equiv a \pmod q}} \log p, \\
\pi(\chi, q, a) &= \sum_{\substack{p \le x \\ p \equiv a \pmod q}} 1 = \#\{p \le x : p \equiv a \pmod q\}.
\end{aligned}$$

- Functional Equation :: Let

$$a = \left\{ \begin{array}{l} 0 \text{ if } \chi \text{ is even,} \\ 1 \text{ if } \chi \text{ is odd.} \end{array} \right\}$$

so that $\chi(-1) = (-1)^a$.

Define, for a primitive character $\chi(\text{mod } q)$ ::

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi).$$

Then

$$\xi(1-s,\overline{\chi}) = \frac{i^a q^{1/2}}{\tau(\chi)}\xi(s,\chi),$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

It turns out that $|\tau(n)| = q^{1/2}$, so that

$$\left|\frac{i^a q^{1/2}}{\tau(n)}\right| = 1.$$

$\Bigg($ One example of a Gauss sum :: Take $q =$ prime,

$$\chi(m) = \left(\frac{m}{q}\right).$$

Then

$$\begin{aligned}\tau(\chi) &= \sum_{m=1}^{q} \left(\frac{m}{q}\right) e^{2\pi i m/q} \\ &= \sum_{m=1}^{q} \left(\#\{b \pmod q : b^2 \equiv m \pmod q) - 1\right) e^{2\pi i m/q} \\ &= \sum_{b=1}^{q} e^{2\pi i b^2/q}. \Bigg)\end{aligned}$$

From the functional equation (together with the Euler product)

$$L(s,\chi) = \prod_p (1-\chi(p)p^{-s})^{-1} \quad \text{for } \sigma > 1,$$

we see that the only zeros of $L(s,\chi)$ outside the critical strip $\{s : 0 \le \sigma \le 1\}$ are when

- $\frac{s+a}{2}$ is a non-positive integer.
- 0, -2, -4, … for χ even.
- -1, -3, -5, … for χ odd.
- Number of zeros

Define

$$N(T,\chi) = \#\{\rho : 0 \leq \mathbf{Re}(\rho) \leq 1, |\mathbf{Im}(\rho)| \leq T, L(s,\chi) = 0\},$$

counted with multiplicity. Then

$$\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\left(\frac{qT}{2\pi e}\right) + O(\log(qT))$$

if χ is primitive modulo q.

Note :: If χ^* (mod q^*) induces χ (mod q), then

$$L(s,\chi) = L(s,\chi^*)\prod_{\substack{p|q \\ (p\nmid q^*)}}(1-\chi^*(p)p^{-s}).$$

(Lecture 32)

If $|z| = 1$, then $0 \leq \mathbf{Re}(3+4z+z^2) \geq 8$.

1. Zero-free region for $L(s,\chi)$

We start from

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1}, \\ \log L(s,\chi) &= \sum_n \kappa(n)\chi(n)n^{-s}, \qquad \sigma \geq 1. \end{aligned}$$

Then

$$\begin{aligned} 0 &\leq \mathbf{Re}\sum_{n=1}^{\infty}\kappa(n)n^{-\sigma}(3+4\chi(n)n^{-it}+\chi^2(n)n^{-2it}) \\ &= \mathbf{Re}(3\log\zeta(\sigma) + 4\log L(\sigma+it,\chi) + \log L(\sigma+2it,\chi^2)). \end{aligned}$$

Exponentiating,

$$1 \leq \zeta(\sigma)^3\left|L(\sigma+it,\chi)^4 L(\sigma+2it,\chi^2)\right|.$$

Since $L(\sigma+2it,\chi^2)$ isn't approaching a pole as $\sigma \to 1^+$, unless $\chi^2 = \chi_0$ and $t = 0$, the same argument as for $\zeta(s)$ yields $L(1+it,\chi) \neq 0$ and in fact.

Let χ be a character (mod q). Then $L(s,\chi)$ has no zeros in the region

$$\sigma > 1 - \frac{c}{\log(q\tau)}, \quad (\text{some } c > 0)$$

with the possible exception of a single real zero σ for a quadratic character χ.

A real number

$$1 > \beta > 1 - \frac{c}{\log q}$$

for which $L(\beta,\chi) = 0$ is called an exceptional zero, or Siegel zero the (quadratic) character χ is called an exceptional character.

We do know :: If $L(\sigma,\chi) = 0$, then

$$\sigma \leq 1 - \frac{c}{\sqrt{q}}.$$

2. Formulas for

$$\chi(x;q,a) = \sum_{\substack{n\leq x \\ n\equiv a \pmod q}} \Lambda(n)$$

(Lets assume $(a,q) = 1$).

- Explicit Formula

$$\begin{aligned}\psi(x;q,a) &= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{n\leq x} \Lambda(n)\chi(n) \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s,\chi)\frac{x^s}{s} ds.\end{aligned}$$

Pulling the conour way to the left ::

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{\text{non-trivial zeros } \rho \text{ of } L(s,\chi)} \frac{x^\rho}{\rho} + O(x^{1/4+\epsilon}).$$

If we use the zero-free region, we get ::

Prime Number Theorem for Arithmetic Progressions

Let $q \leq \exp(c\sqrt{\log x})$. If there's no exceptional character (mod q), then

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x \cdot \exp(-c\sqrt{\log x})).$$

If, however, there's an exceptional zero ρ corresponding to an exceptional character χ_1, then

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)}{\phi(q)}\frac{x^\beta}{\beta} + O(x \cdot \exp(-c\sqrt{\log x})).$$

$$(\overline{\chi_1}(a) = \chi_1(a) \text{ since } \chi_1 \text{ is real})$$

Note :: If $q \leq \log x$, then

$$\rho > 1 - \frac{c}{\sqrt{q}} > 1 - \frac{c}{\sqrt{\log x}}.$$

Corollary :: If $q \leq \log x$, then

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x \cdot \exp(-c\sqrt{\log x})).$$

$\Big($ By comparison, if Generalized Riemann Hypothesis (GRH) is true, then

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x^{1/2}\log^2 x) \quad \text{for all } q \leq x.$$

Note that even this isn't good when $q > \sqrt{x}$.$\Big)$

Facts to tick off

1. $$\sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{1}{p} = \frac{\log\log x}{\phi(q)} + O(1), \quad \text{for } q \leq x.$$

(partial summation from Corollary)

2. **Brun-Titchmarsh Inequality**

$$\pi(x;q,a) = \#\{p \le x : p \equiv a \pmod{q}\} < \frac{2x}{\phi(q)\log(x/q)}, \quad \text{for } 1 \le q \le x.$$

3. **Bombieri-Vinogradov Inequality**

   "GRH is true on average, for $q \le x^{1/2-\epsilon}$."

4. **Siegel-Walfisz Theorem**

   There exists an ineffective constant $C(\epsilon)$ for any $\epsilon > 0$ such that $L(\beta, \chi_1) = 0$, then

$$\beta \le 1 - \frac{C(\epsilon)}{q^{\epsilon}}.$$

5. There exists an ineffective constant H such that if $\zeta(s)$ has non-trivial zeros up to height H, then RH is true.

   Proof. If RH is false, let $\zeta(\sigma + it) = 0$ with $\sigma > 1/2$, and take $H = t$. If RH is true, take $H = 1$. ■

The End.... =)