1. FUNDAMENTAL CONCEPTS

1.1. WHAT IS A GRAPH?

1.1.1. Complete bipartite graphs and complete graphs. The complete bipartite graph \( K_{m,n} \) is a complete graph if and only if \( m = n = 1 \) or \( \{m, n\} = \{1, 0\} \).

1.1.2. Adjacency matrices and incidence matrices for a 3-vertex path.

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

Adjacency matrices for a path and a cycle with six vertices.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

1.1.3. Adjacency matrix for \( K_{m,n} \).

\[
\begin{array}{c|cc}
1 & m & n \\
\hline
m & 0 & 1 \\
1 & n & 0
\end{array}
\]

1.1.4. \( G \cong H \) if and only if \( \overline{G} \cong \overline{H} \). If \( f \) is an isomorphism from \( G \) to \( H \), then \( f \) is a vertex bijection preserving adjacency and nonadjacency, and hence \( f \) preserves non-adjacency and adjacency in \( \overline{G} \) and is an isomorphism from \( \overline{G} \) to \( \overline{H} \). The same argument applies for the converse, since the complement of \( \overline{G} \) is \( G \).

1.1.5. If every vertex of a graph \( G \) has degree 2, then \( G \) is a cycle—FALSE. Such a graph can be a disconnected graph with each component a cycle. (If infinite graphs are allowed, then the graph can be an infinite path.)

1.1.6. The graph below decomposes into copies of \( P_4 \).

1.1.7. A graph with more than six vertices of odd degree cannot be decomposed into three paths. Every vertex of odd degree must be the endpoint of some path in a decomposition into paths. Three paths have only six endpoints.

1.1.8. Decompositions of a graph. The graph below decomposes into copies of \( K_{1,3} \) with centers at the marked vertices. It decomposes into bold and solid copies of \( P_4 \) as shown.

1.1.9. A graph and its complement. With vertices labeled as shown, two vertices are adjacent in the graph on the right if and only if they are not adjacent in the graph on the left.

1.1.10. The complement of a simple disconnected graph must be connected—TRUE. A disconnected graph \( G \) has vertices \( x, y \) that do not belong to a path. Hence \( x \) and \( y \) are adjacent in \( \overline{G} \). Furthermore, \( x \) and \( y \) have no common neighbor in \( G \), since that would yield a path connecting them. Hence
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3 every vertex not in \( \{x, y\} \) is adjacent in \( \overline{G} \) to at least one of \( \{x, y\} \). Hence every vertex can reach every other vertex in \( \overline{G} \) using paths through \( \{x, y\} \).

1.1.11. **Maximum clique and maximum independent set.** Since two vertices have degree 3 and there are only four other vertices, there is no clique of size 5. A complete subgraph with four vertices is shown in bold. Since two vertices are adjacent to all others, an independent set containing either of them has only one vertex. Deleting them leaves \( P_4 \), in which the maximum size of an independent set is two, as marked.

1.1.12. **The Petersen graph.** The Petersen graph contains odd cycles, so it is not bipartite; for example, the vertices 12, 34, 51, 23, 45 form a 5-cycle.

The vertices 12, 13, 14, 15 form an independent set of size 4, since any two of these vertices have 1 as a common element and hence are nonadjacent. Visually, there is an independent set of size 4 marked on the drawing of the Petersen graph on the cover of the book. There are many ways to show that the graph has no larger independent set.

**Proof 1.** Two consecutive vertices on a cycle cannot both appear in an independent set, so every cycle contributes at most half its vertices. Since the vertex set is covered by two disjoint 5-cycles, every independent set has size at most 4.

**Proof 2.** Let \( ab \) be a vertex in an independent set \( S \), where \( a, b \in [5] \). We show that \( S \) has at most three additional vertices. The vertices not adjacent to \( ab \) are \( ac, bd, ae, bc, ad, be \), and they form a cycle in that order. Hence at most half of them can be added to \( S \).

1.1.13. **The graph with vertex set \( \{0, 1\}^k \) and \( x \leftrightarrow y \) when \( x \) and \( y \) differ in one place is bipartite.** The partite sets are determined by the parity of the number of 1s. Adjacent vertices have opposite parity. (This graph is the \( k \)-dimensional hypercube; see Section 1.3.)

1.1.14. **Cutting opposite corner squares from an eight by eight checkerboard leaves a subboard that cannot be partitioned into rectangles consisting of two adjacent unit squares.** 2-coloring the squares of a checkerboard so that adjacent squares have opposite colors shows that the graph having a vertex for each square and an edge for each pair of adjacent squares is bipartite. The squares at opposite corners have the same color; when these are deleted, there are 30 squares of that color and 32 of the other color. Each pair of adjacent squares has one of each color, so the remaining squares cannot be partitioned into sets of this type.

**Generalization:** the two partite sets of a bipartite graph cannot be matched up using pairwise-disjoint edges if the two partite sets have unequal sizes.

1.1.15. **Common graphs in four families:**

- \( A = \{ \text{paths} \} \)
- \( B = \{ \text{cycles} \} \)
- \( C = \{ \text{complete graphs} \} \)
- \( D = \{ \text{bipartite graphs} \} \)

\( A \cap B = \emptyset \): In a cycle, the numbers of vertices and edges are equal, but this is false for a path.

\( A \cap C = \{ K_1, K_2 \} \): To be a path, a graph must contain no cycle.

\( A \cap D = A \): Non-bipartite graphs have odd cycles.

\( B \cap C = K_3 \): Only when \( n = 3 \) does \( \binom{n}{2} = n \).

\( B \cap D = \{ C_{2k} : k \geq 2 \} \): odd cycles are not bipartite.

\( C \cap D = \{ K_1, K_2 \} \): Bipartite graphs cannot have triangles.

1.1.16. **The graphs below are not isomorphic.** The graph on the left has four cliques of size 4, and the graph on the right has only two. Alternatively, the complement of the graph on the left is disconnected (two 4-cycles), while the complement of the graph on the right is connected (one 8-cycle).

1.1.17. **There are exactly two isomorphism classes of 4-regular simple graphs with 7 vertices.** Simple graphs \( G \) and \( H \) are isomorphic if and only if their complements \( \overline{G} \) and \( \overline{H} \) are isomorphic, because an isomorphism \( \phi : V(G) \rightarrow V(H) \) is also an isomorphism from \( \overline{G} \) to \( \overline{H} \), and vice versa. Hence it suffices to count the isomorphism classes of 2-regular simple graphs with 7 vertices. Every component of a finite 2-regular graph is a cycle. In a simple graph, each cycle has at least three vertices. Hence each class is determined by partitioning 7 into integers of size at least 3 to be the sizes of the cycles. The only two graphs that result are \( C_7 \) and \( C_3 + C_4 \) — a single cycle or two cycles of lengths three and four.

1.1.18. **Isomorphism.** Using the correspondence indicated below, the first two graphs are isomorphic; the graphs are bipartite, with \( u_i \leftrightarrow v_i \) if and only if \( i \neq j \). The third graph contains odd cycles and hence is not isomorphic to the others.
1.1.19. Isomorphism of graphs. The rightmost two graphs below are isomorphic. The outside 10-cycle in the rightmost graph corresponds to the intermediate ring in the second graph. Pulling one of the inner 5-cycles of the rightmost graph out to the outside transforms the graph into the same drawing as the second graph.

The graph on the left is bipartite, as shown by marking one partite set. It cannot be isomorphic to the others, since they contain 5-cycles.

1.1.20. Among the graphs below, the first (F) and third (H) are isomorphic, and the middle graph (G) is not isomorphic to either of these.

F and H are isomorphic. We exhibit an isomorphism (a bijection from $V(F)$ to $V(H)$ that preserves the adjacency relation). To do this, we name the vertices of F, write the name of each vertex of F on the corresponding vertex in H, and show that the names of the edges are the same in H and F. This proves that H is a way to redraw F. We have done this below using the first eight letters and the first eight natural numbers as names.

To prove quickly that the adjacency relation is preserved, observe that 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 is a cycle in both drawings, and the additional edges 1c, 2d, 3e, 4f, 5g, 6h, 7a, 8b are also the same in both drawings. Thus the two graphs have the same edges under this vertex correspondence. Equivalently, if we list the vertices in this specified order for the two drawings, the two adjacency matrices are the same, but that is harder to verify.

$G$ is not isomorphic to F or to H. In F and in H, the numbers form an independent set, as do the letters. Thus F and H are bipartite. The graph G cannot be bipartite, since it contains an odd cycle. The vertices above the horizontal axis of the picture induce a cycle of length 7.

It is also true that the middle graph contains a 4-cycle and the others do not, but it is harder to prove the absence of a 4-cycle than to prove the absence of an odd cycle.

1.1.21. Isomorphism. Both graphs are bipartite, as shown below by marking one partite set. In the graph on the right, every vertex appears in eight 4-cycles. In the graph on the left, every vertex appears in only six 4-cycles (it is enough just to check one). Thus they are not isomorphic. Alternatively, for every vertex in the right graph there are five vertices having common neighbors with it, while in the left graph there are six such vertices.

1.1.22. Isomorphism of explicit graphs. Among the graphs below, $\{G_1, G_2, G_5\}$ are pairwise isomorphic. Also $G_3 \cong G_4$, and these are not isomorphic to any of the others. Thus there are exactly two isomorphism classes represented among these graphs.

To prove these statements, one can present explicit bijections between vertex sets and verify that these preserve the adjacency relation (such as by displaying the adjacency matrix, for example). One can also make other structural arguments. For example, one can move the highest vertex in $G_3$ down into the middle of the picture to obtain $G_4$; from this one can list the desired bijection.
One can also recall that two graphs are isomorphic if and only if their complements are isomorphic. The complements of $G_1$, $G_2$, and $G_5$ are cycles of length 7, which are pairwise isomorphic. Each of $G_3$ and $G_4$ consists of two components that are cycles of lengths 3 and 4; these graphs are isomorphic to each other but not to a 7-cycle.

**1.1.23. Smallest pairs of nonisomorphic graphs with the same vertex degrees.** For multigraphs, loopless multigraphs, and simple graphs, the required numbers of vertices are 2, 4, 5; constructions for the upper bounds appear below. We must prove that these constructions are smallest.

![Graphs](image)

- a) general
- b) loopless
- c) simple

a) With 1 vertex, every edge is a loop, and the isomorphism class is determined by the number of edges, which is determined by the vertex degree. Hence nonisomorphic graphs with the same vertex degrees have at least two vertices.

b) Every loopless graph is a graph, so the answer for loopless graphs is at least 2. The isomorphism class of a loopless graph with two vertices is determined by the number of copies of the edge, which is determined by the vertex degrees. The isomorphism class of a loopless graph with three vertices is determined by the number of edges.

Let the three vertex degrees be $a$, $b$, $c$, and let the multiplicities of the opposite edges be $x$, $y$, $z$, where $a = y + z$, $b = x + z$, and $c = x + y$, we can solve for the multiplicities in terms of the degrees by $x = (b + c - a)/2$, $y = (a + c - b)/2$, and $z = (a + b - c)/2$. Hence the multiplicities are determined by the degrees, and all loopless graphs with vertex degrees $a$, $b$, $c$ are pairwise isomorphic. Hence nonisomorphic loopless graphs with the same vertex degrees have at least four vertices.

c) Since a simple graph is a loopless graph, the answer for simple graphs is at least 4. There are 11 isomorphism classes of simple graphs with four vertices. For each of 0, 1, 5, or 6 edges, there is only one isomorphism class. For 2 edges, there are two isomorphism classes, but they have different lists of vertex degrees (similarly for 4 edges). For 3 edges, the three isomorphism classes have degree lists 3111, 2220, and 2211, all different. Hence nonisomorphic simple graphs with the same vertex degrees must have at least five vertices.

**1.1.24. Isomorphisms for the Petersen graph.** Isomorphism is proved by giving an adjacency-preserving bijection between the vertex sets. For pictorial representations of graphs, this is equivalent to labeling the two graphs with the same vertex labels so that the adjacency relation is the same in both pictures; the labels correspond to a permutation of the rows and columns of the adjacency matrices to make them identical. The various drawings of the Petersen graph below illustrate its symmetries; the labelings indicate that these are all “the same” (unlabeled) graph. The number of isomorphisms from one graph to another is the same as the number of isomorphisms from the graph to itself.

**1.1.25. The Petersen graph has no cycle of length 7.** Suppose that the Petersen graph has a cycle $C$ of length 7. Since any two vertices of $C$ are connected by a path of length at most 3 on $C$, any additional edge with endpoints on $C$ would create a cycle of length at most 4. Hence the third neighbor of each vertex on $C$ is not on $C$. 
Thus there are seven edges from \( V(C) \) to the remaining three vertices. By the pigeonhole principle, one of the remaining vertices receives at least three of these edges. This vertex \( x \) not on \( C \) has three neighbors on \( C \). For any three vertices on \( C \), either two are adjacent or two have a common neighbor on \( C \) (again the pigeonhole principle applies). Using \( x \), this completes a cycle of length at most 4. We have shown that the assumption of a 7-cycle leads to a contradiction.

Alternative completion of proof. Let \( u \) be a vertex on \( C \), and let \( v, w \) be the two vertices farthest from \( u \) on \( C \). As argued earlier, no edges join vertices of \( C \) that are not consecutive on \( C \). Thus \( u \) is not adjacent to \( v \) or \( w \). Hence, \( u, v \) have a common neighbor off \( C \), as do \( u, w \). Since \( u \) has only one neighbor off \( C \), these two common neighbors are the same. The resulting vertex \( x \) is adjacent to all of \( u, v, w \), and now \( x, v, w \) is a 3-cycle.

1.1.26. A \( k \)-regular graph of girth four has at least \( 2k \) vertices, with equality only for \( K_{k,k} \). Let \( G \) be \( k \)-regular of girth four, and chose \( xy \in E(G) \). Girth 4 implies that \( G \) is simple and that \( x \) and \( y \) have no common neighbors. Thus the neighborhoods \( N(x) \) and \( N(y) \) are disjoint sets of size \( k \), which forces at least \( 2k \) vertices into \( G \). Possibly there are others.

Note also that \( N(x) \) and \( N(y) \) are independent sets, since \( G \) has no triangle. If \( G \) has no vertices other than these, then the vertices in \( N(x) \) can have neighbors only in \( N(y) \). Since \( G \) is \( k \)-regular, every vertex of \( N(x) \) must be adjacent to every vertex of \( N(y) \). Thus \( G \) is isomorphic to \( K_{k,k} \), with partite sets \( N(x) \) and \( N(y) \). In other words, there is only one such isomorphism class for each value of \( k \).

Comment. One can also start with a vertex \( x \), choose \( y \) from among the \( k \) vertices in \( N(x) \), and observe that \( N(y) \) must have \( k - 1 \) more vertices not in \( N(x) \cup \{ x \} \). The proof then proceeds as above.

(An alternative proof uses the methods of Section 1.3. A triangle-free simple graph with \( n \) vertices has at most \( n^2/2 \) edges. Since \( G \) is \( k \)-regular, this yields \( n^2/4 \geq nk/2 \), and hence \( n \geq 2k \). Furthermore, equality holds in the edge bound only for \( K_{n/2,n/2} \), so this is the only such graph with \( 2k \) vertices. (C. Pikscher))

1.1.27. A simple graph of girth 5 in which every vertex has degree at least \( k \) has at least \( k^2 + 1 \) vertices, with equality achieveable when \( k \in \{ 2, 3 \} \). Let \( G \) be \( k \)-regular of girth five. Let \( S \) be the set consisting of a vertex \( x \) and its neighbors. Since \( G \) has no cycle of length less than five, \( G \) is simple, and any two neighbors of \( x \) are nonadjacent and have no common neighbor other than \( x \). Hence each \( y \in S - \{ x \} \) has at least \( k - 1 \) neighbors that are not in \( S \) and not neighbors of any vertex in \( S \). Hence \( G \) has at least \( k(k - 1) \) vertices outside \( S \) and at least \( k + 1 \) vertices in \( S \) for at least \( k^2 + 1 \) altogether.

The 5-cycle achieves equality when \( k = 2 \). For \( k = 3 \), growing the graph symmetrically from \( x \) permits completing the graph by adding edges among the non-neighbors of \( x \). The result is the Petersen graph. (Comment: For \( k > 3 \), it is known that girth 5 with minimum degree \( k \) and exactly \( k^2 + 1 \) vertices is impossible, except for \( k = 7 \) and possibly for \( k = 57 \).)

1.1.28. The Odd Graph has girth 6. The Odd Graph \( O_k \) is the disjointness graph of the set of \( k \)-element subsets of \( [2k + 1] \).

Vertices with a common neighbor correspond to \( k \)-sets with \( k - 1 \) common elements. Thus they have exactly one common neighbor, and \( O_k \) has no 4-cycle. Two vertices at distance 2 from a single vertex have at least \( k - 2 \) common neighbors. For \( k > 2 \), such vertices cannot be adjacent, and thus \( O_k \) has no 5-cycle when \( k > 2 \). To form a 6-cycle when \( k \geq 2 \), let \( A = \{ 2, \ldots, k \} \), \( B = \{ k + 2, \ldots, 2k \} \), \( a = 1 \), \( b = k + 1 \), \( c = 2k + 1 \). A 6-cycle is \( A \cup \{ a \} \), \( B \cup \{ b \} \), \( A \cup \{ c \} \), \( B \cup \{ a \} \), \( A \cup \{ b \} \), \( B \cup \{ c \} \).

The Odd Graph also is not bipartite. The successive elements \( \{ 1, \ldots, k \} \), \( \{ k + 1, \ldots, 2k \} \), \( \{ 2k + 1, 1, \ldots, k - 1 \} \), \ldots , \( \{ k + 2, \ldots, 2k + 1 \} \) form an odd cycle.

1.1.29. Among any 6 people, there are 3 mutual acquaintances or 3 mutual strangers. Let \( G \) be the graph of the acquaintance relation, and let \( x \) be one of the people. Since \( x \) has 5 potential neighbors, \( x \) has at least 3 neighbors or at least 3 nonneighbors. By symmetry (if we complement \( G \), we still have to prove the same statement), we may assume that \( x \) has at least 3 neighbors. If any pair of these people are acquainted, then with \( x \) we have 3 mutual acquaintances, but if no pair of neighbors of \( x \) is acquainted, then the neighbors of \( x \) are three mutual strangers.

1.1.30. The number of edges incident to \( v_i \) is the \( i \)th diagonal entry in \( MM^T \) and in \( A^2 \). In both \( MM^T \) and \( A^2 \) this is the sum of the squares of the entries
in the $i$th row. For $MM^T$, this follows immediately from the definition of matrix multiplication and transposition, but for $A^2$ this uses the graph-theoretic fact that $A = A^T$; i.e. column $i$ is the same as row $i$. Because $G$ is simple, the entries of the matrix are all 0 or 1, so the sum of the squares in a row equals the number of 1s in the row. In $M$, the 1s in a row denote incident edges; in $A$ they denote vertex neighbors. In either case, the number of 1s is the degree of the vertex.

If $i \neq j$, then the entry in position $(i,j)$ of $A^2$ is the number of common neighbors of $v_i$ and $v_j$. The matrix multiplication puts into position $(i,j)$ the “product” of row $i$ and column $j$; that is $\sum_{k=1}^{n} a_{i,k} a_{k,j}$. When $G$ is simple, entries in $A$ are 0 or 1, depending on whether the corresponding vertices are adjacent. Hence $a_{i,k} a_{k,j} = 1$ if $v_i$ is a common neighbor of $v_j$ and $v_j$; otherwise, the contribution is 0. Thus the number of contributions of 1 to entry $(i,j)$ is the number of common neighbors of $v_i$ and $v_j$.

If $i \neq j$, then the entry in position $(i,j)$ of $MM^T$ is the number of edges joining $v_i$ and $v_j$ (0 or 1 when $G$ has no multiple edges). The $i$th row of $M$ has 1s corresponding to the edges incident to $v_i$. The $j$th column of $M^T$ is the same as the $j$th row of $M$, which has 1s corresponding to the edges incident to $v_j$. Summing the products of corresponding entries will contribute 1 for each edge incident to both $v_i$ and $v_j$; 0 otherwise.

Comment. For graphs without loops, both arguments for $(i,j)$ in general apply when $i = j$ to explain the diagonal entries.

1.1.31. $K_n$ decomposes into two isomorphic (“self-complementary”) subgraphs if and only if $n$ or $n − 1$ is divisible by 4.

a) The number of vertices in a self-complementary graph is congruent to 0 or 1 (mod 4). If $G$ and $\overline{G}$ are isomorphic, then they have the same number of edges, but together they have $\binom{n}{2}$ edges (with none repeated), so the number of edges in each must be $n(n−1)/4$. Since this is an integer and the numbers $n$ and $n−1$ are not both even, one of $\{n,n−1\}$ must be divisible by 4.

b) Construction of self-complementary graphs for all such $n$.

Proof 1 (explicit construction). We generalize the structure of the self-complementary graphs on 4 and 5 vertices, which are $P_4$ and $C_5$. For $n = 4k$, take four vertex sets of size $k$, say $X_1$, $X_2$, $X_3$, $X_4$, and join all vertices of $X_i$ to those of $X_{i+1}$, for $i = 1, 2, 3$. To specify the rest of $G$, within these sets let $X_1$ and $X_4$ induce copies of a graph $H$ with $k$ vertices, and let $X_2$ and $X_3$ induce $\overline{H}$. (For example, $H$ may be $K_4$.) In $\overline{G}$, both $X_2$ and $X_3$ induce $H$, while $X_1$ and $X_4$ induce $\overline{H}$, and the connections between sets are $X_2 \leftrightarrow X_4 \leftrightarrow X_1 \leftrightarrow X_3$. Thus relabeling the subsets defines an isomorphism between $G$ and $\overline{G}$. (There are still other constructions for $G$.)

For $n = 4k + 1$, we add a vertex $x$ to the graph constructed above. Join $x$ to the $2k$ vertices in $X_1$ and $X_4$ to form $G$. The isomorphism showing that $G − x$ is self-complementary also works for $G$ (with $x$ mapped to itself), since this isomorphism maps $N_G(x) = X_1 \cup X_4$ to $N_{\overline{G}}(x) = X_2 \cup X_3$.

Proof 2 (inductive construction). If $G$ is self-complementary, then let $H_1$ be the graph obtained from $G$ and $P_4$ by joining the two ends of $P_4$ to all vertices of $G$. Let $H_2$ be the graph obtained from $G$ and $P_4$ by joining the two center vertices of $P_4$ to all vertices of $G$. Both $H_1$ and $H_2$ are self-complementary. Using this with $G = K_1$ produces the two self-complementary graphs of order 5, namely $C_5$ and the bull.

Self-complementary graphs with order divisible by 4 arise from repeated use of the above using $G = P_4$ as a starting point, and self-complementary graphs of order congruent to 1 modulo 4 arise from repeated use of the above using $G = K_1$ as a starting point. This construction produces many more self-complementary graphs than the explicit construction in Proof 1.

1.1.32. $K_{m,n}$ decomposes into two isomorphic subgraphs if and only if $m$ and $n$ are not both odd. The condition is necessary because the number of edges must be even. It is sufficient because $K_{m,n}$ decomposes into two copies of $K_{m,n/2}$ when $n$ is even.

1.1.33. Decomposition of complete graphs into cycles through all vertices. View the vertex set of $K_n$ as $\mathbb{Z}_n$, the values $0, \ldots , n − 1$ in cyclic order. Since each vertex has degree $n − 1$ and each cycle uses two edges at each vertex, the decomposition has $(n−1)/2$ cycles.

For $n = 5$ and $n = 7$, it suffices to use cycles formed by traversing the vertices with constant difference: $(0, 1, 2, 3, 4)$ and $(0, 2, 4, 1, 3)$ for $n = 5$ and $(0, 1, 2, 3, 4, 5, 6, 0, 2, 4, 6, 1, 3, 5)$, and $(0, 3, 6, 2, 5, 1, 4)$ for $n = 7$.

This construction fails for $n = 9$ since the edges with difference 3 form three 3-cycles. The cyclically symmetric construction below treats the vertex set as $\mathbb{Z}_8$ together with one special vertex.
1.1.34. Decomposition of the Petersen graph into copies of $P_4$. Consider the drawing of the Petersen graph with an inner 5-cycle and outer 5-cycle. Each $P_4$ consists of one edge from each cycle and one cross edge joining them. Extend each cross edge $e$ to a copy of $P_4$ by taking the edge on each of the two 5-cycles that goes in a clockwise direction from $e$. In this way, the edges on the outside 5-cycle are used in distinct copies of $P_4$, and the same holds for the edges on the inside 5-cycle.

Decomposition of the Petersen graph into three pairwise-isomorphic connected subgraphs. Three such decompositions are shown below. We restricted the search by seeking a decomposition that is unchanged by 120° rotations in a drawing of the Petersen graph with 3-fold rotational symmetry. The edges in this drawing form classes of size 3 that are unchanged under rotations of 120°; each subgraph in the decomposition uses exactly one edge from each class.

1.1.35. $K_n$ decomposes into three pairwise-isomorphic subgraphs if and only if $n + 1$ is not divisible by 3. The number of edges is $n(n - 1)/2$. If $n + 1$ is divisible by 3, then $n$ and $n - 1$ are not divisible by 3. Thus decomposition into three subgraphs of equal size is impossible in this case.

If $n + 1$ is not divisible by 3, then $e(K_n)$ is divisible by 3, since $n$ or $n - 1$ is divisible by 3. We construct a decomposition into three subgraphs that are pairwise isomorphic (there are many such decompositions).

When $n$ is a multiple of 3, we partition the vertex set into three subsets $V_1, V_2, V_3$ of equal size. Edges now have two types: within a set or joining two sets. Let the $i$th subgraph $G_i$ consist of all the edges within $V_i$ and all the edges joining the two other subsets. Each edge of $K_n$ appears in exactly one of these subgraphs, and each $G_i$ is isomorphic to the disjoint union of $K_{n/3}$ and $K_{n/3, n/3}$.

When $n \equiv 1 \pmod{3}$, consider one vertex $w$. Since $n - 1$ is a multiple of 3, we can form the subgraphs $G_i$ as above on the remaining $n - 1$ vertices. Modify $G_i$ to form $H_i$ by joining $w$ to every vertex of $V_i$. Each edge involving $w$ has been added to exactly one of the three subgraphs. Each $H_i$ is isomorphic to the disjoint union of $K_{1+(n-1)/3}$ and $K_{(n-1)/3, (n-1)/3}$.

1.1.36. If $K_n$ decomposes into triangles, then $n - 1$ or $n - 3$ is divisible by 6. Such a decomposition requires that the degree of each vertex is even and the number of edges is divisible by 3. To have even degree, $n$ must be odd. Also $n(n - 1)/2$ is a multiple of 3, so 3 divides $n$ or $n - 1$. If 3 divides $n$ and $n$ is odd, then $n - 3$ is divisible by 6. If 3 divides $n - 1$ and $n$ is odd, then $n - 1$ is divisible by 6.

1.1.37. A graph in which every vertex has degree 3 has no decomposition into paths with at least five vertices each. Suppose that $G$ has such a decomposition. Since every vertex has degree 3, each vertex is an endpoint of at least one of the paths and is an internal vertex on at most one of them. Since every path in the decomposition has two endpoints and has at least three internal vertices, we conclude that the number of paths in the decomposition is at least $n(G)/2$ and is at most $n(G)/3$, which is impossible.

Alternatively, let $k$ be the number of paths. There are $2k$ endpoints of paths. On the other hand, since each internal vertex on a path in the decomposition must be an endpoint of some other path in the decomposition, there must be at least $3k$ endpoints of paths. The contradiction implies that there cannot be such a decomposition.

1.1.38. A 3-regular graph $G$ has a decomposition into claws if and only if $G$ is bipartite. When $G$ is bipartite, we produce a decomposition into claws. We use all claws obtained by taking the three edges incident with a single vertex in the first partite set. Each claw uses all the edges incident to its central vertex. Since each edge has exactly one endpoint in the first partite set, each edge appears in exactly one of these claws.

When $G$ has a decomposition into claws, we partition $V(G)$ into two independent sets. Let $X$ be the set of centers of the claws in the decomposition. Since every vertex has degree 3, each claw in the decomposition
uses all edges incident to its center. Since each edge is in at most one claw, this makes $X$ an independent set. The remaining vertices also form an independent set, because every edge is in some claw in the decomposition, which means that one of its endpoints must be the center of that claw.


Triangle—No. A graph decomposing into triangles must have even degree at each vertex. (This excludes all decompositions into cycles.)

Paw, $P_6$—No. $K_6$ has 15 edges, but each paw or $P_6$ has four edges.

House, Boutie, Dart—No. $K_6$ has 15 edges, but each house, boutie, or dart has six edges.

Claw—Yes. Put five vertices 0, 1, 2, 3, 4 on a circle and the other vertex $z$ in the center. For $i \in \{0, 1, 2, 3, 4\}$, use a claw with edges from $i$ to $i + 1$, $i + 2$, and $z$. Each edge appears in exactly one of these claws.

Kite—Yes. Put all six vertices on a circle. Each kite uses two opposite edges on the outside, one diagonal, and two opposite edges of “length” 2. Three rotations of the picture complete the decomposition.

Bull—Yes. The bull has five edges, so we need three bulls. Each bull uses degrees 3, 3, 2, 1, 1. 0 at the six vertices. Each bull misses one vertex, and each vertex of $K_6$ has five incident edges, so three of the vertices will receive degrees 3, 2, 0 from the three bulls, and the other three will receive degrees 3, 1, 1. Thus we use vertices of two types, which leads us to position them on the inside and outside as on the right below. The bold, solid, and dashed bulls obtained by rotation complete the decomposition.

1.1.40. Automorphisms of $P_n$, $C_n$, and $K_n$. A path can be left alone or flipped, a cycle can be rotated or flipped, and a complete graph can be permuted arbitrarily. The numbers of automorphisms are 2, 2n, n!, respectively. Correspondingly, the numbers of distinct labelings using vertex set [n] are $n! / 2$, $(n - 1)! / 2$, 1, respectively. For $P_n$, these formulas require $n > 1$.

1.1.41. Graphs with one and three automorphisms. The two graphs on the left have six vertices and only the identity automorphism. The two graphs on the right have three automorphisms.

1.1.42. The set of automorphisms of a graph $G$ satisfies the following:

a) The composition of two automorphisms is an automorphism.

b) The identity permutation is an automorphism.

c) The inverse of an automorphism is also an automorphism.

d) Composition of automorphisms satisfies the associative property.

The first three properties are essentially the same as the transitive, reflexive, and symmetric properties for the isomorphism relation; see the discussion of these in the text. The fourth property holds because composition of functions always satisfies the associative property (see the discussion of composition in Appendix A).

1.1.43. Every automorphism of the Petersen graph maps the 5-cycle $(12,34,51,23,45)$ into a 5-cycle with vertices $ab, cd, ea, bc, de$ by a permutation of [5] taking $1, 2, 3, 4, 5$ to $a, b, c, d, e$, respectively. Let $\sigma$ denote the automorphism, and let the vertex $ab$ be the image of the vertex 12 under $\sigma$. The image of 34 must be a pair disjoint from $ab$, so we may let $cd = \sigma(34)$. The third vertex must be disjoint from the second and share an element with the first. We may select $a$ to be the common element in the first and third vertices. Similarly, we may select $c$ to be the common element in the second and fourth vertices. Since nonadjacent vertices correspond to sets with a common element, the other element of the fourth vertex must be $b$, and the fifth vertex can’t have $a$ or $b$ and must have $d$ and $e$. Thus every 5-cycle must have this form and is the image of $(12,34,51,23,45)$ under the specified permutation $\sigma$.

The Petersen graph has 120 automorphisms. Every permutation of [5] preserves the disjointness relation on 2-element subsets and therefore defines an automorphism of the Petersen graph. Thus there are at least 120 automorphism. To show that there are no others, consider an arbitrary automorphism $\sigma$. By the preceding paragraph, the 5-cycle $C$ maps to some 5-cycle $(ab, cd, ea, bc, de)$. This defines a permutation $f$ mapping $1, 2, 3, 4, 5$ to $a, b, c, d, e$, respectively. It suffices to show that the other vertices must also have images under $\sigma$ that are described by $f$.

The remaining vertices are pairs consisting of two nonconsecutive values modulo 5. By symmetry, it suffices to consider just one of them, say 24. The only vertex of $C$ that 24 is adjacent to (disjoint from) is 51. Since
1.1.44. For each pair of 3-edge paths $P = (u_0, u_1, u_2, u_3)$ and $Q = (v_0, v_1, v_2, v_3)$ in the Petersen graph, there is an automorphism of the Petersen graph that turns $P$ into $Q$. In the disjointness representation of the Petersen graph, suppose the pairs corresponding to the vertices of $P$ are $ab$, $cd$, $ef$, $gh$, respectively. Since consecutive pairs are disjoint and the edges are unordered pairs, we may write the pairs so that $a, b, c, d, e$ are distinct, $f = a$, $g = b$, and $h = c$. Putting the vertex names of $Q$ in the same format $AB, CD, EF, GH$, we chose the isomorphism generated by the permutation of $[5]$ that turns $a, b, c, d, e$ into $A, B, C, D, E$, respectively.

1.1.45. A graph with 12 vertices in which every vertex has degree 3 and the only automorphism is the identity:

![Graph with 12 vertices](image)

There are many ways to prove that an automorphism must fix all the vertices. The graph has only two triangles ($abc$ and $uvw$). Now an automorphism must fix $p$, since it is the only vertex having no neighbor on a triangle, and also $e$, since it is the only vertex with neighbors on both triangles. Now $d$ is the unique common neighbor of $p$ and $e$. The remaining vertices can be fixed iteratively in the same way, by finding each as the only unlabeled vertex with a specified neighborhood among the vertices already fixed. (This construction was provided by Luis Dissett, and the argument forbidding nontrivial automorphisms was shortened by Fred Galvin. Another such graph with three triangles was found by a student of Fred Galvin.)

1.1.46. Vertex-transitivity and edge-transitivity. The graph on the left in Exercise 1.1.21 is isomorphic to the 4-dimensional hypercube (see Section 1.3), which is vertex-transitive and edge-transitive via the permutation of coordinates. For the graph on the right, rotation and inside-out exchange takes care of vertex-transitivity. One further generating operation is needed to get edge-transitivity; the two bottom outside vertices can be switched with the two bottom inside vertices.

1.1.47. Edge-transitive versus vertex-transitive. a) If $G$ is obtained from $K_n$ with $n \geq 4$ by replacing each edge of $K_n$ with a path of two edges through a new vertex of degree 2, then $G$ is edge-transitive but not vertex-transitive. Every edge consists of an old vertex and a new vertex. The $n!$ permutations of old vertices yield automorphism. Let $x \& y$ denote the new vertex on the path replacing the old edge $xy$; note that $xy = y \& x$. The edge joining $x$ and $x \& y$ is mapped to the edge joining $u$ and $uv$ by any automorphism that maps $x$ to $u$ and $y$ to $v$. The graph is not vertex-transitive, since $xy$ has degree 2, while $x$ has degree $n - 1$.

b) If $G$ is edge-transitive but not vertex-transitive and has no isolated vertices, then $G$ is bipartite. Let $uv$ be an arbitrary edge of $G$. Let $S$ be the set of vertices to which $u$ is mapped by automorphisms of $G$, and let $T$ be the set of vertices to which $v$ is mapped. Since $G$ is edge-transitive and has no isolated vertex, $S \cup T = V(G)$. Since $G$ is not vertex-transitive, $S \neq V(G)$. Together, these statements yield $S \cap T = \emptyset$, since the composition of two automorphisms is an automorphism. By edge-transitivity, every edge of $G$ contains one vertex of $S$ and one vertex of $T$. Since $S \cap T = \emptyset$, this implies that $G$ is bipartite with vertex bipartition $S, T$.

c) The graph below is vertex-transitive but not edge-transitive. A composition of left-right reflections and vertical rotations can take each vertex to any other. The graph has some edges on triangles and some edges not on triangles, so it cannot be edge-transitive.

1.2. PATHS, CYCLES, AND TRAILS

1.2.1. Statements about connection.

a) Every disconnected graph has an isolated vertex—FALSE. A simple 4-vertex graph in which every vertex has degree 1 is disconnected and has no isolated vertex.

b) A graph is connected if and only if some vertex is connected to all other vertices—TRUE. A vertex is “connected to” another if they lie in a common path. If $G$ is connected, then by definition each vertex is connected to every other. If some vertex $x$ is connected to every other, then because a $u, x$-path and $x, v$-path together contain a $u, v$-path, every vertex is connected to every other, and $G$ is connected.