1. Introduction

This survey regards the following conjecture which was independently proposed by Szekeres [15] and Seymour [13] (Szekeres’ statement has a small error).

Conjecture 1.1 (Circuit Double Cover). For every finite coloop-free graph $G$, there exists a list of its circuits such that every edge of $G$ appears in exactly two circuits in the list.

1.1. Cycles. Graphs are finite unless otherwise specified. It is convenient to adopt terminology used for binary matroids. A cycle is the edge set of a subgraph of $G$ in which all vertices have even degree. The binary sum or symmetric difference $D_1 \oplus D_2$ of two cycles $D_1, D_2$ results in another cycle; this operation corresponds to addition of their 0,1-characteristic vectors $\chi_{D_i}$ ($i = 1, 2$) in the space $\mathbb{F}_2^E$ of functions from $E(G)$ to the 2-element field. Such vectors form a subspace of $\mathbb{F}_2^E$ called the (binary) cycle space of $G$. The dimension of the cycle space is Betti number or corank $\text{e}(G) - n(G) + c(G)$, where $c(G)$ is the number of connected components in $G$. A circuit is a minimal nonempty cycle. A coloop is an edge of $G$ contained in no circuit. A $k$-cycle double cover (abbreviated $k$-CDC) of a graph is a list $\mathcal{D} = (D_1, \ldots, D_k)$ of cycles such that every edge in $G$ appears in exactly two cycles of the list. We may suppress the prefix “$k$-” from $k$-CDC. A CDC whose members are circuits is a circuit double cover (cDC). We generally use $C$ instead of $D$ to denote a cDC. By decomposing some of the cycles in a CDC $\mathcal{D}$ into nonempty subcycles, we obtain a new CDC which is a refinement of $\mathcal{D}$. Every CDC can be refined to a cDC. A cDC is $k$-colourable if it is a refinement of a $k$-CDC.

1.2. Embeddings. Let $v$ be a vertex in a graph $G$ with a cDC $\mathcal{C}$. There exists a permutation $\pi_v$ of the edges in the vertex boundary $\delta(v)$ such that, for every $e \in \delta(v)$, some circuit in $\mathcal{C}$ contains both $e$ and $\pi_v(e)$. A vertex $v$ has rotation (with respect to $\mathcal{C}$) if $\pi_v$ is a cyclic permutation. For each circuit $C \in \mathcal{C}$ we may erect a topological disc whose boundary is $C$, thereby obtaining an embedding of $G$ in a pseudosurface. A pseudosurface is obtained from a closed 2-manifold by a finite number of point identifications (such identified points correspond to those vertices which fail to have rotation). A CDC is strong if it can be refined to a cDC for which every vertex has rotation. Every strong cDC of a graph $G$ corresponds to an embedding of $G$ on some surface in which every region is bounded by a circuit of the graph. Such an embedding is called a strong embedding of $G$. If $G$ has maximum degree 3, then every cDC of $G$ yields a strong embedding of $G$. A graph has strong $k$-CDC if and only if it has a strong embedding whose regions can be properly coloured with $k$ colours.

1.3. Flows. Let $\Gamma$ be an additive abelian group. A $\Gamma$-flow of a graph $G$ is a pair $(\vec{E}, \phi)$ where $\vec{E}$ is an orientation of $E(G)$ and $\phi : \vec{E} \to \Gamma$ satisfies $\sum \{ \phi(uv) \mid uv \in \vec{E} \} = \sum \{ \phi(vu) \mid uv \in \vec{E} \}$, for every vertex $v$. By associating each $g \in \Gamma$ with its negative, the orientation $\vec{E}$ in a $\Gamma$-flow becomes less relevant, so we often write $\phi$ instead of $(\vec{E}, \phi)$. We typically seek a $\Gamma$-flow $\phi$ whose support, $\text{supp} \phi$, is all of $E(G)$; such a flow is a nowhere-zero $\Gamma$-flow ($\Gamma$-NZF).

Let $\mathbb{Z}/k\mathbb{Z}$ denote the cyclic group $\mathbb{Z}/k\mathbb{Z}$. Some authors use “mod $k$-flow” to refer to a $\mathbb{Z}_k$-flow. The set of $\mathbb{Z}_2$-flows in $G$ coincides with the binary cycle space of $G$. We write $\mathbb{Z}_k^d$ for the $d$th cartesian power of $\mathbb{Z}_k$. Thus a $\mathbb{Z}_k^d$-NZF corresponds to an ordered list of $d$ cycles in $G$ whose union is $E(G)$.

A $k$-flow is a $\mathbb{Z}$-flow, $\phi$, for which $|\phi(e)| < k$ for every edge $e$. A nowhere-zero $k$-flow ($k$-NZF) has an obvious definition. Tutte [16] showed that a graph has a $\Gamma$-NZF if and only if it has a $|\Gamma|$-NZF. The flow number of $G$ is the least $k \in \mathbb{Z}$ for which $G$ has a $k$-NZF. Much of the work in this topic has been motivated by the observation that the flow number of a plane graph equals the chromatic number of its dual graph.
Seymour [14] showed that every coloop-free graph has a 6-NZF. Tutte’s 5-flow conjecture [17] asserts that all coloop-free graphs have a 5-NZF. A cubic graph has a 4-NZF if and only if it has a proper 3-edge colouring. Any graph obtained from such a cubic graph by subdividing edges, and then identifying some vertices also has a 4-NZF. This construction is reversible: every graph with a 4-NZF can be obtained from a 3-edge colourable cubic graph and some loops by subdividing some edges and then identifying some vertices. The 4-colour theorem asserts that every plane coloop-free graph has a 4-NZF. Tutte [?] conjectured that all graphs not containing a subdivision of Petersen’s graph has a 4-NZF, and this conjecture has been verified for cubic graphs [?]. A cubic graph has a 3-NZF if and only if it is bipartite. Accordingly: every graph with a 3-NZF can be obtained from a bipartite cubic graph and a set of loops by subdividing some edges and identifying some vertices. We refer the reader to [?] for further information on flows.

1.4. Orientability. The cycles of a graph \( G \) are precisely the supports of its 2-flows. A CDC \( \mathcal{D} \) is orientable if every cycle \( D \in \mathcal{D} \) is supported by a 2-flow \( \phi_D \) and \( \sum \{ \phi_D \mid D \in \mathcal{D} \} = 0 \). Let \( \overrightarrow{G} \) be the directed graph obtained from \( G \) by replacing each edge with two opposite arcs. Then an orientable CDC of a simple graph \( G \) corresponds to a decomposition of \( \overrightarrow{G} \) into directed circuits of length \( \geq 3 \). A strong orientable CDC of \( G \) corresponds to a strong embedding of \( G \) in an orientable surface.

1.5. Stronger Conjectures. Several authors have proposed stronger versions of the CDC conjecture, most of which can be stated by using adjectives such as “strong”, “orientable” and “\( k \)” on “cycle double cover”. Perhaps the strongest of these was proposed by Celmins [2].

**Conjecture 1.2.** Every graph without coloops has a strong orientable 5-CDC.

If true, this would imply Tutte’s 5-flow conjecture via the following.

**Lemma 1.3.** If \( G \) has an orientable \( k \)-CDC, then \( G \) has a \( k \)-NZF. The converse of this statement holds true for \( k = 2, 3, 4 \).

**Proof.** Let \( \phi_1, \ldots, \phi_k \) be a list of 2-flows associated with a an orientable \( k \)-CDC of \( G \). Then \( \sum_{0 \leq i < k} i \phi_i \) is a \( \mathbb{Z} \)-flow which is easily seen to be a \( k \)-NZF.

The converse statement is obvious for \( k = 2 \). For \( k = 3 \), let \( \phi \) be a \( 3 \)-NZF of \( G \). Then \( \phi^{-1}(\pm 1) \) is a cycle in \( G \), and this cycle is the support of some 2-flow \( \theta \). It is easy to verify that \( (-\theta, \frac{\theta + \phi}{2}, \frac{\theta - \phi}{2}) \) is a list of three 2-flows which sum to 0, and whose supports form a 3-CDC of \( G \).

Finally, suppose \( G \) has a 4-NZF. Then \( G \) has a \( \mathbb{Z}_2^2 \)-NZF \( \phi \), so \( E(G) \) is the union of two cycles \( D_1, D_2 \). Let \( D_3 = D_1 \oplus D_2 \). Then \( (D_1, D_2, D_3) \) is a 3-CDC of \( G \). Let \( \phi_i \) be a 2-flow with support \( D_i \) \( (i = 1, 2, 3) \). By checking cases, we find find that each of the flows

\[
\frac{\phi_1 + \phi_2 + \phi_3}{2}, \quad \frac{\phi_1 - \phi_2 - \phi_3}{2}, \quad \frac{-\phi_1 + \phi_2 - \phi_3}{2}, \quad \frac{-\phi_1 - \phi_2 + \phi_3}{2}
\]

is a 2-flow, their sum is 0, and every edge of \( G \) is supported by exactly two of these four 2-flows. \( \square \)

The here-conjectured converse of Lemma 1.3 for \( k = 5 \) would be very interesting, if true, in view of Tutte’s 5-flow conjecture.

Bondy [1] has proposed that every graph without coloops has a small circuit double cover, a circuit double cover using at most \( n(G) - 1 \) circuits. This has been proved for 4-connected planar graphs [12], for various line graphs [4, 11], and for simple graphs \( G \) having a vertex of degree \( n(G) - 1 \) [?]. Celmins [?] proposed that every graph without coloops has a CDC using any prescribed circuit of \( G \). This conjecture holds if \( G \) has a 4-NZF; if \( C \) is a circuit and \( (D_1, D_2, D_3) \) is a 3-CDC of \( G \) (as in the proof of Lemma 1.3), then \( (D_1 \oplus C, D_2 \oplus C, D_3 \oplus C, C) \) is a CDC of \( G \).
1.6. **Cycle $k$-fold covers.** One might wonder whether there is list of circuits which covers each edge of a graph $G$ precisely $k$ times, for fixed integers $k > 2$. If $k$ is odd, then this exists if and only if $E(G)$ is a cycle. The case where $k > 2$ is even, is settled by the following.

**Theorem 1.4** (7-cycle 4-fold cover). If $G$ has no coloops, then there exists a list of 7 cycles which covers each edge of $G$ exactly 4 times.

**Theorem 1.5** (10-cycle 6-fold cover). If $G$ has no coloops, then there exists a list of 10 cycles which covers each edge of $G$ exactly 6 times.

Theorem 1.4, first proved by Jaeger [10], follows from the existence of a $Z^3$-flow in $G$. For $i = 1, 2, 3$, let $D_i$ be the set of edges supported by the $i$th coordinate of $\phi(e)$. Then each edge of $G$ belongs to exactly four of the cycles

$$D_1, \quad D_2, \quad D_3, \quad D_1 \oplus D_2, \quad D_2 \oplus D_3, \quad D_3 \oplus D_1, \quad D_1 \oplus D_2 \oplus D_3.$$ 

A conjectured strengthening of Theorem 1.4 asserts that every graph has a 6-cycle 4-fold cover. This is equivalent to the Berge-Fulkerson conjecture [9]: every 2-connected cubic graph has a list of 6 perfect matchings which together cover every edge exactly twice.

Theorem 1.5, proved by Fan [3], follows from the existence of a $(Z_3 \times Z_3)$-flow $\phi$ in $G$. Let $\theta_1$ be the $Z_3$-flow which is the projection of $\phi$ onto the first coordinate. Let $D$ be the cycle supported by the second coordinate of $\phi$. Let $\psi$ be a $Z_3$-flow whose support equals $D$. Then each of $\theta_1$, $\theta_1 + \psi$, $\theta_1 + 2\psi$ is a $Z_3$-flow in $G$. Let $E_0$, $E_1$, $E_2$ denote the respective supports of these three $Z_3$-flows. By Lemma 1.3, each of the subgraphs $G[E_i]$ ($i = 1, 2, 3$) has an orientable 3-CDC, say $D_i$. One easily checks that every edge of $G$ is in exactly three of the four sets $E_1$, $E_2$, $E_3$, $D$. Therefore every edge of $G$ appears exactly six times in the 11 cycles in the multiset $D_1 \cup D_2 \cup D_3 \cup \{D, D\}$. Since each $D_i$ is an orientable CDC, these 11 cycles constitute an orientable 11-cycle 6-fold cover of $G$. We may eliminate one of these cycles at the expense of losing orientability. By replacing every cycle in $D_1 \cup D_2 \cup D_3$ by its binary sum with $D$, we obtain a set of 9 cycles which still covers every edge in $E(G) - D$ six times, and which covers every edge in $D$ exactly $9 - 4 = 5$ times. By appending $D$, we obtain a 10-cycle 6-fold cover of $G$.

It is unknown whether every graph without coloops has an orientable cycle 4-fold cover. Indeed the existence of an orientable cycle 10-fold cover is also unknown.

2. **Faithful cycle covers**

Let $w : E(G) \to \mathbb{Z}_+$ be a prescribed weighting with nonnegative integers. It is natural to ask whether there exists a list of cycles in a graph $G = (V, E)$ such that every edge $e$ appears in exactly $w(e)$ members of the list. Such a list is called a faithful $w$-cover of $G$, or a faithful cover of $(G, w)$. Two obvious necessary conditions on $(G, w)$ are that for any bond (minimal edge cut) $B$ of $G$ we have that $w(B)$ is even and that $w(e) \leq w(B - e)$ for every $e \in B$. (We use the abbreviations $w(B) = \sum w(e) \mid e \in B$ and $B - e = B \setminus \{e\}$.) For which pairs $(G, w)$ are these two conditions sufficient for the existence of a faithful cover? It is perhaps best to pose this in a more general context.

2.1. **Lattice Hilbert bases.** We consider an edge weighting of $G$ to be a vector in the space $\mathbb{Q}^E$ of rational vectors indexed by $E(G)$. Let $\chi_F \in \{0, 1\}^E$ be the characteristic vector of $F \subseteq E$. Let $D$ denote the set of cycles in $G$. Let $\mathbb{Q}_+$ and $\mathbb{Z}_+$ be the nonnegative rationals and integers. We define the following “linear combinations of cycles” of $G$.

- $Q(G) = \{ \sum_{D \in D} \alpha_D \chi_D \mid \alpha_D \in \mathbb{Q}_+, \text{ for } D \in D \}$
- $Z(G) = \{ \sum_{D \in D} \alpha_D \chi_D \mid \alpha_D \in \mathbb{Z}_+, \text{ for } D \in D \}$
Theorem 2.1. For every graph $G = (V, E)$ and $w \in \mathbb{Q}^E$ we have the following.

1. $w \in \mathbb{Q}(G)$ if and only if $w(e) = 0$ for every coloop $e$ and $w(f) = w(f')$ for every cocircuit \{f, f'\}.

2. $w \in \mathbb{Z}(G)$ if and only if $w \in \mathbb{Q}(G) \cap \mathbb{Z}^E$ and $w(B)$ is even for every cocircuit $B$ of $G$.

3. $w \in \mathbb{Z}_+(G)$ if and only if $0 \leq w(e) \leq w(B - e)$ for every edge $e$ and every cocircuit $B$ of $G$ containing $e$.

Proof. The necessity of these conditions are easy to see, since they are necessary for all characteristic vectors of cycles in $G$. To prove sufficiency, we may assume $G$ is 3-edge connected (contract any coloop and contract all but one edge in every series class of edges). Let $e \in E(G)$. By Menger’s theorem, there exist two circuits $C, C'$ in $G$ with $C \cap C' = \{e\}$. We have $2\chi\{e\} = \chi_C + \chi_{C'} - \chi_{C \cap C'}$. Therefore $\chi\{e\} \in \mathbb{Q}(G)$, which proves claim (1). To prove claim (2), we suppose $w \in \mathbb{Z}^E$ and $w(B)$ is even for every cocircuit. Then the set of edges having odd weight is a cycle $D$ in $G$. Every entry of $w + \chi_D$ is even, so this vector is an integer combination of $\{2\chi\{e\} \mid e \in E\}$. Therefore $w \in \mathbb{Z}(G)$.

Claim (3) is proved by Seymour [?] in a very pretty argument. A tight pair $(e, B)$ is a cocircuit $B$ and an edge $e \in B$ for which the inequality $w(e) \leq w(B - e)$ holds with equality. First he reduces to the case where no edge has weight zero (by deleting such edges), and where every cocircuit in a tight pair is the boundary $B = \delta(v)$ of some vertex $v$ (by contracting, in turn, each side of any $B$ in a tight pair which is not of this form, applying induction, then piecing together the two resulting linear combinations). He then shows the existence of a circuit $C$ in $G$ which uses every edge $e$ for which there is a tight pair $(\delta(v), e)$ with $v \in V(G)$. For a small enough $\alpha > 0$, the reduced weighting $w - \alpha\chi_C$ still satisfies the inequalities in (3). Selecting $\alpha$ just large enough, we find that the reduced weighting has either a strictly smaller support than $w$ does, or a new tight pair $(B, e)$. Either way, we are done by induction.

By using network flows, one can devise polytime algorithms to decide, for an input $(G, w)$, whether an edge weighting $w$ belongs to $\mathbb{Z}(G)$ or $\mathbb{Z}_+(G)$. Deciding whether $w \in \mathbb{Z}_+(G)$ for a given pair $(G, w)$ (the faithful cycle cover problem) is of unknown computational complexity. We have

$$\mathbb{Z}_+(G) \subseteq \mathbb{Z}(G) \cap \mathbb{Q}_+(G).$$

We say that the cycles of $G$ form a lattice Hilbert base if equality holds here. By the above comments, we are interested in graphs whose cycles form a lattice Hilbert base.

2.2. Cycles in Petersen Free Graphs. The following result is proved in [?]. Let $P$ denote Petersen’s graph. A Petersen minor of $G$ is a minor of $G$ isomorphic to $P$.

Theorem 2.2. The cycles of $G$ form a lattice Hilbert base if and only if $G$ has no Petersen minor.

Proof. Let $w_P$ be a $\{1, 2\}$-valued weighting of $P$ in which $w_P^{-1}(2)$ forms a perfect matching. One easily checks that $w_P \in \mathbb{Z}(P)$. There are 5 circuits of length 8 in $P$ which form a faithful cover of $(P, 2w_P)$, so $w_P \in \mathbb{Z}_+(P)$. For any circuit $C$ in $P$, the weight $w_P - \chi_C$ supports a subgraph of $P$ which has a coloop, so $w_P \notin \mathbb{Z}_+(P)$. Any graph $G$ containing a subdivision of $P$ has a weighting derived from $w_P$ belonging to $\mathbb{Q}_+(G) \cap \mathbb{Z}(G) - \mathbb{Z}_+(G)$. This proves the necessity of the condition in Theorem 2.2.
Sufficiency is proved by establishing the following sequence of claims. We say that \((G, w)\) is bad if \(w \in \mathbb{Z}(G) \cap \mathbb{Q}_+(G) - \mathbb{Z}_+(G)\), and \((G, w)\) is minimally bad if \(e(G) + w(E(G))\) is as small as possible among the bad pairs \((G, w)\). Let \(x, y\) be vertices of \(G\). A list \((C_1, \ldots, C_\ell)\) of circuits is an \(x, y\)-chain if \(C_i\) is the only circuit containing \(x\) as a vertex, \(C_\ell\) is the only circuit containing \(y\) as a vertex, and for \(1 \leq i < j \leq \ell\), we have \(V(C_i) \cap V(C_j) \neq \emptyset\) if and only if \(j = i + 1\).

**Claim 1:** If \((G, w)\) is minimally bad, then \(w\) is \(\{1, 2\}\) valued. For every \(e_0 = xy \in w^{-1}(2)\), the pair \((G, w - 2\chi_{\{e_0\}})\) has a faithful circuit cover. Moreover, every faithful circuit cover of \((G, w - 2\chi_{\{e_0\}})\) is an \(x, y\)-chain.

**Claim 2:** If \((G, w)\) is minimally bad, then \(w^{-1}(1)\) induces a cycle in \(G\) with exactly two connected components, and every edge in \(w^{-1}(2)\) joins these two components.

**Claim 3:** If \((G, w)\) is minimally bad, then \(G\) is a cubic graph.

**Claim 4:** If \((G, w)\) is minimally bad, then deleting some edges of weight 2 results in a subdivision of \(P_5\).

A critical aspect of this strategy is that these claims do not need the hypothesis that \(G\) does not have a Petersen minor.

Let \((G, w)\) be minimally bad. As with the proof of Theorem 2.1(3), we conclude that no edge has zero weight and \(G\) has no nontrivial tight pairs \((e, B)\). Also, \((G, w)\) does not contain two tight pairs, \((e, B), (f, B')\), with \(f \in B - e\); for otherwise, restricting \(w\) to the contracted graph \(G/f\) results in another bad pair, a contradiction. We select \(e_0 \in E(G)\) to have minimum weight subject to the conditions \(w(e_0) \geq 2\), and that no tight pair \((e, B)\) with \(e_0 \in B - \{e\}\). The edge \(e_0\) exists, since the heaviest edge in \((G, w)\) satisfies the conditions. We have the following.

Every edge \(e\) with \(2 \leq w(e) < w(e_0)\) is adjacent to an edge of weight \(\geq w(e_0)\). (*)

By the choice of \(e_0\), the reduced weight \(w' = w - 2\chi_e\) still belongs to \(\mathbb{Z}(G) \cap \mathbb{Q}_+(G)\). By minimality, \((G, w')\) has a faithful circuit cover, say \(L_1 \cup L_2\), where \(L_2\) consists of those circuits in the cover which contain \(e_0\). Following an argument of Seymour [13], we form an auxiliary directed graph \(H\) with \(V(H) = V(G)\); we define an arc \(uv \in E(H)\) if either some circuit in \(L_1\) contains \(u, v\) as vertices, or some circuit in \(L_2\) traverses \(x, v, u, y\) in that order. Using Menger’s theorem, and the fact every \(x, y\)-separating cut \(B\) satisfies \(w(e_0) \leq w(B - e_0)\), one concludes that \(H\) has a directed \(x, y\)-path. Let \(P = v_0v_1 \ldots v_\ell\) be a shortest such path, so \(v_0 = x\) and \(v_\ell = y\). The sequence of arcs along \(P\) corresponds to a list \(C = (C_1, \ldots, C_\ell)\) of circuits in \(L_1 \cup L_2\), where \(\{v_{i-1}, v_i\} \subseteq V(C_i)\) \((1 \leq i \leq \ell)\). This list may contain repetitions; let \(\{C\}\) denote the set of distinct circuits appearing in \(C\).

Our first goal is to show \(\{C\} \cap L_2 = \emptyset\), whence \(w(e_0) = 2\). From the choice of \(P\) one can derive the following facts regarding this list.

1. If \(C_i \in \{C\} \cap L_1\) then \(C_i\) appears just once in the list \(C\) and \(V(C_i) \cap V(P) = \{v_{i-1}, v_i\}\). If \(C_i, C_j \in L_1\) and \(i \leq j - 2\), then \(C_i\) and \(C_j\) are vertex disjoint. If \(C_i, C_{i+1}, \ldots, C_j\) is a maximal subsequence of consecutive circuits in \(C\), all belonging to \(L_1\), then this subsequence of circuits is called a chain. The two vertices \(v_{i-1}, v_j\) are ends of that chain, and \(v_j\) is the terminal end of that chain.

2. If \(C \in \{C\} \cap L_2\) then \(C\) may appear several times in \(C\), although nonconsecutively; say \(C = C_{i_0} = C_{i_1} = \cdots = C_{i_r} \in L_2\), where \(i_{k-1} \leq i_k - 2\) for \(1 \leq k \leq r\). This circuit traverses the vertices of \(P\) in the following order

\[x = v_0, \quad v_{i_1}, v_{i_1 - 1}, \quad v_{i_2}, v_{i_2 - 1}, \ldots \quad v_{i_r}, v_{i_r - 1}, \quad v_\ell = y\]

These vertices partition the \(x, y\)-path \(C - e_0\) into subpaths called segments. Segments which connect \(v_{i_k}\) to \(v_{i_{k-1}}\), for some \(k\), are called reverse segments. The remaining segments are
forward segments. The segments comprising the \( x, y \)-path alternate: forward – reverse – \( \cdots \) – forward.

We define another auxiliary directed graph \( J \) whose vertices are all the ends of all the chains described in (1), and all the endpoints of all the segments described in (2). For every forward segment from \( v_i \) to \( v_j \), \( J \) has an arc \( v_iv_j \). For every reverse segment from \( v_{ik} \) to \( v_{ik-1} \), \( J \) has an arc \( v_{ik-1}v_{ik} \). For every chain in \( \mathcal{C} \) whose ends are \( v_i, v_j, i < j \), \( J \) has two parallel arcs \( v_iv_j \) called chain arcs of \( J \). Arcs arising from segments are called segment arcs.

\[ \text{Figure 1. Cycles } \mathcal{C} = (C_1, \ldots, C_{10}) \text{ and the auxiliary directed graph } J. \]

For every \( v_s \in V(J) - \{v_0\} \) we consider the set \( \delta(s) \) of arcs \( v_iv_j \in E(J) \) with \( i < s \leq j \). Each \( v_s \in V(J) - \{v_0\} \) is either the terminal end of a chain in \( \mathcal{C} \), or \( v_s \) is the first vertex in a reverse segment of the circuit \( C_s \in \mathcal{C} \cap L_2 \). These two cases are exclusive. In the first case, each circuit in \( \{\mathcal{C}\} \cap L_2 \) contributes exactly one segment arc to \( \delta(s) \), whereas the chain contributes exactly two chain arcs to \( \delta(s) \). In the second case, the circuit \( C_s \) contributes exactly three arcs to \( \delta(s) \), whereas all other circuits in \( \{\mathcal{C}\} \cap L_2 \) contribute one arc to \( \delta(s) \). We conclude that \( \delta(s) \) contains exactly \( w'(e) + 2 = w(e) \) arcs for every \( v_s \in V(J) - \{x\} \). From Menger’s theorem, the arcs of \( J \) can be decomposed into exactly \( w(e_0) \) \( x, y \)-paths.

For each chain \( Q = (C_i, C_{i+1}, \ldots, C_j) \) in \( \mathcal{C} \), we form a new weighted graph \( (G_Q, w_Q) \), where \( G_Q \) is obtained from \( \bigcup Q \) by joining the two ends of \( Q \) with a new edge \( e_Q \). We define \( w_Q(e_Q) = 2 \) and for every other edge \( e \), we define \( w_Q(e) \) to be the number of circuits in \( Q \) containing \( e \). Then \( w_Q \) is \( \{1, 2\} \)-valued with \( w^{-1}(1) \) being the cycle \( C_i \oplus \cdots \oplus C_j \). As \( G_Q \) has no coloops, we conclude that \( w_Q \) is in the cone and lattice of circuits of \( G_Q \).

We may now achieve our first goal as follows. If \( \{\mathcal{C}\} \cap L_2 \neq \emptyset \), then for each chain \( Q \), \( G_Q \) is a proper minor of \( G \) so, by choice of \( (G, w) \), \( G_Q \) has a faithful circuit \( w_Q \)-cover. Two circuits in this cover become paths in \( G_Q - e_Q \) called chain paths. Each chain path (among all the chains in \( \mathcal{C} \)) is bijectively associated with a chain arc in \( J \), and every segment arc in \( J \) is associated with a segment of a circuit in \( \{\mathcal{C}\} \cap L_2 \). Using the abovementioned path decomposition of \( J \), we can piece together all of these chain paths and segments to produce \( w(e_0) \) \( x, y \)-walks in \( G \). Using (*), one can show that each of these \( x, y \)-walks \( W \) has no repeated edges, so \( W + e_0 \) is a cycle in \( G \). Adding these \( w(e_0) \) cycles to the list \( L_1 \cup L_2 - \mathcal{C} \), we obtain a faithful cycle cover of \( (G, w) \), a contradiction. It follows that \( L_2 = \emptyset \) so \( \mathcal{C} \) consists of a single chain \( Q \), and \( (G, w) = (G_Q, w_Q) \). In particular, \( w \) is \( \{1, 2\} \)-valued and \( w(e_0) = 2 \). Any edge in \( w^{-1}(2) \) serves as a valid choice for \( e_0 \) in the above arguments. We have proved Claim 1.
Let $E_i = w^{-1}(i), i = 1, 2$. Then $E_1$ a cycle in $G$. Let $e_0 \in E_2$. By Claim 1, every faithful circuit cover of $(G, w - 2\chi_{e_0})$ consists of a single chain whose ends are the two endpoints of $e_0$. If we colour these circuits red-blue alternately, then the union of the red (blue) circuits is a cycle, say $D_1$ (resp. $D_2$). The edge set $D_1 - E_1$ is a cycle in the contracted graph $G/E_1$ ($i = 1, 2$). Therefore $(D_1 - E_1, D_2 - E_1)$ is a 2-CDC of $(G/E_1) - e_0$. This can happen only if each cycle $D_i - E_1$ consists of all the edges in $(G/E_1) - e_0$. We have shown that $E_2 - e_0$ is a cycle in $G/E_1$ for every $e_0 \in E_2$. One immediately concludes that $G/E_1$ is either a single vertex with some loops, or $G/E_1$ has exactly two vertices, with no loops and an odd number of edges. In the first case we select any $e_0 = xy \in E_2$, and find a faithful cycle cover $(D_1, D_2)$ of $(G, w - 2\chi_{e_0})$ as above. Let $P$ be an $x, y$-path in $G - e_0$ whose edges are all in $E_1$. Letting $C$ be the circuit $P + e_0$, we find that $(D_1 \oplus C, D_2 \oplus C)$ is a faithful cycle cover of $(G, w)$, a contradiction. Therefore the second case holds and we have proved Claim 2.

To prove Claim 3, suppose $v$ has degree different from 3 in a minimally bad example $(G, w)$. Then $v$ has degree $\geq 4$, since $G$ is 3-edge connected. Let $e_0 \in E_2$. By Claim 1, $v$ belongs to two consecutive circuits in a chain $(C_1, \ldots, C_t)$ which forms a faithful circuit cover of $(G, w - 2\chi_{e_0})$. Therefore $v$ has degree 4 and its four incident edges have weight 1 in $w$. A lifting at $v$ is a graph obtained from $G - v$ by adding two new edges which form a perfect matching of the four neighbours of $v$. By assigning weights 1 to the new edges, we obtain a new weighed graph $(G', w')$. A result of Fleischner asserts that, at least one of the three possible graphs $G'$ has no coloops. Therefore $w'$ is in the cone and the lattice of circuits of this graph $G'$. By minimality of $(G, w)$, there exists a faithful cycle cover of $(G', w')$. This is easily modified into a faithful circuit cover of $(G, w)$. This contradiction establishes Claim 3.

We have shown that if $(G, w)$ is minimally bad, then $E(G) = E_1 \cup E_2$ where $E_1$ induces two disjoint circuits and every edge in $E_2$ has an endpoint in both circuits. Such a graph is called a quasiprism. Suppose $G$ has a Hamilton cycle, then $G$ has a proper edge colouring with colours 1, 2, 3. Let $D_i$ be the cycle obtained by deleting edges of colour $i$ ($i = 1, 2, 3$). Then $(D_1 \oplus E_1, D_2 \oplus E_1, D_3 \oplus E_1)$ is a faithful cycle cover of $(G, w)$, a contradiction. Therefore $G$ is not Hamiltonian. Ellingham [?] has shown that if a quasiprism is not Hamiltonian, then one may obtain a subdivision of $P$ by deleting some edges in $E_2$. This proves Claim 4, and completes the proof. 

\[ \square \]

**Corollary 2.3.** Every coloop-free graph with no Petersen minor has a CDC.

We comment that this corollary would be implied by Tutte’s 4-flow conjecture: indeed such graphs would have a 3-CDC. The 4-flow conjecture has been proved for cubic graphs [?].

A related, but easier result [?], states that if $G$ is series-parallel, and $w \in \mathbb{Z}(G) \cap \mathbb{Q}_+(G)$, then $G$ has an orientable faithful circuit $w$-cover. A \{1, 2\}-valued weighting $w$ of $K_4$, in which $w^{-1}(1)$ is a 4-circuit, has no orientable faithful cover.

Let $X$ be a set of vectors in a finite-dimensional real vector space, and let $\mathbb{Z}_+(X)$ be the set of non-negative integer combinations of vectors in $X$ (so $\mathbb{Z}_+(X)$ is the integer cone generated by $X$). It is known [?] that there exists a unique minimal subset of $X$ which forms a lattice Hilbert base for $\mathbb{Z}_+(X)$. Let $H(G)$ denote the minimal lattice Hilbert base for $\mathbb{Z}_+(G)$, the integer cone of circuits in $G$. All characteristic vectors of all circuits in $G$ belong to $H(G)$. We describe another class of vectors contained in $H(G)$. To blister an edge $e = uv$ with weight $w(e) \in \mathbb{Z}_+$ is to subdivide $e$ with a new vertex $x$, and then to replace the new edge $xv$ with $w(e)$ parallel edges. Each new
parallel edge \( xy \) is assigned weight 1, whereas the new edge \( ux \) is assigned weight \( w(e) \). A Petersen weight of \( G \) is a weighting in \( \{0, 1, 2\}^{E(G)} \) whose support can be obtained from the weighted Petersen \((P, wp)\) by successively blistering its edges. It is easy to see that every Petersen weight of \( G \) must belong to \( H(G) \). We propose the following.

Conjecture 2.4. For any graph \( G \), the minimal Hilbert base \( H(G) \) consists of the characteristic functions of circuits together with the Petersen weights of \( G \).

3. Necessary Conditions

Here we assume that the CDC conjecture is false, and let \( G \) be a counterexample with the fewest possible edges. We derive properties which \( G \) necessarily holds, with the ultimate hope of showing \( G \) does not exist. A snark is a cyclically 4-connected cubic graph with flow number greater than 4.

Theorem 3.1. A minimal counterexample \( G \) to to Conjecture 1.1 is a snark having girth at least 12. Furthermore, \( G - e \) has flow number greater than 4 for every edge \( e \).

4. Sufficient Conditions

Here we find conditions under which a graph \( G \) has a CDC. The hope here is that to find sufficient conditions which are broad enough that every graph without coloops has a CDC.

Theorem 4.1. If \( G \) has a cycle \( D \) such that \( G/D \) is a cycle, then \( G \) has a 6-CDC

More generally, there is a procedure \([5, 7, 8, 9]\) for generating a family of graphs called frames with the following property.

Theorem 4.2. If \( G \) has a spanning subgraph \( H \) which is a subdivision of a frame, then \( G \) has a CDC.

References

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