Removable Circuits in Binary Matroids

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Abstract  
We show that if $M$ is a connected binary matroid of cogirth at least five which does not have both an $F_7$-minor and an $F_7^*$-minor, then $M$ has a circuit $C$ such that $M - C$ is connected and $r(M - C) = r(M)$.

1 Introduction

We shall consider the problem of finding sufficient conditions for the existence of a circuit in a given matroid $M$ whose deletion leaves the rank or connectivity of $M$ unchanged. The existence of such a circuit in graphs has been considered by various authors. The most general result for simple graphs can be deduced from a theorem of W. Mader [5, Satz 1].

**Theorem 1** Let $k$ be a positive integer and $G$ be a simple $k$-connected graph of minimum degree at least $k + 2$. Then $G$ has a circuit $C$ such that $G - E(C)$ is $k$-connected.

Stronger results for the special case when $G$ is simple and $k = 2$ can be found in Jackson [4] and Thomassen and Toft [10].
It seems natural to ask if Theorem 1 can be extended to a graph $G$, which may contain multiple edges. We can obtain a partial result by applying Theorem 1 to the underlying simple graph of $G$, if $G$ has no edges of multiplicity greater than two, and otherwise choosing $C$ to be a 2-circuit of $G$ belonging to an edge of multiplicity at least three, to deduce

**Corollary 2** Let $k$ be a positive integer and $G$ be a $k$-connected graph of minimum degree at least $k + 3$. Then $G$ has a circuit $C$ such that $G - E(C)$ is $k$-connected.

It follows from a result of Sinclair [9] that the bound $k + 3$ in Corollary 2 can be reduced to $k + 2$ for the special case when $k = 1$. This is not true when $k = 2$, however, as can be seen from an example constructed by N. Robertson and later B. Jackson (see [4]). However, replacing $k + 3$ by $k + 2$ when $k = 2$ in Corollary 2 is valid for graphs which do not contain a vertex of degree four incident with two edge-disjoint 2-circuits by [9], for planar graphs by [1], and, more generally, graphs with no Petersen minor, by [2].

Oxley asked in [7, Problem 14.4.8] if the following partial extension of Theorem 1 when $k = 2$ is valid for binary matroids: does every connected binary matroid of girth at least three and cogirth at least four have a circuit $C$ such that $M - C$ is connected? L. Lemos (see [2]) has constructed a cographic matroid of cogirth four which shows that the answer to Oxley's question is no. It remains an open problem, however, to decide if there exists an integer $t \geq 5$ such that all connected binary matroids $M$ of cogirth at least $t$ have a circuit $C$ such that $M - C$ is connected. We shall show in Theorem 7 that this assertion is true with $t = 5$ for binary matroids $M$ which do not have both an $F_7$- and an $F_7^*$-minor. This gives a partial generalisation of Corollary 2 when $k = 2$. Our proof uses the decomposition theory of Seymour in [8] which implies that a 3-connected, vertically 4-connected binary matroid which does not have both an $F_7$-minor and an $F_7^*$-minor is either graphic or cographic, or is isomorphic to $R_{10}$, $F_7$ or $F_7^*$. We shall first show that our result holds for graphic and cographic matroids. We then proceed by contradiction and show that a smallest counterexample to the result would be vertically 4-connected. It then only remains to check that the result holds for matroids obtained from $R_{10}$, $F_7$ or $F_7^*$ by parallel extensions.
2 Graphs

We shall consider finite graphs which may contain multiple edges, but no loops. We consider a graph $G$ to be 2-connected if $G - v$ is connected for all $v \in V(G)$. We shall use $E_G(v)$ to denote the set of edges of $G$ incident with a vertex $v$ and put $d_G(v) = |E_G(v)|$. We will suppress the subscript $G$ when it is clear to which graph we are referring. Given a circuit $C$ of $G$, put $|C| = |E(C)|$.

We first obtain, in Lemma 4 below, a slight extension of the case $k = 2$ of Corollary 2. We need this extension for our inductive proof on matroids. Lemma 4 itself follows from a result of Sinclair [9]. We include a proof in this paper for the sake of completeness. We shall use the following elementary result.

**Lemma 3** Let $G$ be a graph on $n$ vertices and $C_0$ be a circuit of $G$ such that $|C_0| \leq 3$ and $n > |C_0|$. Suppose that for all $v \in V(G) - V(C_0)$ we have $d_G(v) \geq 4$. Then $G$ has a circuit $C$ such that $E(C_0) \cap E(C) = \emptyset$.

**Proof.** If $G$ is not 2-connected then choosing $C$ to be any circuit in an end-block of $G$ which does not contain $C_0$ we have $E(C_0) \cap E(C) = \emptyset$. Hence we may suppose that $G$ is 2-connected.

Let $H = G - E(C_0)$. Suppose $H$ is a forest. Then $|E(H)| \leq n - 1$. Let $t$ be the number of edges between $V(C_0)$ and $V(G) - V(C_0)$. Then $|E(H)| = \frac{1}{2}(t + \sum_{v \in V(G) - V(C_0)} d_G(v))$. Since $G$ is 2-connected, $t \geq 2$, and since $d_G(v) \geq 4$ for all $v \in V(G) - V(C_0)$, we have $|E(H)| \geq 2n - 2|C_0| + 1$. Thus $n \leq 2|C_0| - 2$. Since $n \geq |C_0|$, we have $|C_0| = 3$, and $n = 4$. Let $V(G) - V(C_0) = \{v\}$. Using the assumption that $H$ is a forest, we have $d_G(v) \leq 3$. This contradicts an hypothesis on $G$ and so the assumption that $H$ is a forest must be false.

**Lemma 4** Let $G$ be a 2-connected graph on $n$ vertices and $C_0$ be a circuit of $G$ such that $|C_0| \leq 3$ and $n > |C_0|$. Suppose that for all $v \in V(G) - V(C_0)$ we have $d_G(v) \geq 5$. Then $G - E(C_0)$ has a circuit $C$ such that $G - E(C)$ is 2-connected.

**Proof.** Suppose the theorem is false and let $G$ be a counterexample. By Lemma 3, we can choose a circuit $C$ in $G - E(C_0)$. Let $H = G - E(C)$, let $B_0$ be the block of $H$ which contains $C_0$ and $B$ be an end-block of $H$.
distinct from \( B_0 \). We may suppose that \( C \) has been chosen such that \(|E(B)|\) is minimal. Let \( e \) be an edge of \( B \) chosen such that, if \( B \) contains a cut-vertex \( x \) of \( H \), then \( e \) is incident with \( x \). Since \( d_G(v) \geq 5 \) for all \( v \in V(G) - V(C_0) \), at most one vertex of \( B - e \) has degree less than two. Thus we may choose a circuit \( C' \) contained in \( B - e \). Using the minimality of \(|E(B)|\) and the fact that \( G \) is 2-connected we see that each end-block of \( H - E(C') \) is incident with \( C \) and each component of \( H - E(C') \) is incident with at least two vertices of \( C \). Thus \( G - E(C') = (H - E(C')) \cup E(C) \) is 2-connected. This contradicts the choice of \( G \) as a counterexample to the theorem. \( \blacksquare \)

Given a graph \( G \) and \( U \subseteq V(G) \), we use \( N_G(U) \) to denote the set of vertices of \( V(G) - U \) adjacent to a vertex of \( U \) and \( G[U] \) to denote the subgraph of \( G \) induced by \( U \). For \( S \subseteq E(G) \), let \( G/S \) be the graph obtained from \( G \) by contracting the edges in \( S \), and \( V(S) \) the set of vertices of \( G \) incident with \( S \).

We next show, in Lemma 6 below, that the case \( k = 2 \) of Corollary 2 can be extended to cographic matroids. We shall use the following elementary result.

**Lemma 5** Let \( G \) be a connected graph on \( n \) vertices and \( X_0 \) be a cocircuit of \( G \) such that \( |X_0| \leq 3 \) and \( |E(G)| \geq n + |X_0| - 1 \). Suppose that \( G - X_0 \) has girth at least four. Then \( V(G) \neq V(X_0) \).

**Proof.** Let \( H_1 \) and \( H_2 \) be the two components of \( G - X_0 \). Suppose \( V(G) = V(X_0) \). Then \( |V(H_i)| \leq 3 \) and since \( G - X_0 \) has girth at least four, \( H_i \) is a tree for \( 1 \leq i \leq 2 \). Thus

\[
|E(G)| = |V(H_1)| - 1 + |V(H_2)| - 1 + |X_0| = n + |X_0| - 2.
\]

This contradicts the hypothesis on \(|E(G)|\). \( \blacksquare \)

**Lemma 6** Let \( G \) be a 2-connected graph on \( n \) vertices and \( X_0 \) be a cocircuit of \( G \) such that \( |X_0| \leq 3 \) and \( |E(G)| \geq n + |X_0| - 1 \). Suppose that \( G - X_0 \) has girth at least five. Then there exists \( v \in V(G) - V(X_0) \) such that \( G/E(v) \) is 2-connected.

**Proof.** Suppose the theorem is false and let \( G \) be a counterexample. By Lemma 5, we can choose a vertex \( v \) in \( V(G) - V(X_0) \). Let \( H = G/E(v) \) and
$x$ be the vertex of $H$ corresponding to $N_G(v) \cup \{v\}$. Then $x$ is the unique cut vertex of $H$. Since $X_0 \cap E(v) = \emptyset$, $X_0$ is a cocircuit of $H$ and hence is contained in a block $B$ of $H$. Let $U = V(B) - x$. We may suppose that $v$ has been chosen such that $|U|$ is maximal. Note that $N_G(U) \subseteq \{v\} \cup N_G(v)$. Furthermore, since $G$ is 2-connected, $|N_G(U)| \geq 2$ and $G[U \cup N_G(U) \cup \{v\}]$ is 2-connected. Choose $v' \in V(H) - V(B)$. Then $v' \in V(G) - V(X_0)$. Let $H' = G_E(v')$ and $x'$ be the vertex of $H$ corresponding to $N_G(v') \cup \{v'\}$. Let $B'$ be the block of $H'$ containing $X_0$ and $U' = V(B') - x'$. Then $U \cup (N_G(U) - N_G(v'))$ is properly contained in $V(B')$. By the maximality of $|U|$ we must have $N_G(U) \subseteq N_G(v')$. Now the facts that $N_G(U) \subseteq \{v\} \cup N_G(v)$ and $|N_G(U)| \geq 2$ imply that $E(v) \cup E(v')$ contains a circuit of $G$ of length at most four. This contradicts the fact that $G - X_0$ has girth at least five.

3 Binary Matroids

We shall use the following operation on binary matroids from Seymour [8]. Given binary matroids $M_1$ and $M_2$ let $M_1 \triangle M_2$ be the binary matroid with $E(M) = E(M_1) \triangle E(M_2)$ and circuits all minimal non-empty subsets of $E(M)$ of the form $C_1 \triangle C_2$, where $C_i$ is a circuit of $M_i$. We refer the reader to [7] for other definitions on matroids. Our main result is

**Theorem 7** Let $M$ be a connected binary matroid which does not have both an $F_7$-minor and an $F_2$-minor. Let $C_0$ be a circuit of $M$ such that $|C_0| \leq 3$ and $r(M) > r(C_0)$. Suppose $|X| \geq 5$ for all cocircuits $X$ of $M$ such that $X \cap C_0 = \emptyset$. Then $M - C_0$ has a circuit $C$ such that $M - C$ is connected and $r(M - C) = r(M)$.

**Proof.** We proceed by contradiction. Suppose the theorem is false and let $M$ be a counterexample chosen such that $r(M)$ is as small as possible.

**Claim 1** $M$ is vertically 3-connected.

**Proof.** Suppose that $M$ has a vertical 2-separation $(S_1, S_2)$. Choose $(S_1, S_2)$ such that $|S_1 \cap C_0|$ is minimal. Since $r(S_i) \geq 2$ we have $|S_i| \geq 2$. By [8, 2, 6], $M = M_1 \triangle M_2$ for minors $M_1$ and $M_2$ of $M$ such that $2 \leq r(M_i) < r(M)$, $E(M_1) \cap E(M_2) = C'_0$ for some 2-circuit $C'_0 = \{f, g\}$ of $M_i$, and $E(M_i) - C'_0 = S_i$ for $1 \leq i \leq 2$. Since $M$ is connected each $M_i$ is connected. Since $M_i$ is
a minor of $M$, $M_i$ is binary and does not have both an $F_7$-minor and an $F_7^*$-minor. Since $C'_0 \cap E(M) = \emptyset$ we have $C'_0 \cap C_0 = \emptyset$. Since $|C_0| \leq 3$, $|C_0 \cap E(M_1)| \leq 1$.

Suppose $C_0 \cap E(M_1) = \{e\}$. Then $C_0 = C_1 \triangle C_2$ for some circuits $C_i$ of $M_i$, $1 \leq i \leq 2$. Thus $|C_i| = 2$ and $e$ is parallel to $f$ and $g$ in $M_1$. Let $h \in S_1 - e$ and $Y$ be a circuit of $M$ which meets both $S_1$ and $S_2$. Then $Y = Y_1 \triangle Y_2 \triangle$ for some circuits $Y_i$ of $M_i$ such that $|Y_i \cap C'_0| = 1$, $1 \leq i \leq 2$. Thus $Y_1 - C'_0 + e$ is a circuit of both $M_1$ and $M$, and $r(S_1 - e) = r(S_1) \geq 2$. Similarly since $e \in C_0 \subseteq S_2 + e$ we have $r(S_2 + e) = r(S_2) \geq 2$. Thus $(S_1 - e, S_2 + e)$ is a vertical 2-separation of $M$. This contradicts the minimality of $|S_1 \cap C_0|$. Hence we must have $C_0 \cap E(M_1) = \emptyset$.

Let $X_1$ be a cocircuit of $M_1$ such that $X_1 \cap C'_0 = \emptyset$. Then $X_1$ is a cocircuit of $M$ such that $X_1 \cap C_0 = \emptyset$ so by an hypothesis of the theorem we have $|X_1| \geq 5$. Using the minimality of $r(M)$ we deduce that $M_1 - C'_0$ has a circuit $C$ such that $M_1 - C$ is connected and $r(M_1 - C) = r(M_1)$. Since $M - C = (M_1 - C) \triangle M_2$ we have $C$ is a circuit of $M - C_0$ such that $M - C$ is connected and $r(M - C) = r(M)$. This contradicts the choice of $M$. Thus $M$ has no vertical 2-separation and hence $M$ is vertically 3-connected.

**Claim 2** $M$ is vertically 4-connected.

**Proof.** Suppose that $M$ has a vertical 3-separation $(S_1, S_2)$. Choose $(S_1, S_2)$ such that $|S_1 \cap C_0|$ is minimal. Since $|C_0| \leq 3$, $|C_0 \cap E(M_1)| \leq 1$. We first show that $|S_i| \geq 4$ for $1 \leq i \leq 2$.

Suppose $|S_i| = 3$ for some $i \in \{1, 2\}$. Since $r(S_i) \geq 3$ we must have $r(S_i) = 3$. Since $r(S_1) + r(S_2) - r(M) = 2$ we have $r(S_j) = r(M) - 1$, for $j = 3 - i$. Thus the closure of $S_j$ is a hyperplane of $M$. The complement of this hyperplane will be a cocircuit $X_0$ of $M$ contained in $S_i$. Since $|X_0| \leq |S_i| = 3$, it follows from an hypothesis of the theorem that $X_0 \cap C_0 \neq \emptyset$. Since $M$ is binary we must have $|X_0 \cap C_0| = 2$. Since $S_i$ is independent we must have $|C'_0| = 3$ and $|S_j \cap C_0| = 1$. By the minimality of $|S_1 \cap C_0|$, we must have $i = 2$. Choosing $e_0 \in S_1 \cap C_0$ we have $r(S_1 - e) \leq r(S_1)$ and, since $e_0 \in C_0 \subseteq S_2 + e_0$, $r(S_2 + e_0) = r(S_2) = 3$. Thus $(S_1 - e_0, S_2 + e_0)$ is either a vertical 2-separation of $M$, contradicting Claim 1, or it is a vertical 3-separation of $M$, contradicting the minimality of $|S_1 \cap C_0|$. Thus $|S_i| \geq 4$ for $i \in \{1, 2\}$.

By $[8, 2.9]$, $M = M_1 \triangle M_2$ for minors $M_1$ and $M_2$ of $M$ such that $3 \leq r(M_i) < r(M)$, $E(M_1) \cap E(M_2) = C'_0$ for some 3-circuit $C'_0 = \{f, g, h\}$ of
$M_i$, and $E(M_i) - C'_0 = S_i$ for $1 \leq i \leq 2$. Since $M$ is connected, each $M_i$ is connected. Since $M_i$ is a minor of $M$, $M_i$ is binary and does not have both an $F_7$- and an $F_7^*$-minor. Since $C'_0 \cap E(M) = \emptyset$ we have $C'_0 \cap C_0 = \emptyset$.

Suppose $e \in C_0 \cap E(M_1)$. Then $C_0 = C_1 \Delta C_2$ for some circuit $C_i$ of $M_i$, $1 \leq i \leq 2$. Thus $C_1 - C'_0 = \{e\}$ and $1 \leq \left| C_1 \cap C'_0 \right| \leq 2$. If $\left| C_1 \cap C'_0 \right| = 2$ then replacing $C_1$ by $C'_1 = C_1 \Delta C'_0$ we have $\left| C'_1 \cap C'_0 \right| = 1$. Thus we may assume without loss of generality that $e$ is parallel to $f$ in $M_1$. Let $M'_1$ be the simple matroid obtained by replacing all parallel classes of $M_1$ by single elements and let $f$, $g$ and $h$ represent their own parallel classes in $M'_1$. Using Claim 1 it follows that $M'_1$ is 3-connected. If $f$ is a coloop $M'_1 - \{g, h\}$ then $C'_0$ would contain a cocircuit of $M'_1$. Since $M'_1$ is binary this cocircuit would have size two and hence would contradict the fact that $M'_1$ is 3-connected. Thus $f$ is contained in some circuit of $M'_1 - \{g, h\}$. Since $e$ is parallel to $f$ we deduce that $M$ has a circuit which contains $e$ and is contained in $S_1$. Hence $r(S_1 - e) = r(S_1) \geq 2$. Similarly since $e \in C_0 \subseteq S_2 + e$ we have $r(S_2 + e) = r(S_2) \geq 2$. Thus $(S_1 - e, S_2 + e)$ is a vertical 3-separation of $M$ which contradicts the minimality of $\left| S_1 \cap C_0 \right|$. Hence we must have $C_0 \cap E(M_1) = \emptyset$.

Let $X_1$ be a cocircuit of $M_1$ such that $X_1 \cap C'_0 = \emptyset$. Then $X_1$ is a cocircuit of $M$ such that $X_1 \cap C_0 = \emptyset$ so by an hypothesis of the theorem we have $\left| X_1 \right| \geq 5$. Using the minimality of $r(M)$ we deduce that $M_1 - C'_0$ has a circuit $C$ such that $M_1 - C$ is connected and $r(M_1 - C) = r(M_1)$. Since $M - C = (M_1 - C) \Delta M_2$ it follows that $C$ is a circuit of $M$ such that $M - C$ is connected and $r(M - C) = r(M)$. This contradicts the choice of $M$. Thus $M$ has no vertical 3-separation and hence $M$ is vertically 4-connected. ■

We are now ready to complete the proof of the theorem. Let $M'$ be the simple matroid obtained by replacing all parallel classes of $M$ by single elements. By Claims 1 and 2, $M'$ is a 3-connected vertically 4-connected binary matroid. By [8, 7.6 and 14.3], $M'$ is either graphic or cographic, or is isomorphic to $R_{10}$, $F_7$ or $F^*_7$. Thus $M$ is either graphic or cographic, or can be obtained from $R_{10}$, $F_7$ or $F^*_7$ by a sequence of parallel extensions. If the latter alternative holds then since $R_{10}$, $F_7$ and $F^*_7$ have many cocircuits of size four, $M - C_0$ must contain a circuit $C$ of size two. The 3-connectivity of $M'$ now implies that $M - C$ is connected and $r(M - C) = r(M)$. Hence $M$ is graphic or cographic. Lemmas 4 and 6 now give a contradiction to the choice of $M$ as a counterexample to the theorem. ■
4 Closing Remarks

Remark 1 It follows from Corollary 2 that every connected graph $G$ of minimum degree at least three has a circuit $C$ such that $G - E(C)$ is connected. Thus every graphic matroid $M$ of cogirth at least three has a circuit $C$ such that $r(M) = r(M - C)$. The same result holds for a cographic matroid $M$ of cogirth at least three. (This can be seen by considering the graph $G$ for which $M$ is the cographic matroid. Then $G$ has girth at least three and the set of edges incident with any non-cutvertex of $G$ will give the required circuit $C$ of $M$.) The result does not extend to regular matroids of cogirth at least three since it does not hold for $R_{10}$ (which has cogirth four). However, if $M$ is a binary matroid which does not have both an $F_7$- and an $F_7^*$-minor, and has cogirth at least five, then we may apply Theorem 7 to a component of $M$ to deduce that $M$ has a circuit $C$ such that $r(M) = r(M - C)$.

One may hope that all binary matroids $M$ of sufficiently high girth have a circuit $C$ such that $r(M) = r(M - C)$. This is not the case. To see this note that $r(M) = r(M - C)$ if and only if $C$ does not contain any cocircuit of $M$. Thus, if $M$ is identically self-dual (and in particular if $M = R_{10}$) then no such circuit can exist. The assertion now follows since there exist identically self-dual binary matroids of arbitrarily high cogirth. The column matroid of the parity check matrix of the binary Reed-Muller code $R(s, 2s + 1)$, for example, is identically self dual and has cogirth $2s+1$.

Remark 2 It is not true that every connected matroid of sufficiently high girth has a circuit $C$ such that $M - C$ is connected. This can be seen by considering the uniform matroid $U_{m,2m}$. It is still conceivable, however, that this may hold for binary matroids.

Problem 1 Does there exist an integer $t$ such that every connected binary matroid $M$ of cogirth at least $t$ has a circuit $C$ such that $M - C$ is connected?

Remark 3 We could also ask for sufficient conditions for the existence of a cocircuit in a matroid $M$ the deletion of which preserves the connectivity of $M$. The following result of P.D. Seymour (see [6, Lemma 6]) is in the spirit of this paper. It is a matroid analogue of an earlier graph theoretic result of Kaugars (see [3, p. 31]).

Lemma 8 Let $M$ be a connected binary matroid of girth and cogirth at least three. Then $M$ has a cocircuit $X$ such that $M - X$ is connected.
References


