Generalized thrackles and graph embeddings

by

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Abstract

A *thrackle on a surface* X is a graph of size e and order n drawn on X such that every two distinct edges of G meet exactly once either at their common endpoint, or at a proper crossing. An unsolved conjecture of Conway (1969) asserts that $e \leq n$ for every thrackle on a sphere. Until now, the best known bound is $e \leq 1.428n$. By using discharging rules we show that $e \leq 1.4n$. Furthermore we show that the following are equivalent:

- G has a drawing on X where every two edges meet an odd number of times (a *generalized thrackle*)
- G has a drawing on X where every two edges meet exactly once (a *one-thrackle*)
- G has a special embedding on a surface whose genus differs from the genus of X by at most one.

Keywords: thrackle; one-thrackle; generalized thrackle; crossing number; graphs on surfaces; surface homology

This thesis is dedicated to my parents,

Mr. Baogang Xu, and Ms. Li Liu.
Their payment and supports inspire me in pursuit of knowledge.

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Chapter 1

Introduction

1.1 Basic Concepts in Graph Theory

Let $G = (V, E)$ be a finite simple graph with n vertices and e edges.

A graph can be represented on the plane, with vertices being considered as points or shown as small circles, and edges are simple curves joining the points corresponding to their ends. Curves representing the edges are allowed to cross each other but their interiors do not contain any vertices of the graph. Such a representing of a graph G is call a *drawing of G on the plane*. A graph is said to be *planar*, if it can be drawn on the plane so that its edges intersect only at their ends. Such a drawing is called a *plane embedding* of the graph. Figure 1.1(b) shows a plane embedding of K_4.

The planar drawing of G is *outerplanar* if it has an embedding on a closed disc with all its vertices on the boundary. Basic concepts in graph theory and embedded graphs can be found in [2] and [13].

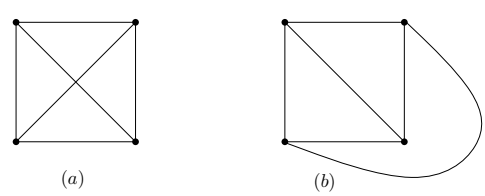


Figure 1.1: (a) K_4, (b) a drawing of K_4 on the plane

In late 1960s, a little bit before 1969, J. H. Conway defined a new kind of graph embedding: a thrackle. A *thrackle* of G is a drawing of G on the plane such that every two distinct edges of G either

- share an endpoint, and then they have no other point in common; or
- do not share an endpoint, in which case they meet exactly once at a proper crossing.

For the definition of proper crossing see [11]. A graph that can be drawn as a thrackle is said to be *thrackable*. Here is a C_6 drawn as a thrackle.

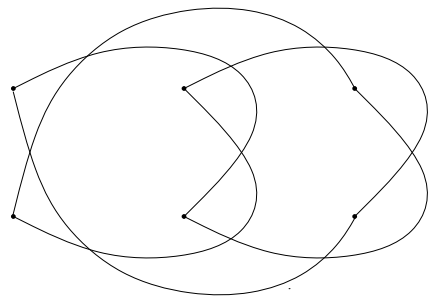


Figure 1.2: C_6 drawn as a thrackle.

Conway's Thrackle Conjecture: Every thrackable graph has at most as many edges as vertices.

This conjecture together with some partial solutions were first mentioned by Richard Guy and D. R. Woodall in the IMA conference in Oxford, July 1969. It is very difficult and still remains unresolved. In the past 40 years, many researches have worked on this problem and some progress has been made ([16], [11], [6], [8]).

1.2 Forbidden Configurations for Thrackable Graphs

Conway noted that the 4-cycle is not thrackable. Woodall determined the set of thrackable graphs assuming that the conjecture is true.

Theorem 1.2.1 *[16] If Conway's Thrackle Conjecture is true, then a finite graph is thrackable if and only if it has at most one odd cycle, it has no cycle of length four, and each of its connected components contains at most one cycle.*

It is straightforward to check the necessity of these conditions using Lemma 1.2.1.

Lemma 1.2.1 *[11] A 4-cycle is not thracklable, and no thrackable graph contains two vertex disjoint odd cycles.*

By various methods, Fulek and Pach determined three other configurations which are not thrackable, similar to Lemma 1.2.1.

Lemma 1.2.2 *[8, 11] No thrackable graph contains two 6-cycles where their intersection is a path of length l, $l = 0, 1, 2, 3$.*

Lemma 1.2.3 *[11] No thrackable bipartite graph contains a subdivision of K_5 or of $K_{3,3}$.*

Corollary 1.2.1 *Every thrackable bipartite graph is planar.*

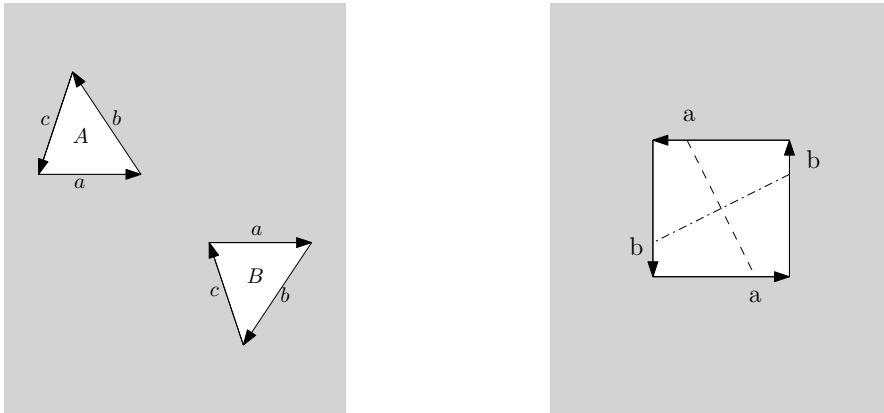
This permits us to invoke properties of planar graphs when studying both bipartite and non-bipartite thrackable graphs.

1.3 Thrackles and Surfaces

A *surface* is a compact connected Hausdorff topological space in which every point has a neighbourhood that is homeomorphic to the plane $\mathbb{R}^2$. A graph G can be represented on some surface X, with vertices being considered as points on X (or shown as small circles), and edges are simple (polygonal) curves on X joining the points corresponding to their ends. Curves representing the edges are allowed to cross each other but their interiors do not contain any vertices of G. Such a representation of a graph G on surface X is called a *drawing of G on X* . An *embedding* of a graph G on some surface X is an isomorphism of G with a graph G' embedded on X, where G' is a *representation* of G on X. We say G can be *embedded on X* if there is an embedding of it on X. A graph G is said to be *2-cell embedded* on X if it is embedded on X with the interior of each face homeomorphic to an open disc. An embedding of a graph G is determined, up to surface homeomorphism, by the following combinatorial description, $\Pi = (\pi, \lambda)$, where $\pi = \{\pi_v : v \in V(G)\}$ is a *rotation system*. Here π_v is a cyclic permutation of the edges incident with v and λ is a *signature mapping* which assigns to each edge $e \in E(G)$ a sign $\lambda(e) \in \{-1, 1\}$ ([13]). A cycle c of G is *1-sided* if it has an odd number of edges with negative sign, otherwise c is *2-sided*. An embedding Π of a graph G is a *parity embedding* if every even cycle of G is a 2-sided cycle and every odd cycle is a 1-sided cycle.

Let A and B be two disjoint triangles on the sphere $\mathbb{S}_0$. Cut the interiors of the triangles and identify pairs of edges on the boundary such that the directions of the edges in each pair agree, as in Figure 1.3(1). We say that the resulting surface S' is obtained by *adding a handle* to $\mathbb{S}_0$. Now let C be a quadrangle on $\mathbb{S}_0$. Cut its interior and identify the diametrically opposite points on the boundary as shown in Figure 1.3(2). In this case, the new surface is obtained by *adding a crosscap* on $\mathbb{S}_0$.

We consider all surfaces obtained from $\mathbb{S}_0$ by adding handles and crosscaps, respectively. If we add g handles to $\mathbb{S}_0$, we obtain the surface $\mathbb{S}_g$ which we refer to as the *orientable surface* of *Euler genus* $2g$. If we add k crosscaps to $\mathbb{S}_0$, we obtain the surface $\mathbb{N}_k$ which we refer to as the *nonorientable surface* of *Euler genus* k. The order in which we add the handles and crosscaps is



(1) (2)

Figure 1.3: Adding a handle or a crosscap to $\mathbb{S}_0$

not important. More related concepts and background can be found in [13]. The surfaces $\mathbb{S}_1$, $\mathbb{S}_2$ $\mathbb{N}_1$ and $\mathbb{N}_2$ are called the *torus*, *double torus*, *projective plane* and *Klein Bottle* respectively.

Classification of Surfaces Every surface is homeomorphic to precisely one of the surfaces $\mathbb{S}_g$ ($g \geq 0$), or $\mathbb{N}_k$ ($k \geq 1$).

The *Euler genus* $\hat{g}(S)$ of a surface S is defined as

$$\hat{g}(S) = \begin{cases} 2g & \text{if } S = \mathbb{S}_g \\ k & \text{if } S = \mathbb{N}_k. \end{cases}$$

Euler's Formula Let G be a multigragh which is 2-cell embedded on the surface X. If G has n vertices, e edges and f faces, then

$$n - e + f = 2 - \hat{g}(X). \tag{1.1}$$

Woodall [16] introduced the following generalized notion which has been used by several authors [11, 4, 6, 8, 7].

Generalized Thrackle A graph drawing is a *generalized thrackle* if any two distinct edges meet an odd number of times, either at a common end point, or at a proper crossing.

Notice that every thrackle is also a generalized thrackle, but the converse is false. For instance, C_4 is not thrackable, but it can be drawn as a generalized thrackle as shown in Figure 1.4.

Theorem 1.3.1 *[11, 6] A bipartite graph can be drawn as a generalized thrackle on the plane if and only if it is planar; a non-bipartite graph can be drawn as a generralized thrackle on plane if and only if it has a parity 2-cell embedding on* $\mathbb{N}_1$.

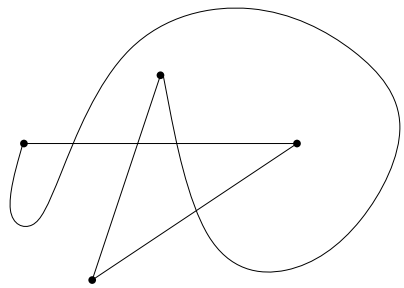


Figure 1.4: C_4 drawn as a generalized thrackle.

This was generalized by Cairns and Nikolayevsky. They extended Theorem 1.3.1 to orientable sufaces and present an analogous theorem for non-bipartite graphs. In this thesis, we extend the following two theorems to generalized thrackles on nonorientable surfaces.

Theorem 1.3.2 *[4] A bipartite graph G can be drawn as a generalized thrackle on a closed orientable connected surface $\mathbb{S}_g$ with genus $2g$ if and only if G can be embedded on $\mathbb{S}_g$.*

Recall that a parity embedding of a graph G on X is an embedding such that every even cycle of G is embedded as a 2-sided cycle and every odd cycle is embedded as a 1-sided cycle.

Theorem 1.3.3 *[6] A connected non-bipartite graph G can be drawn as a generalized thrackle on an oriented closed surface $\mathbb{S}_g$ if and only if G admits a parity embedding on a nonorientable closed surface $\mathbb{N}_{2g+1}$.*

1.4 Variations of Thrackability

Recently, Dan Archdeacon and Kirsten Stor have proved a surprising characterization of generalized thrackles on the plane.

Theorem 1.4.1 *For all graphs G, the following are equivalent:*

- *G has generalized thrackle embedding on the plane;*
- *G has a drawing such that every two distinct edges (including adjacent pairs) properly cross exactly once;*
- *There exits a point P on the plane such that G has a drawing where every two distinct edges (including adjacent ones) properly cross exactly once with the crossing at P;*
- *G has a parity embedding on $\mathbb{S}_0$ (if G is bipartite) or $\mathbb{N}_1$ (if G is non-bipartite).*

Here are a few more variations of thrackability that appeared in the literature.

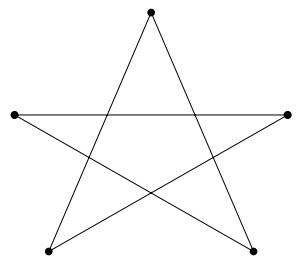


Figure 1.5: C_5 drawn as a straight thrackle.

Straight Thrackle ([16]) A thrackle is *straight* if each of its Jordan arcs is a segment of a straight line.

For example, C_5 is a straight thrackle as shown in Figure 1.5

A *caterpillar* is a tree in which each vertex is joined to at most two interior vertices of the tree. The interior vertices are those who are not leaves of a tree. The graph shown in Figure 1.6 is a caterpillar of order 13.

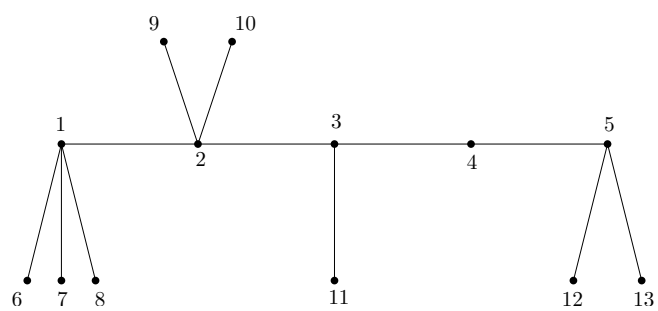


Figure 1.6: A caterpillar of order 13

Theorem 1.4.2 *[16] A finite graph G has a straight thrackle embedding on the plane if and only if it is a union of disjoint caterpillars or G is an odd cycle together possibly with extra vertices some or all of which are joined to a single vertex of the odd cycle.*

x-monotone ([14]) A thrackle is *x-monotone* if each curve representing an edge meets every vertical line in at most one point. Figure 1.5 shows an x-monotone thrackle. In particular, any straight thrackle with no vertical edge is x-monotone.

Theorem 1.4.3 *[14] Conway's thrackle conjecture holds for x-monotone thrackles.*

Other variations such as *musquash*, *outerplanar thrackles* and *superthrackles*, together with some partial results appear in [16], [3], [7], [1].

1.5 Upper Bounds on $|E|$

Conway described a technique, which is called Conway's Doubling Method, that converts any thrackle G on $\mathbb{S}_0$ to a bipartite thrackle G', by modifying a single odd cycle of G. By applying this, Woodall ([16]) proved the following.

Theorem 1.5.1 *Conway's thrackle conjecture is equivalent to the statement that every graph consisting of two even cycles sharing one common vertex is not thrackable.*

Applying Euler's Formula helps us to find upper bounds on $|E|$. Theorems stated in Section 1.3 and Section 1.4 make it possible to reduce upper bounds on $|E|$ as in the following four theorems.

Theorem 1.5.2 *[11] Every thrackle with n vertices has at most $2n-3$ edges. Every generalized thrackle with n vertices has at most $3n-4$ edges.*

Based on the same idea, but extended to general surfaces, Cairns and Nikolayevsky ([4]) stated the theorem below improving the bound in Theorem 1.5.2.

Theorem 1.5.3 *Every thrackle with n vertices has at most $\frac{3}{2}(n-1)$ edges. Every generalized thrackle with n vertices has at most $2n-2$ edges.*

Following the same idea, together with Conway's Doubling Method, Fulek and Pach [8] reduced the upper bound to $1.428n$.

Theorem 1.5.4 *Every thrackle with n vertices has at most $1.428n$ edges.*

However, Fulek and Pach didn't use the projective planar property of a thrackable graph G. And their use of Conway's Doubling introduced unnecessary complexity to the proof, which also degraded the upper bound they obtained in Theorem 1.5.4. In Chapter 3 we show the following.

Theorem 1.5.5 *Let G be a thrackable graph with n vertices and e edges. Then $e \leq 1.4n$.*

Chapter 2

Generalized Thrackles on Other Surfaces

Defination 2.0.1 *A one-thrackle of a simple graph G on surface X is a drawing of G on X where every two edges properly cross exactly once. Thus, adjacent edges in a one-thrackle meet at two points, once at a crossing and once at an endpoint.*

We recall that a generalized thrackle is a drawing such that any two distinct edges meet an odd number of times, either at a common end point, or at a proper crossing.

The purpose of this chapter is to show the following equivalences.

- G has a generalized thrackle drawing on X;
- G has a one-thrackle drawing on X;
- G has a special embedding on a surface whose genus differs from the genus of X by at most one.

2.1 Introduction

Refer to [13] for basic definitions.

Let G be a connected graph with $|V(G)| = n$ and $|E(G)| = e$. A *cycle* is the edge set of a subgraph $H \subseteq G$ where $deg_H(v)$ is even for every vertex in H. A *circuit* is a minimal non-empty cycle. Let $\mathcal{C}$ be the set of cycles of G, and $c, c' \in \mathcal{C}$. Then the symmetric difference $c \triangle c'$ is also in $\mathcal{C}$. Indeed, $(\mathcal{C}, \triangle)$ is a subspace of the vector space $(E(G), \triangle) \cong Z_2^e$. We call $\mathcal{C}$ the *cycle space* of G. The dimension of $\mathcal{C}$ is the *Beti Number*, $\beta = e - n + 1$. Let W be a closed walk in G. Define *odd(W)* be the set of edges in W traversed odd number of times in W. Then $c \in \mathcal{C}$ if and only if G has a closed walk W where $c = odd(W)$.

Let $\mathcal{M} : G \to X$ be a 2-cell embedding of G on a surface X. The *face space* of $\mathcal{M}$ is the subspace $\mathcal{F}$ of $\mathcal{C}$ generated by facial boundaries of $\mathcal{M}$. Let $\hat{g}$ be the Euler genus of X. The dimension of $\mathcal{F}$ is one less than the number of faces in $\mathcal{M}$, which equals $e - n - \hat{g} + 1$ by Euler's formula. It is a fact that $c \in \mathcal{F}$ if and only if c is a disjoint union of circuits $c_1, c_2, \ldots, c_k$ where each $\mathcal{M}(c_i)$ is a surface-separating simple curve in X.

The homology group of X is easily characterized in terms of a $2-$cell embedding $\mathcal{M} : G \to X$. We define Z_2*-homology group* H_1 *of the embedding* $\mathcal{M}$ to be the quotient group $\mathcal{C}/\mathcal{F}$. Thus, $H_1 \cong Z_2^{\beta-(e-n+f-\hat{g})} = Z_2^{\hat{g}}$. Each element of H_1 is an equivalence class $[c]$ of cycles in G, where $c' \in [c]$ if and only if $c' = c \triangle f$ where $f \in \mathcal{F}$. Addition in H_1 is defined by

$$[c] + [c'] = [c \triangle c']. \tag{2.1}$$

We extend each homology class $[c]$ to include all finite sets $\mathcal{W} = \{W_i\}$ of closed walks in G, so that $[c] = [\mathcal{W}]$ if and only if $c = \triangle_i odd(W_i) + f$ for some $f \in \mathcal{F}$. By connecting the termini of walks W_i with paths in G that are each traversed twice, we may obtain a single closed walk $W' \in [\mathcal{W}]$. Thus, each homology class contains a closed walk in G.

We further extend $[\mathcal{W}]$ to include all finite sets $\gamma = \{\gamma_i\}$ of closed curves, $\gamma_i : [0,1] \longrightarrow X$, where each γ_i is continuous and $\gamma_i(0) = \gamma_i(1)$. Here $\gamma \in [\mathcal{W}]$ if each γ_i can be continuously deformed to W_i. Since G is 2-cell embedded, every γ_i can be deformed to a walk in G. Addition of such homology classes is again well defined, and no longer depends on the graph G, so we may write $H_1 = H_1(X)$, where $H_1(X)$ is the usual homology group of X over GF_2. It is known ([10]) that $H_1(\mathbb{N}_k) \cong Z_2^k$ and $H_1(\mathbb{S}_g) \cong Z_2^{2g}$.

Let γ_1, γ_2 be two closed curves on X. We say γ_1, γ_2 are in *general position* if $\gamma_1 \bigcap \gamma_2$ is a finite set of points at which γ_1, γ_2 properly cross. We define the Z_2*-intersection form*

$$\Omega_X : H_1(X) \times H_1(X) \to Z_2$$

as follows. If γ_1, γ_2 are in general position, then $\Omega_X(\gamma_1, \gamma_2) = |\gamma_1 \cap \gamma_2|$ mod 2 (we write $\Omega_X(\gamma_1, \gamma_2)$ instead of $\Omega_X([\gamma_1], [\gamma_2])$). Otherwise, $\Omega_X(\gamma_1, \gamma_2) = \Omega_X(\gamma_1, \gamma_2')$ where γ_2' is obtained from γ_2 by a continuous deformation so that γ_1, γ_2' are in general position. The function Ω_X is symmetric, and bilinear, i.e., for $c, c', c'' \in H_1$, $\Omega_X(c \triangle c', c'') = \Omega_X(c'', c \triangle c') = \Omega_X(c, c'') + \Omega_X(c', c'')$. In addition, we have

$$\Omega_X(\gamma, \gamma) = \begin{cases} 0 & \text{if } \gamma \text{ is 2-sided} \\ 1 & \text{if } \gamma \text{ is 1-sided.} \end{cases}$$

To see this, we slightly deform one copy of γ to γ' and note that an odd number of crossings are required if γ is one-sided.

Suppose $\mathcal{D} : G \to X$ is a drawing of G to X. Let $\omega_{\mathcal{D}}$ be the *pull back* of Ω_X on $\mathcal{C}$. That is, for $c, c' \in \mathcal{C}(G)$, $\omega_D(c, c') = \Omega_X(\mathcal{D}(c), \mathcal{D}(c'))$. In general, $\{[\mathcal{D}(c)] : c \in \mathcal{C}\} \leq H_1(X)$. For any ordered

basis $\mathcal{B} = \{c_1, c_2, \ldots, c_\beta\}$ of $\mathcal{C}$, we get a *standard matrix* $J = J_\mathcal{B}$ which represents the bilinear form $\omega_\mathcal{D}$ with respect to $\mathcal{B}$, Thus J is defined by

$$(J)_{ij} = \omega_\mathcal{D}(c_i, c_j). \tag{2.2}$$

Let $c = \sum \alpha_i c_i$, $c' = \sum \alpha'_i c_i$ be any two curves, and $\alpha = (\alpha_1, \ldots, \alpha_\beta)^T$, $\alpha' = (\alpha'_1, \ldots, \alpha'_\beta)^T$. Then $\omega_\mathcal{D}(c, c') = \alpha^T J \alpha'$. The *rank* of $\omega_\mathcal{D}$ is defined by $rk(\omega_\mathcal{D}) = rk(J_\mathcal{B})$. In general, $rk(\omega_\mathcal{D}) \leq \hat{g}(X)$. The matrix $J_\mathcal{B}$ is the adjacency matrix of a graph. Let $G_\mathcal{B}$ be the graph with $V(G_\mathcal{B}) = \mathcal{B}$, and where $c_i c_j$ is an edge of $G_\mathcal{B}$ if and only if $\omega_\mathcal{D}(c_i, c_j) = 1$, for $1 \leq i \leq j \leq \beta$. Thus c_i has a loop in $G_\mathcal{B}$ if and only if $\omega_\mathcal{D}(c_i, c_i) = 1$.

We may visualize the effect of changing a basis from $\mathcal{B}$ to $\mathcal{B}'$ by comparing the graphs $G_\mathcal{B}$ and $G_{\mathcal{B}'}$. The effect of an elementary operation $c_i \mapsto c_i + c_j$ on $J_\mathcal{B}$ is that $J_{\mathcal{B}'}$ is obtained from $J_\mathcal{B}$ by adding row j to row i, then adding column j to column i.

Let $E_0(G_\mathcal{B})$ be the set of nonloop edges of $G_\mathcal{B}$. Let $S(G_\mathcal{B})$ be the set of vertices of $G_\mathcal{B}$ that have a loop. Thus, $u \in S$ if and only if $\mathcal{D}(u)$ is a 1-sided curve in X. Therefore,

$$E(G_\mathcal{B}) = E_0(G_\mathcal{B}) \cup \{uu : u \in S\}.$$

Let $N(u) = \{z : uz \in E(G_\mathcal{B})\}$. Thus $u \in N(u)$ if and only if $u \in S$. To *switch* a pair of vertices u, z is to make a new graph G' where $V(G') = V(G_\mathcal{B})$, and $E(G')$ and $S(G')$ are defined as follows:

- If $u \neq z$, then $E_0(G') = E_0(G_\mathcal{B}) \triangle \{uz\}$, and $S(G') = S(G_\mathcal{B})$;
- If $u = z$, then $E_0(G') = E_0(G_\mathcal{B})$, and $S(G') = S(G_\mathcal{B}) \triangle \{u\}$.

If $\mathcal{B}'$ is obtained from $\mathcal{B}$ by replacing $u \in \mathcal{B}$ by $u + v$, then $G_{\mathcal{B}'}$ is obtained from $G_\mathcal{B}$ by

- switching each pair u, z where $z \in N(v)$, then
- switching the loop uu if $v \in S(G_\mathcal{B})$.

We illustrate this with two examples in Figure 2.1

2.2 Canonical Basis Arising from a Bilinear Form

Let X be a surface, and $\mathcal{D} : G \to X$ be a drawing. Let $\omega_\mathcal{D}$ be the bilinear Z_2-pull back form as defined above. We aim to find a canonical basis $\mathcal{B}$ of $\mathcal{C}$, with respect to $\omega_\mathcal{D}$.

A basis $\mathcal{B}$ is *canonical with respect to* $\omega_\mathcal{D}$ if among all bases of $\mathcal{C}$, $\mathcal{B}$ satisfies the following:

(1) $|S(G_\mathcal{B})|$ is minimal;

(2) Subject to (1), the number of components of $G_\mathcal{B}$ is maximal.

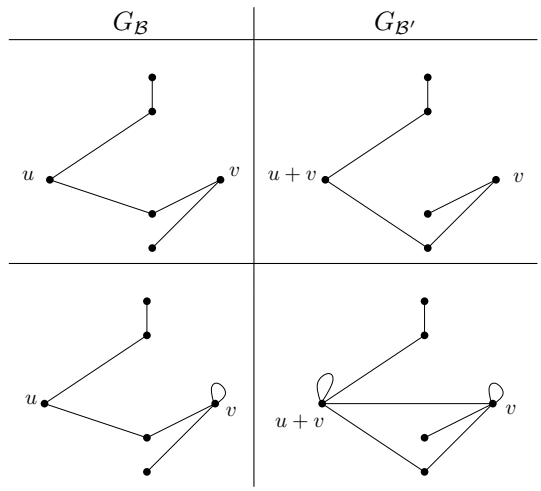


Figure 2.1: The effect on $G_{\mathcal{B}}$ of adding v to u

Let $\mathcal{B}$ be canonical with respect to $\omega_{\mathcal{D}}$, and let $G_{\mathcal{B}}, S$ be as above.

Claim 1: $|S(G_{\mathcal{B}})| \leq 1$.

To see this, suppose $|S(G_{\mathcal{B}})| \geq 2$, then adding one vertex in S to one of the others decreases $|S(G_{\mathcal{B}})|$ and (1) has been violated.

Claim 2: Each component of $G_{\mathcal{B}}$ has at most two vertices.

Proof. Let H be a component of size at least three. If H has an edge uv with no loops on its ends, then add v to $N(u) - \{v\}$ in $G_{\mathcal{B}}$. And then add u to $N(v) - \{u\}$. The resulting graph has more components, because the vertex set $\{u, v\}$ induces a component in the resulting graph, and no other component of $G_{\mathcal{B}}$ is affected, so (2) is violated.

We may assume $S = \{s\}$ where $s \in V(G_{\mathcal{B}})$ and H is a star centered at s, with at least two leaves. Adding one of the leaves to one of the others results in a graph with more components, again violating (2). ∎

We have proved the following:

Lemma 2.2.1 *If $\mathcal{B}$ is a canonical basis with respect to $\omega_{\mathcal{D}}$, then $G_{\mathcal{B}}$ is isomorphic to a graph in one of the three infinite classes shown in Figure 2.2, for some $k \geq 0$.*

Remark: One can easily check that if $\mathcal{B}$ is canonical, $rk(\omega_{\mathcal{D}})$ is the number of vertices of $G_{\mathcal{B}}$ incident to at least one edge or loop. Thus,

$$rk(\omega_{\mathcal{D}}) = \begin{cases} 2k+1 & \text{if } G_{\mathcal{B}} \text{ is of type A} \\ 2k+2 & \text{if } G_{\mathcal{B}} \text{ is of type B} \\ 2k & \text{if } G_{\mathcal{B}} \text{ is of type C.} \end{cases}$$

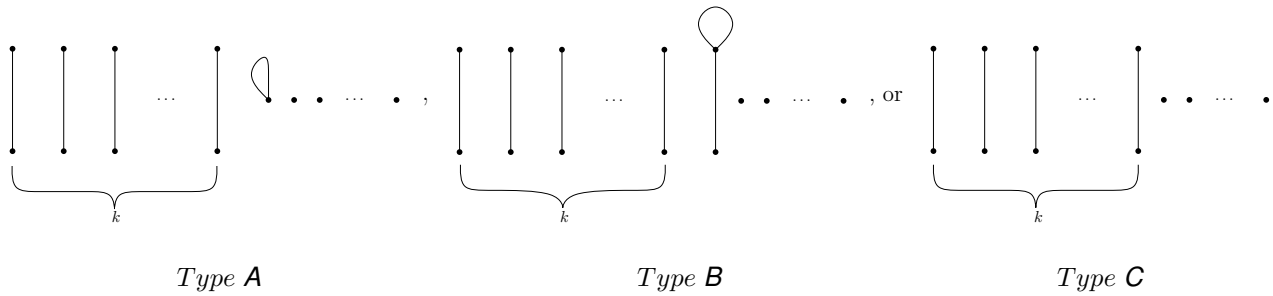


Figure 2.2: Three infinite classes of $G_{\mathcal{B}}$

If $\mathcal{D}$ is a $2-$cell embedding of G in X, then a canonical basis corresponds to a presentation of X as the direct sum of $k \geq 0$ handles and at most one Klein handle or crosscap. Here, we have

$$\Omega_X \cong \omega_{\mathcal{D}},$$

and

$$X = \begin{cases} \mathbb{N}_{2k+1} & \text{if } G_{\mathcal{B}} \text{ is of type A} \\ \mathbb{N}_{2k+2} & \text{if } G_{\mathcal{B}} \text{ is of type B} \\ \mathbb{S}_k & \text{if } G_{\mathcal{B}} \text{ is of type C.} \end{cases}$$

Proposition 2.2.1 *If $\mathcal{D} : G \to X$ is a drawing, then*

$$rk(\omega_{\mathcal{D}}) \leq \hat{g}(X). \tag{2.3}$$

Moreover, $rk(\omega_{\mathcal{D}}) = \hat{g}(X)$ if $\mathcal{D}$ is a $2-$cell embedding.

2.3 Canonical Basis Arising from a Bilinear Form and a Linear Form

Let $l : \mathcal{C} \to Z_2$ be a linear form. In our application, $l(c)$ will be the length of c mod 2. Let $\mathcal{D} : G \to X$ be a drawing of G on a surface, and let $\omega_{\mathcal{D}} : \mathcal{C} \times \mathcal{C} \to Z_2$ be the pull-back bilinear form of $\mathcal{D}$. We aim to find a canonical basis $\mathcal{B}$ of $\mathcal{C}$ with respect to the *pair* $(\omega_{\mathcal{D}}, l)$. For any basis $\mathcal{B}$ of $\mathcal{C}$, we define the graph $G_{\mathcal{B}} = (V, E)$ where $V(G_{\mathcal{B}}) = \mathcal{B}$ and $E(G_{\mathcal{B}}) = E_0(G_{\mathcal{B}}) \cup \{uu : u \in S\}$, and the set $S \subseteq V(G_{\mathcal{B}})$ is defined as before. Let $L = \{c \in \mathcal{B} : l(c) = 1\}$.

A basis $\mathcal{B}$ is *canonical with respect to* $(\omega_{\mathcal{D}}, l)$ if, among all bases of $\mathcal{C}$, $\mathcal{B}$ satisfies the following:

(1) $|L|$ is minimal;

(2) Subject to (1), $|S \setminus L|$ is minimal;

(3) Subject to (1) and (2), the number of components of $G_{\mathcal{B}}$ is maximal.

Let $\mathcal{B}$ be canonical with respect to $(\omega_{\mathcal{D}}, l)$, and let $G_{\mathcal{B}}, S, L$ be as above.

Claim 1**:** $|L| \leq 1$.
Otherwise add any cycle in L to another cycle in L. This decreases $|L|$, violating (1).

Claim 2**:** $|S \setminus L| \leq 1$.
Otherwise add any cycle in $S \setminus L$ to another cycle in $S \setminus L$. This does not change L, but decreases $|S \setminus L|$, violating (2).

Claim 3**:** If $v \in V \setminus (S \cup L)$ and $\deg(v) = 1$, then v is in a connected component of $G_{\mathcal{B}}$ with exactly two vertices.

Suppose v is in a component with at least three vertices. Let u be the neighbour of v, and add v to every neighbour of u that is different from v. This operation does not change S or L. Only edges adjacent to u will be switched, but $\{u, v\}$ will induce a connected component in the new graph, so the new graph has more components than $G_{\mathcal{B}}$, violating (3) (note that this claim holds, even if $u \in S \cup L$).

Claim 4**:** If $uv \in E_0(G_{\mathcal{B}})$ and $u, v \in V \setminus (S \cup V)$, then $\{u, v\}$ induces a connected component of $G_{\mathcal{B}}$ isomorphic to K_2.

Suppose v is in a component with at least three vertices. By adding u to every other neighbour of v, as in Claim 3, we may assume vertex v has degree 1 in the resulting graph $G_{\mathcal{B}'}$. Then we add v to every neighbour of u in $G_{\mathcal{B}'}$ that is different from v, to obtain $G_{\mathcal{B}''}$. Neither of these operations changes S or L, but $\{u, v\}$ will induce a connected component of $G_{\mathcal{B}''}$, so $G_{\mathcal{B}''}$ has more components than $G_{\mathcal{B}}$, which violates (3).

Claim 5**:** If $L = \{w\}$ and let H be the component of $G_{\mathcal{B}}$ containing w, then either

- $V(H) = \{w\}$, or
- $V(H) = \{w, v\}$ for some $v \in V \setminus (S \cup L)$.

Proof of Claim 5:

By Claim 4, every edge in H has at least one end in L or S. Thus either H is a star with center vertex w and every other vertex of H is in $V \setminus (S \cup L)$, or H also contains the only vertex, say s, belonging to $S \setminus L$, and every edge has at least one end in $\{w, s\}$.

In the first case, we are done by Claim 3, so we assume the second case.

If some vertex in $V(H) \setminus \{w, s\}$ has degree 1, then we are done by Claim 3.

It follows that every vertex in $V(H) \setminus \{w, s\}$ is adjacent to both w and s and has degree exactly 2. If $V(H) \setminus \{w, s\}$ contains distinct vertices, say u and v, then we may add u to v, and v will become an isolated vertex, contradicting to (3).

Thus we may assume that H has exactly three vertices w, s and v with $L = \{w\}$, $v \in V \setminus (S \cup L)$, and $s \in S \subseteq \{w, s\}$, and H contains at least two edges wv and vs. By adding v to w if necessary, we may assume that ws is also an edge of $G_{\mathcal{B}}$, so that H is a triangle with a loop at s and possibly a loop at w. Now adding s to w results in w becoming an isolated vertex (with a possible loop), and we have contradicted (3). ∎

Claim 6**:** If $S = \{s\}$ and H is the component containing s, then either

- $V(H) = \{s\}$, or
- $V(H) = \{s, v\}$ for some $v \in V \setminus (S \cup L)$.

The argument here is almost exactly as for Claim 5, but with w and s interchanged in the first paragraph.

We have shown that for a canonical basis $\mathcal{B}$ with respect to $(\omega_{\mathcal{D}}, l)$, every connected component of $G_{\mathcal{B}}$ has one or two vertices, and no component contains both a vertex in $S \setminus L$ and a vertex in L. Thus $G_{\mathcal{B}}$ is characterized as follows.

Theorem 2.3.1 *Let $\mathcal{D} : G \to X$. Let $\mathcal{B}$ be a canonical basis with respect to $(\omega_{\mathcal{D}}, l)$. Let $G_{\mathcal{B}}, S, L$ be as above. Let G' be the graph obtained from $G_{\mathcal{B}}$ by deleting every connected component H that is isomorphic to K_1 or K_2, and $V(H) \cap L = V(H) \cap S = \emptyset$. Then G' is one of 15 possible graphs shown in Figure 2.3.*

The cases are partitioned into nine *Types* $i \in \{1, 2, \ldots, 8, 9\}$, and further partitioned into 15 *sub-Types* as below

$$1, 2, 3, \{i(a) : i = 4, 5, \ldots, 8, 9\}, \text{ and } \{i(b) : i = 4, 5, \ldots, 8, 9\}.$$

If $\mathcal{D} : G \to X$ is a $2-$cell embedding and $l(c) = |c| \pmod 2$, where $c \in \mathcal{C}$, then the sub-Type of $\mathcal{D}$ determines whether the surface X is orientable and whether G is bipartite. This is summarized in the table of Figure 2.4.

2.4 Thrackable Graphs and 2-cell Embeddings

Recall that a drawing $\mathcal{T} : G \to X$ is a generalized thrackle if any two distinct edges meet an odd number of times, either at a common end point, or at a proper crossing. We prove that if G

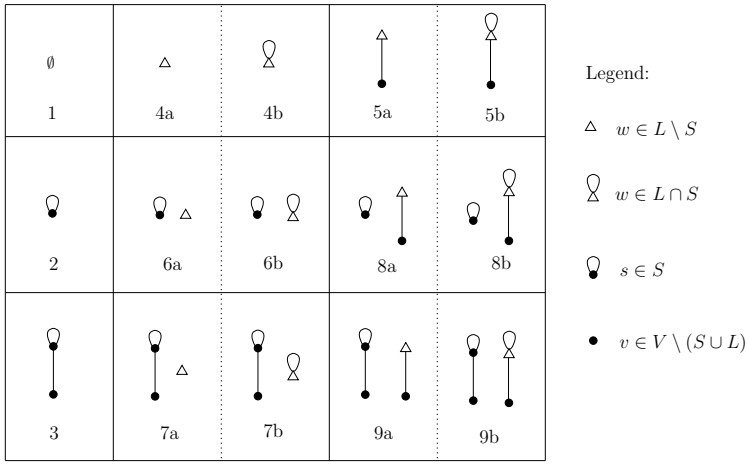


Figure 2.3: Fifteen cases for G'

(sub-)Type	G	X
Type 1	Bipartite	Orientable
Types $2, 3$	Bipartite	Non-orientable
sub-Types $4(a), 5(a)$	Non-bipartite	Orientable
Types $6, 7, 8, 9$, sub-Types $4(b), 5(b)$	Non-bipartite	Non-orientable.

Figure 2.4: Properties of G and X determined by the Type of $\mathcal{D}$, when $\mathcal{D}$ is a $2-$cell embedding

has a generalized thrackle drawing on X, then G has a $2-$cell embedding on a surface Y where $\hat{g}(Y) \leq \hat{g}(X) + 1$.

Let $e \neq e'$ be two distinct edges of G. For each pair e, e' we add a handle to X in a small neighbourhood of the crossing point. Let one of the two edges pass through the handle. Let X' be the resulting surface (Figure 2.5).

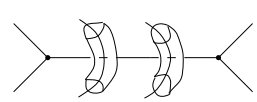


Figure 2.5: A new surface X' obtained from X by adding handles to X

Take a closed neighbourhood Z of $\mathcal{T}(G)$ such that $\mathcal{T}(G) \subseteq Z \subseteq X'$ consists of small discs around the vertices of G joined by narrow bands along the edges. Thus Z is a compact surface with boundary. Gluing discs to the boundary components of Z results in a new surface U. Thus, we

obtain a $2-$cell embedding $\mathcal{P} : G \to U$, with pull-back $\omega_{\mathcal{P}}$ onto the cycle space $\mathcal{C}$.

Define functions $l \otimes l : \mathcal{C} \times \mathcal{C} \to Z_2$ and $\tau : \mathcal{C} \times \mathcal{C} \to Z_2$ by

$$(l \otimes l)(c, c') = l(c)l(c'), \tag{2.4}$$

$$\tau(c, c') = |c \cap c'| \pmod 2. \tag{2.5}$$

Each of $\omega_{\mathcal{T}}, l \otimes l$, and τ is a bilinear form.

Lemma 2.4.1 *([6]) Let $\mathcal{T} : G \to X$ be a generalized thrackle. Let $\mathcal{P} : G \to U$, $l \otimes l$ and τ be as above. Then $\omega_{\mathcal{T}} = \omega_{\mathcal{P}} + l \otimes l + \tau$.*

Proof. Let c, c' be two circuits in $\mathcal{C}$. We are going to show that

$$\omega_{\mathcal{T}}(c, c') = \omega_{\mathcal{P}}(c, c') + l(c)l(c') + \tau(c, c') \pmod 2.$$

The result follows by the bilinearity of $\omega_{\mathcal{T}}, \omega_{\mathcal{P}}, l \otimes l$ and τ when c, c' are two cycles in $\mathcal{C}$. Divide the edge set of c into four disjoint parts: k_1 edges in $c \bigcap c'$; k_2 edges each of which is incident at one endpoint to an edge of c'; k_3 edges each of which is not contained in $c \bigcap c'$ but have both end vertices incident to some edges of c'; and finally k_4 edges not adjacent to any edge of c'. Every crossing of c, c' in $\mathcal{T}$ that is not already accounted for by $\omega_{\mathcal{P}}(c, c')$ is a proper crossing of distinct edges $e \in c$, $e' \in c'$ at a common interior points. Thus,

$$\begin{aligned}
\omega_{\mathcal{T}}(c, c') &= \omega_{\mathcal{P}}(c, c') + k_1(l(c') - 3) + k_2(l(c') - 2) + k_3(l(c') - 4) + k_4 l(c') \pmod 2 \\
&= \omega_{\mathcal{P}}(c, c') + l(c)l(c') + k_1 \pmod 2 \\
&= \omega_{\mathcal{P}}(c, c') + l(c)l(c') + \tau(c, c').
\end{aligned}$$

∎

For each edge e of G, we break its corresponding band on Z, and glue it back with a half turn, and seal discs to the resulting boundary components. We thus obtain a $2-$cell embedding $\mathcal{E} : G \to Y$, where Y is a new surface. Let $H_1(Y)$ be the Z_2-homology group of Y over Z_2 as defined in Section 2.1. Let $\Omega_Y : H_1(Y) \times H_1(Y) \to Z_2$ be the bilinear intersection form, and let $\omega_{\mathcal{E}}$ be the pull-back onto $\mathcal{C}$. Since $\mathcal{E}$ is a $2-$cell embedding, we have $\hat{g}(Y) = rk(\omega_{\mathcal{E}})$.

Lemma 2.4.2 *([6]) Let $\mathcal{T} : G \to X$ be a generalized thrackle. Let $\mathcal{P} : G \to U$, $l \otimes l$ and τ be as above. Then $\omega_{\mathcal{E}} = \omega_{\mathcal{P}} + \tau$.*

Proof. Let c, c' be two circuits in $\mathcal{C}$. We will show that

$$\omega_{\mathcal{E}}(c, c') = \omega_{\mathcal{P}}(c, c') + \tau(c, c') \pmod 2.$$

The result follows by the bilinearity of $\omega_{\mathcal{T}}, \omega_{\mathcal{P}}, l \otimes l$ and τ when c, c' are two cycles in $\mathcal{C}$.

Since $\omega_{\mathcal{T}}(c,c') = \omega_{\mathcal{P}}(c,c') + l(c)l(c') + |c \cap c'|$, then $\omega_{\mathcal{T}}(c,c') + l(c)l(c') = \omega_{\mathcal{P}}(c,c') + |c \cap c'|$. When changing from the surface Z to Y, we half turn each edge of G, which implies there are $|c \cap c'|$ more crossings counted under $\omega_{\mathcal{E}}$ than under $\omega_{\mathcal{P}}$. Therefore, $\omega_{\mathcal{E}}(c,c') = \omega_{\mathcal{P}}(c,c') + \tau(c,c')$. ∎

The following theorem immediately comes from Lemma 2.4.2 and Lemma 2.4.1.

Theorem 2.4.1 *Let $\mathcal{T} : G \to X$ be a generalized thrackle. Let $\mathcal{P} : G \to U$, $l \otimes l$ and τ be as above. Then $\omega_{\mathcal{E}} = \omega_{\mathcal{T}} + l \otimes l$.*

Let $\mathcal{B}$ be a canonical basis of $\mathcal{C}$ with respect to $(\omega_{\mathcal{T}}, l)$, and let $G_{\mathcal{B}}$ be the associated graph defined in Section 2.3. Let $G_{\mathcal{B}}^{\mathcal{E}}$ be the graph where $V(G_{\mathcal{B}}^{\mathcal{E}}) = \mathcal{B}$, and $c_i c_j$ is an edge of $G_{\mathcal{B}}^{\mathcal{E}}$ if and only if $\omega_{\mathcal{E}}(c_i, c_j) = 1$. Let $E_0(G_{\mathcal{B}}^{\mathcal{E}})$ be the set of nonloop edges of $G_{\mathcal{B}}^{\mathcal{E}}$, $S(G_{\mathcal{B}}^{\mathcal{E}})$ be the set of vertices of $G_{\mathcal{B}}^{\mathcal{E}}$ that have a loop. Therefore, $E(G_{\mathcal{B}}^{\mathcal{E}}) = E_0(G_{\mathcal{B}}^{\mathcal{E}}) \cup \{vv : v \in S\}$. Let $L(G_{\mathcal{B}}^{\mathcal{E}})$ be the set of cycles c in $\mathcal{B}$ with $l(c) = 1$. Recall that $|L(G_{\mathcal{B}})| \leq 1$, and $|S(G_{\mathcal{B}}) - L(G_{\mathcal{B}})| \leq 1$. Furthermore, we have

$$E_0(G_{\mathcal{B}}^{\mathcal{E}}) = E_0(G_{\mathcal{B}}), \tag{2.6}$$

$$L(G_{\mathcal{B}}^{\mathcal{E}}) = L(G_{\mathcal{B}}), \tag{2.7}$$

and

$$S(G_{\mathcal{B}}^{\mathcal{E}}) = \begin{cases} S(G_{\mathcal{B}}) \triangle \{w\} & \text{if } L = \{w\} \\ S(G_{\mathcal{B}}) & \text{if } L = \emptyset. \end{cases}$$

As in Theorem 2.3.1, $G_{\mathcal{B}}^{\mathcal{E}}$ has one of the 15 possible Types as does $G_{\mathcal{B}}$. For each Type, we compare $G_{\mathcal{B}}$ and $G_{\mathcal{B}}^{\mathcal{E}}$, and the ranks of $\omega_{\mathcal{T}}$, $\omega_{\mathcal{E}}$ as below.

- Type $1, 2, 3$: since there are no odd cycles, $G_{\mathcal{B}} = G_{\mathcal{B}}^{\mathcal{E}}$, and $\omega_{\mathcal{T}} = \omega_{\mathcal{E}}$.
- For $i \in \{4, 6, 7\}$,

Type i		
$i(a)$	$i(b)$	rank
$G_{\mathcal{B}}$	$G_{\mathcal{B}}^{\mathcal{E}}$	$rk(\omega_{\mathcal{E}}) = rk(\omega_{\mathcal{T}}) + 1$
$G_{\mathcal{B}}^{\mathcal{E}}$	$G_{\mathcal{B}}$	$rk(\omega_{\mathcal{E}}) = rk(\omega_{\mathcal{T}}) - 1$

- For $j \in \{5, 8, 9\}$

Type j		
$j(a)$	$j(b)$	rank
$G_{\mathcal{B}}$	$G_{\mathcal{B}}^{\mathcal{E}}$	$rk(\omega_{\mathcal{E}}) = rk(\omega_{\mathcal{T}})$
$G_{\mathcal{B}}^{\mathcal{E}}$	$G_{\mathcal{B}}$	$rk(\omega_{\mathcal{E}}) = rk(\omega_{\mathcal{T}})$

Theorem 2.4.2 *For any generalized thrackle $\mathcal{T} : G \to X$, let $\mathcal{E}$, $G_{\mathcal{B}}$ and $G_{\mathcal{B}}^{\mathcal{E}}$ be as above. Then $Type(G_{\mathcal{B}})$=$Type(G_{\mathcal{B}}^{\mathcal{E}})$=$i$ for $i \in \{1, 2, \ldots, 8, 9\}$. Furthermore, if $i \geq 4$, we have $\{Type(G_{\mathcal{B}}), Type(G_{\mathcal{B}}^{\mathcal{E}})\} = \{i(a), i(b)\}$.*

2.5 Dual Euler Walks of an Even Faced Embedded Graph

Let $\mathcal{T} : G \to X$ be a generalized thrackle. Let $\mathcal{E} : G \to Y$ be the $2-$cell embedding we obtained in the last section, with an intersection form Ω_Y and its pull-back $\omega_{\mathcal{E}}$ onto $\mathcal{C}(G)$. In this section, we assume that $\mathcal{E}$ is even-faced. That is, the boundary of every face is an even-length walk. Thus the sub-Type of $G^{\mathcal{E}}_{\mathcal{B}}$ is not $4(a)$, $6(a)$, or $7(a)$. We define a surface dual embedding $\mathcal{E}^* : G^* \to Y$ such that $V(G^*) = F(G)$, where $F(G)$ is the set of faces of G under $\mathcal{E}$. Let $\mathcal{B}$ be a canonical basis of $\mathcal{C}$ with respect to $(\omega_{\mathcal{T}}, l)$, and l is the linear form where $l(c)$ is the length of c mod 2.

Recall from Section 2.1 that the Z_2-homology group H_1 of the embedding $\mathcal{E}$ is the quotient group $\mathcal{C}/\mathcal{F}$. We extended each homology class to include all finite sets of closed curves on Y, including the closed walks in G^*.

Since G^* is connected with no odd vertices, G^* has an Euler walk d^* traversing every edge of G^* exactly once. We claim that the homology class $[d^*] \in H_1(Y)$ depends only on the sub-Type of $\mathcal{E}$, so any two Euler walks have equal homology type.

Let d^* be a closed walk in G^* which uses every edge of G^* at least once. The homology class of *dual walk of G* can be expressed as a linear combination of the homology classes of cycles in $\mathcal{B}$.

$$[d^*] = \sum_{i=1}^{\beta} p_i[\mathcal{E}(c_i)] \; with \; p_i \in Z_2. \tag{2.8}$$

Let $G^{\mathcal{E}}_{\mathcal{B}}, S, L$ and $E_0(G^{\mathcal{E}}_{\mathcal{B}})$ be as above, and let $L = \{w\}$ and $S \setminus L = \{s\}$ if $L, S \setminus L$ are not empty. Each edge of G crosses d^* exactly once on Y. For each cycle $c_i \in \mathcal{B}$,

$$\Omega_Y([d^*], [\mathcal{E}(c_i)]) = \begin{cases} 0 & \text{if } l(c_i) \text{ is even} \\ 1 & \text{if } l(c_i) \text{ is odd.} \end{cases}$$

Let $N(c_i) = \{z : c_i z \in E(G^{\mathcal{E}}_{\mathcal{B}})\}$. Recall that β is the dimension of the cycle space $\mathcal{C}$. Then

$$\begin{aligned} l(c_i) &= \Omega_Y(d^*, \mathcal{E}(c_i)) \\ &= \Omega_Y(\sum_{j=1}^{\beta} p_j \mathcal{E}(c_j), \mathcal{E}(c_i)) \\ &= \sum_{j=1}^{\beta} p_j \Omega_Y(\mathcal{E}(c_j), \mathcal{E}(c_i)) \\ &= \sum_{z \in N(c_i)} p_z. \end{aligned}$$

Let $sign(\gamma)$ of a closed curve γ on Y be the *signature* of γ, thus

$$sign(\gamma) = \begin{cases} 0 & \text{if } \gamma \text{ is 2-sided} \\ 1 & \text{if } \gamma \text{ is 1-sided.} \end{cases}$$

Since $p_i \in Z_2$, then we have

$$\begin{aligned} sign(d^*) &= \Omega_Y(d^*, d^*) \\ &= \Omega_Y(\sum_{i=1}^{\beta} p_i \mathcal{E}(c_i), \sum_{i=1}^{\beta} p_i \mathcal{E}(c_i)) \\ &= \sum_{i=1}^{\beta} \sum_{j=1}^{\beta} p_i p_j \Omega_Y(\mathcal{E}(c_i), \mathcal{E}(c_j)) \\ &= \sum_{i=1}^{\beta} p_i^2 \Omega_Y(\mathcal{E}(c_i), \mathcal{E}(c_i)) + 2 \sum_{1 \le i < j \le \beta} p_i p_j \Omega_Y(\mathcal{E}(c_i), \mathcal{E}(c_j)) \\ &= \sum_{i=1}^{\beta} p_i \Omega_Y(\mathcal{E}(c_i), \mathcal{E}(c_i)). \end{aligned}$$

We have used the fact $p_i^2 = p_i$ modulo 2.

When $G_{\mathcal{B}}^{\mathcal{E}}$ is among the sub-Types $4(b)$, $6(b)$ and $7(b)$, we have $N(w) = \{w\}$. Thus,

$$l(w) = p_w = 1. \tag{2.9}$$

Therefore,

$$sign(d^*) = p_w \Omega_Y(\mathcal{E}(w), \mathcal{E}(w)) = 1, \tag{2.10}$$

implying that d^* is 1-sided.

When $G_{\mathcal{B}}^{\mathcal{E}}$ is among the sub-Types $5(a)$, $8(a)$ and $9(a)$, $N(w) = \{v\} \subseteq V \setminus (S \cup L)$. Thus,

$$l(w) = p_v = 1, \tag{2.11}$$

$$l(v) = p_w = 0. \tag{2.12}$$

Therefore,

$$sign(d^*) = p_w \Omega_Y(\mathcal{E}(w), \mathcal{E}(w)) + p_v \Omega_Y(\mathcal{E}(v), \mathcal{E}(v)) = 0, \tag{2.13}$$

implying that d^* is 2-sided.

When $G_{\mathcal{B}}^{\mathcal{E}}$ is among the sub-Types $5(b)$, $8(b)$ and $9(b)$, $N(w) = \{v, w\}$ where $v \in V \setminus (S \cup L)$, and $w \in S \cup L$. Thus,

$$l(v) = p_w = 0, \tag{2.14}$$

$$l(w) = p_v + p_w = 1, \tag{2.15}$$

which implies $p_v = 1$. Therefore,

$$sign(d^*) = p_w \Omega_Y(\mathcal{E}(w), \mathcal{E}(w)) + p_v \Omega_Y(\mathcal{E}(v), \mathcal{E}(v)) = 0, \tag{2.16}$$

implying that d^* is 2-sided.

When $G^{\mathcal{E}}_{\mathcal{B}}$ is Type 2, $N(s) = \{s\}$ where $s \in S \setminus L$. Thus,

$$l(s) = p_s = 0. \tag{2.17}$$

Therefore,

$$sign(d^*) = p_s \Omega_Y(\mathcal{E}(s), \mathcal{E}(s)) = 0, \tag{2.18}$$

implying that d^* is 2-sided.

When $G^{\mathcal{E}}_{\mathcal{B}}$ is Type 3, $N(s) = \{v, s\}$ where $v \in V \setminus (S \cup L)$, and $s \in S \setminus L$. Thus,

$$l(v) = p_s = 0, \tag{2.19}$$

$$l(s) = p_v + p_s = 0, \tag{2.20}$$

which implies $p_v = 0$. Therefore,

$$sign(d^*) = p_s \Omega_Y(\mathcal{E}(s), \mathcal{E}(s)) + p_v \Omega_Y(\mathcal{E}(v), \mathcal{E}(v)) = 0, \tag{2.21}$$

implying that d^* is 2-sided.

When $G^{\mathcal{E}}_{\mathcal{B}}$ is Type 1, where $L = S \setminus L = \emptyset$, every cycle is 2-sided. Thus,

$$sign(d^*) = 0, \tag{2.22}$$

implying that d^* is 2-sided.

We summarize the above discussion as following.

Theorem 2.5.1 *Let $\mathcal{T}\colon G \to X$ be a generalized thrackable drawing of a graph G on some surface X. Then there exists a $2-$cell embedding $\mathcal{E}\colon G \to Y$ satisfying exactly one of the following properties:*

(1) $\hat{g}(Y) = rk(\omega_{\mathcal{T}}) + 1 \leq \hat{g}(X) + 1$, and $\mathcal{E}$ is even-faced and every Euler walk is 1-sided (sub-Type $4(b)$, $6(b)$ and $7(b)$);

(2) $\hat{g}(Y) = rk(\omega_{\mathcal{T}}) \leq \hat{g}(X)$, and $\mathcal{E}$ is even-faced and every Euler walk is 2-sided (Type $1, 2, 3, 5, 8, 9$);

(3) $\hat{g}(Y) = rk(\omega_{\mathcal{T}}) - 1 \leq \hat{g}(X) - 1$ and G has some odd faces on Y (sub-Type $4(a)$, $6(a)$ and $7(a)$).

Let $\mathcal{E} : G \to Y$ be a $2-$cell embedding obtained from a generalized thrackle $\mathcal{T}$ as above. Then $\mathcal{E}$ has *class (1)*, *class (2)*, or *class (3)* according to the corresponding case in Theorem 2.5.1.

2.6 The Converse of Theorem 2.5.1

Theorem 2.5.1 shows how to go from a generalized thrackle on X to a special embedding on a nearby surface Y. In this section, we present a construction which takes such an embedding on Y back to a one-thrackle on X.

Let $\mathcal{T} : G \to X$ be a generalized thrackle of G on some surface X, with $\omega_{\mathcal{T}}$ as the pull-back onto the cycle space $\mathcal{C}$. Let $\mathcal{E}^* : G^* \to Y$ be the dual embedding of the dual graph G^* on Y. In the Theorem 2.5.1, we showed that $\mathcal{E}^*$ belongs to one of three classes $(1), (2)$ and (3).

Let $d^* = f_1 e_1^* f_2 e_2^* \dots e_m^* f_1$ be an Euler walk of G^*, where f_i are the faces of G, and e_i^* is an edge of G^*, which crosses the boundary of a face. A *crossing* of d^* is a pair of faces (f_i, f_j), where $1 \le i < j \le m$, such that $f_i = f_j$, and $e_i, e_j, e_{i+1}, e_{j+1}$ are distinct edges of G appearing along the boundary of f_i in that order (Figure 2.6).

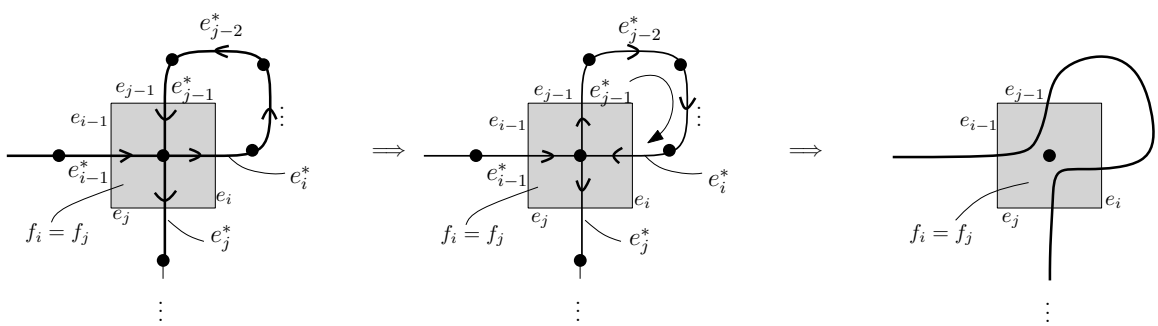


Figure 2.6: Reduce a crossing of d^*

Lemma 2.6.1 *If $\mathcal{E} : G \to Y$ is even faced, then G^* has an Euler walk with no crossings.*

Proof. Since $\mathcal{E}$ is even faced, G^* has an euler walk. Let d^* be an Euler walk of G^* with the fewest possible crossings. Suppose d^* has a crossing (f_i, f_j) with $i < j$. By traversing the portion of the walk $f_i e_i^* \dots f_{j-1} e_{j-1}^* f_j$ in reverse direction, we obtain an Euler walk with one fewer crossing, a contradiction. ∎

Theorem 2.6.1 *Let $\mathcal{T} : G \to X$ be a generalized thrackle. Let $\mathcal{B}$ be the canonical basis of the cycle space $\mathcal{C}(G)$ with respect to $(\omega_{\mathcal{T}}, l)$. There exists a one-thrackle drawing $\mathcal{T}' : G \to X'$ such that $\omega_{\mathcal{T}'} = \omega_{\mathcal{T}}$. Furthermore,*

(1) $\hat{g}(X') = \hat{g}(Y) - 1$ for class (1);

(2) $\hat{g}(X') = \hat{g}(Y)$ for class (2);

(3) $\hat{g}(X') = \hat{g}(Y) + 1$ for class (3).

Proof. Let $\mathcal{E} : G \to Y$ be the $2-$cell embedding of class (1), class (2), or class (3) obtained from $\mathcal{T}$ as in Theorem 2.5.1.

(1): Suppose $\mathcal{E}$ has class (1). Then G^* has an Euler walk, and every Euler walk is 1-sided. By Lemma 2.6.1, G^* has an Euler walk d^* with no crossings. Thus, we may represent d^* by a 1-sided simple closed curve $\delta : [0, 1] \to Y$ that is not self intersecting and where every edge of G crosses δ properly exactly once. Let I be a small neighbourhood around δ. Since δ is 1-sided, I is a Möbius band, as shown in Figure 2.7.

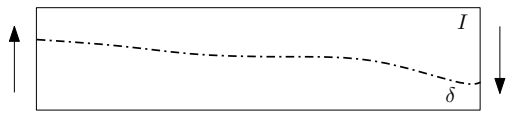


Figure 2.7: A Möbius band I around δ

Every edge e of G intersects I in a segment. We reroute the segments in the Möbius band as shown in Figure 2.8. Let γ_e be the new segment for the edge e. We may assume each segment γ_e is disjoint from the curve δ.

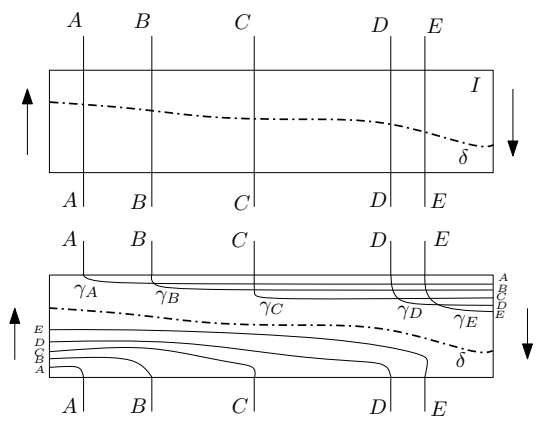


Figure 2.8: Rerouting the segments of the edges when δ is 1-sided.

Let $\mathcal{E}' : G \to Y$ be the resulting drawing of G. Recall that a one-thrackle of a simple graph G on some surface is a drawing where every two edges cross exactly once, so adjacent edges meet at two points, once at a crossing and once at an endpoint. Let e, f be two distinct edges of G. Then e crosses f exactly once in $\mathcal{E}'$ regardless of whether or not e and f are adjacent in G. Therefore $\mathcal{E}'$ is a one-thrackle.

Let $Y - \delta$ be the bordered surface obtained by cutting Y along δ. The boundary, $bd(Y - \delta)$, is a single simple closed curve. By gluing a disc to $bd(Y - \delta)$, we obtain a new surface X' without border. To recover Y from X', we remove the disc and identify opposite points on the boundary of the disc. So Y is the direct sum of X' and $\mathbb{N}_1$. Thus, $\hat{g}(Y) = \hat{g}(X') + 1$.

Thus, there is a one-thrackle $\mathcal{T}' : G \to X'$ where $Y = X' + \mathbb{N}_1$. Let c, c' be two cycles in the cycle space $\mathcal{C}$. Since every two edges of G meet at exactly one proper crossing on X', we have,

$$\omega_{\mathcal{T}'}(c, c') = \omega_{\mathcal{E}}(c, c') + l(c)l(c'). \tag{2.23}$$

By Theorem 2.4.1, we have

$$\omega_{\mathcal{T}'}(c, c') = \omega_{\mathcal{E}}(c, c') + l(c)l(c') = \omega_{\mathcal{T}}(c, c') + l(c)l(c') + l(c)l(c') = \omega_{\mathcal{T}}(c, c'). \tag{2.24}$$

Hence, $\omega_{\mathcal{T}'} = \omega_{\mathcal{T}}$.

(2): Suppose $\mathcal{E}$ has class (2). Then G^* has an Euler walk, and every Euler walk is 2-sided. By Lemma 2.6.1, G^* has an Euler walk d^* with no crossings. Thus, we may represent d^* by a 2-sided simple closed curve $\delta : [0, 1] \to Y$ that is not self intersecting and where every edge of G crosses δ properly exactly once.

Since δ is 2-sided, we can stretch Y to deform some small neighbourhood $\delta \subset W \subseteq Y$ to be a cylinder. The edges of G intersect W in disjoint segments. Cut Y along δ to obtain a bordered surface with exactly two boundary components with neighbourhoods C_1, C_2. We redraw the segments within C_1 as shown in Figure 2.9.

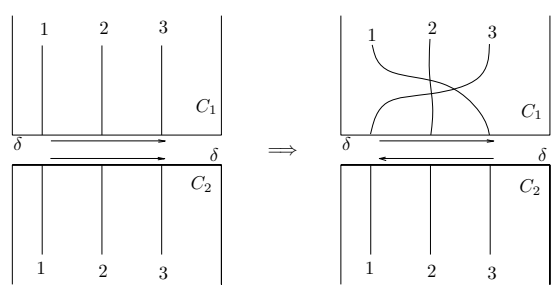


Figure 2.9: Redraw the segments in C_1 when δ is 2-sided

Then re-identify the two boundary components with reverse orientation, so that corresponding segments align. Thus we obtain a drawing $\mathcal{T}' : G \to X'$. Since X' is obtained from Y by replacing a handle by a Klein handle, or vice versa, we have that $\hat{g}(X') = \hat{g}(Y)$.

Let e, f be two edges of G. Then e crosses f exactly once regardless of whether or not e and f are adjacent in G. Thus, we obtain a one-thrackle $\mathcal{T}' : G \to X'$.

Let c, c' be two cycles in the cycle space $\mathcal{C}$. Since every two distinct edges of G meet at exactly one proper crossing on X', c, c' cross $l(c)l(c')$ more times on X' than on $Y \pmod 2$. Thus,

$$\omega_{\mathcal{T}'}(c, c') = \omega_{\mathcal{E}}(c, c') + l(c)l(c'). \tag{2.25}$$

By Theorem 2.4.1, we have

$$\omega_{\mathcal{T}'}(c, c') = \omega_{\mathcal{E}}(c, c') + l(c)l(c') = \omega_{\mathcal{T}}(c, c') + l(c)l(c') + l(c)l(c') = \omega_{\mathcal{T}}(c, c'). \tag{2.26}$$

Hence, $\omega_{\mathcal{T}'} = \omega_{\mathcal{T}}$.

(3): Suppose $\mathcal{E}$ has class (3). Then $\mathcal{E}$ is not even faced. Let G^* be the surface dual. Let $\mathcal{E}^* : G^* \to Y$ be a $2-$cell embedding such that

- for every face f of $\mathcal{E}$, the corresponding vertex $v^* \in V(G^*)$ satisfies

  $$\mathcal{E}^*(v^*) \in int(f),$$

  where $int(f)$ is the interior of f;

- for every $e \in E(G)$ the corresponding edge $e^* \in E(G^*)$ satisfies,

  $$|\mathcal{E}(e) \cap \mathcal{E}^*(e^*)| = 1.$$

Let $T \subseteq E(G^*)$ be a spanning tree of G^*. Every edge $e^* \in E(G^*)$ is represented by a curve

$$\gamma_{e^*} : [0, 1] \to Y.$$

For all $e^* \in E(G^*) - T$ and $\varepsilon \in (0, 1)$, replace γ_{e^*} with the truncated curve

$$\gamma'_{e^*} : [0, 1 - \varepsilon] \to Y$$

to obtain a drawing $\mathcal{E}'$, where $\mathcal{E}'(G^*)$ is a tree T', some of whose leaves are "incomplete edges", and which is contractible (Figure 2.10). Let A be a small neighbourhood of $\mathcal{E}^*(T')$. Let $C = bd(A)$,

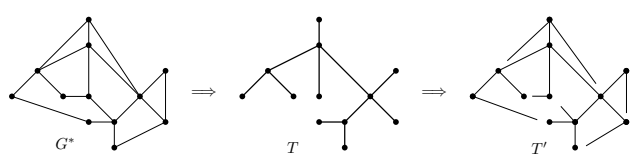


Figure 2.10: Surface dual and $T' = \mathcal{E}'(G^*)$

thus C is a contractible simple closed curve on Y.

We may assume that C is a unit circle in the plane and that A is a closed disc centred at $\mathcal{O}$. Let e be an edge of G. Then e intersects A in a single segment s_e. Let A_e, B_e be the end points of s_e on C. Let D be the boundary of a small disc in A of radius r_0 centered at $\mathcal{O}$. We may assume D is disjoint from every segment s_e. We replace the interior of D with a crosscap, so that opposite points of D are identified.

We deform s_e to s'_e in such a way that s'_e consists of a radial line segment $A_eA'_e$ at angle θ_0, where $A'_e \in D$, together with a linear spiral of the form

$$r - r_0 = k(\theta - (\theta_0 + \pi)),$$

with $0 \leq |\theta - (\theta_0 + \pi)| \leq \alpha$, where $\alpha = \angle A_e\mathcal{O}B_e$. This linear spiral connects B_e to the point opposite to A'_e on D (Figure 2.11).

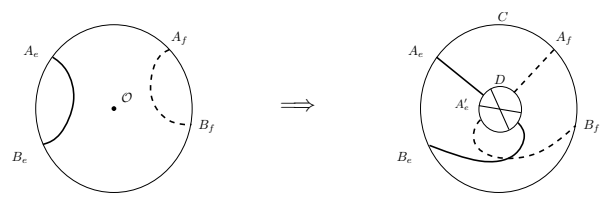


Figure 2.11: Reroute the segments of the edges when $\mathcal{E}$ is not even faced

Now for any two edges $e, f \in E(G)$, $s'_e \cap s'_f$ consists of exactly one proper crossing. Thus we have obtained a one-thrackle

$$\mathcal{T}' : G \to X'$$

where $X' = Y + \mathbb{N}_1$. ∎

We remark that, in general, $\hat{g}(X') \leq \hat{g}(X)$. By adding crosscaps, or handles, we can extend the surface X' to X, such that G can be drawn as a one-thrackle on X. Then by flipping a small neighbourhood around each vertex of G (Figure 2.12), we get back to a generalized thrackle on the surface X.

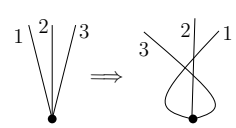


Figure 2.12: Flip a small neighbourhood around each vertex of G

Chapter 3

Proof of Theorem 1.5.5

The purpose of this chapter is to prove the following.

Theorem 1.5.5 *Let G be a thrackable graph with n vertices and e edges. Then $e \leq 1.4n$.*

3.1 Definition of Dumbells

Defination 3.1.1 *([8]) Given three integers $c', c'' > 2$, $l \geq -\min(|c'|, |c''|)$, the dumbell $DB(c', c'', l)$ is a simple graph consisting of two distinct cycles of length c' and c'' such that*

- $l = 0$*, the two cycles share a vertex.*
- $l > 0$*, the two cycles are connected by a path of length l.*
- $-\min(c', c'') < l < 0$*, the two cycles share a path of length $-l$.*

For Example:

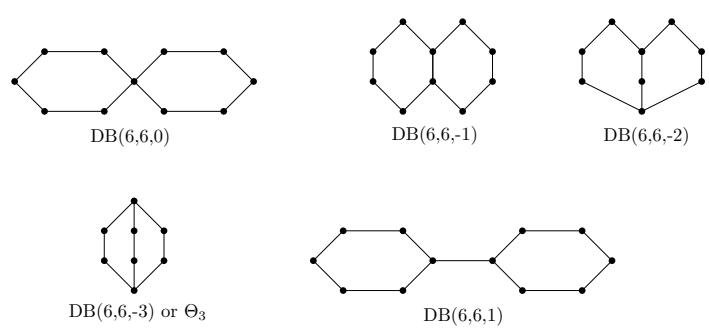


Figure 3.1: Examples of dumbells

The Conway's Thrackle Conjecture has been verified for every graph of order at most 11 (see [6]). In particular, we have the following.

Lemma 3.1.1 *[8] $DB(6,6,l)$ for $l \in \{-3,-2,-1,0\}$ are not thrackable.*

3.2 Discharging Rules

By Corollary 1.2.1, every bipartite thrackable graph is planar. Let G be a plane embedding of a bipartite thrackable graph with n vertices and m edges. Recall that the face space $\mathcal{F}$ is a subgroup of the cycle space generated by the facial boundaries, where the facial boundary of each face is a closed walk bounds the face. Let F be a face of G on the plane, and $bd(F)$ be its facial boundary. The size of $bd(F)$ is called the degree of F, denoted by $d(F)$. A k-face is a face of degree k, and a k^+-face is a face of degree at least k.

Lemma 3.2.1 *Suppose G is not isomorphic to C_6. Then G has no two adjacent distinct 6-faces.*

Proof. Let H be a subgraph of G induced by two adjacent distinct 6-faces. Then $n(H) \leq 10$, and $n(H) < e(H)$. By a conclusion in [6] that the Conway's Thrackle Conjecture is verified for every graph of order at most 11, H is not thrackable unless H has at most one circuit. But this hypothesis holds only if $H \cong C_6$, a contradiction. ∎

Since the Conway's Thrackle Conjecture is verified for every graph of order at most 11, every face of degree at most 10 is bounded by a circuit.

An edge $e \in E(G)$ is a *bad edge* if and only if it is incident with a 6-face. The following proposition immediately comes from Lemma 3.2.1.

Proposition 3.2.1 *Let $e \in E(G)$ be a bad edge. Then e must be incident with an 8^+-face and a 6-face.*

Lemma 3.2.2 *Each 8-face is incident with at most six bad edges. Furthermore, if an 8-face is incident with six bad edges, then it must be incident with two distinct 6-faces as shown in Figure 3.2.*

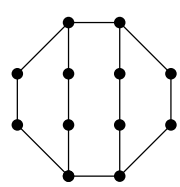


Figure 3.2: An 8-face incident with six bad edges

Proof. Let F be an 8-face of G , and let P be the subgraph induced by a set of edges in $bd(F)$ each of which is incident with a 6-face.

Suppose $|E(P)| \geq 7$. Since each edge in P is incident with a 6-face, by Lemma 3.1.1 and Lemma 3.2.1, consecutive edges in P must be incident with the same 6-face. Thus, all edges in P must be incident with exactly one 6-face. Since $|E(P)| \geq 7$, this is impossible. Therefore, $|E(P)| \leq 6$.

Suppose $|E(P)| = 6$. Then the 8-face F must be adjacent with at least two distinct 6-faces of G. Let F' be one of the 6-faces adjacent with F. Let $P_{F'}$ be a subgraph induced by a set of edges in $bd(F)$, where each of them is incident with the 6-face F'. Thus, $P_{F'}$ is a union of finite vertex disjoint paths $p_1, p_2, \ldots, p_k$, where $E(p_j) \subseteq E(P_{F'})$. Let v be an endvertex of a path p_j. Then v is incident with two distinct edges e_1, e_2, where $e_1 \in E(p_j)$ and $e_2 \in E(bd(F)) \setminus E(P_{F'})$. By Lemma 3.1.1 and Lemma 3.2.1, e_2 is not incident with any 6-face of G.

If $k \geq 3$, then F must be incident with at least three different edges each of which is not incident with any 6-face of G, implying that $|E(P)| \leq 5$, a contradiction.

Thus, $P_{F'}$ is a single path. Furthermore, there are only three possibilities when $|E(P)| = 6$, as shown below (Figure 3.3). Case 2 contains a $DB(6, 6, -2)$ (Figure 3.4), contradicting Lemma 3.1.1,

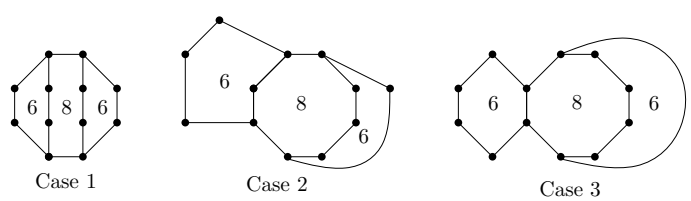


Figure 3.3: Three possibilities when $|E(P)| = 6$

and Case 3 contains a 4-cycle (Figure 3.5), contradicting Lemma 1.2.1.

Hence, when $|E(P)| = 6$, F must be adjacent with two distinct 6-faces F_1, F_2, which is the Case 1 in Figure 3.3. ∎

Recall that a bad edge is an edge incident with a 6-face. By Proposition 3.2.1, our discharging rules are as follows. Let $w(F) = d(F)$ be the original weight of each face of G, where $d(F)$ is the

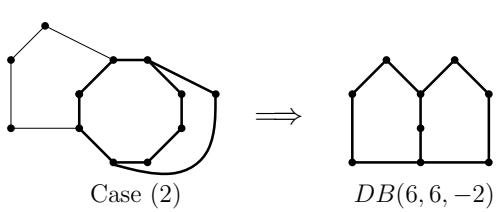


Figure 3.4: Case (2) contains a $DB(6, 6, -2)$

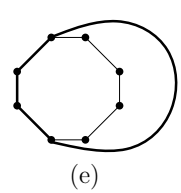


Figure 3.5: A $4-$cycle in Case (3)

degree of F.

Discharging Rule: Suppose G is not isomorphic to C_6. Let F be a face of G. Let e be a bad edge of G, thus e is incident with an 8^+-face and a 6-face. Transfer $\frac{1}{6}$ across e from the 8^+-face to the 6-face to obtain a new face weighting w^*.

Let r be the number of bad edges in G incident with F. When F is an 8^+-face, the new weight of F is $w^*(F) = w(F) - \frac{r}{6}$. When F is a 6-face, $w^*(F) = 6 + \frac{6}{6} = 7$.

If F is an 8-face, then by Lemma 3.2.2, $w^*(F) \geq 8 - 1 = 7$. If F is a 10^+-face of G, $w^*(F) \geq d(F) - \frac{d(F)}{6} \geq \frac{50}{6} > 7$. Therefore, after applying the discharging rules, $w^* \geq 7$ for each face of G.

Now we are quite close to finishing the proof of Theorem 1.5.5. However, we need to be a little bit more careful. Notice that during the discussion above, we lost the generality by only discussing bipartite graphs. The next section will correct this.

3.2.1 Non-bipartite Thrackable Graph on Projective Plane

Recall that a thrackle of a graph is a drawing on the plane such that for every two distinct edges either

- share an endpoint, and then they have no other point in common; or
- do not share an endpoint, in which case they meet exactly once at a proper crossing.

And a graph drawing is a generalized thrackle if any two distinct edges meet an odd number of times, either at a common end point, or at a proper crossing. Thus, a thrackle is also a generalized thrackle.

Let G be a non-bipartite graph. Let $\mathcal{C}$ be its cycle space. Let $\mathcal{T} : G \to \mathbb{S}_0$ be a thrackle drawing of G on the sphere $\mathbb{S}_0$. Let $\hat{g}$ be the Euler genus of $\mathbb{S}_0$, thus $\hat{g}(\mathbb{S}_0) = 0$. Let $\omega_{\mathcal{T}} : \mathcal{C} \times \mathcal{C} \to Z_2$ be the pull-back bilinear form onto $\mathcal{C}$. Let $l : \mathcal{C} \times \mathcal{C} \to Z_2$ be a linear form where for each cycle $c \in \mathcal{C}$, $l(c)$ is its length mod 2. Let $\mathcal{B}$ be a canonical basis with respect to $(\omega_{\mathcal{T}}, l)$. By Theorem 2.5.1, there is a 2-cell embedding $\mathcal{E} : G \to Y$, where $\hat{g}(Y) \leq \hat{g}(X) + 1$. Since G is non-bipartite, based on the construction of the surface Y, $\mathcal{E}$ is a 2-cell parity embedding, which is also even-faced. Thus Y is a nonorientable surface. Hence, $\hat{g}(Y) = 1$, where Y is the projective plane.

Lemma 3.2.3 *Let G be a non-bipartite thrackable graph on the plane. Then there exits an even-faced 2-cell embedding $\mathcal{E} : G \to \mathbb{N}_1$ of G on the projective plane.*

Recall that every face of degree at most 10 is bounded by a circuit, since the Conway's Thrackle Conjecture is verified for every graph of order at most 11. Since $\mathcal{E}$ is even-faced and 2-cell, and G contains no 4-cycle, the discharging rules defined in Section 3.2 can be applied to all non-bipartite thrackable graphs after they have been embedded on the projective plane.

3.2.2 A New Upper Bound on $|E(G)|$

Now we are ready to complete the proof of Theorem 1.5.5.

Let G be a graph with n vertices and e edges. Suppose $G \cong C_6$. Apparently, $e \leq n$ without applying the discharging rules, which satisfies Theorem 1.5.5. Thus, we may assume that G is not isomorphic to a C_6.

Let $v \in V(G)$ be a vertex of degree 1. We apply induction to the graph $G - v$, and observe that

$$\frac{e(G-v)}{n(G-v)} = \frac{e-1}{n-1}.$$

Thus

$$\begin{aligned} \tfrac{e(G)}{n(G)} &\leq \max(1, \tfrac{e-1}{n-1}) \\ &\leq \max(1, \tfrac{e(G-v)}{n(G-v)}) \\ &\leq 1.4. \end{aligned}$$

So we may also assume that each vertex of G has degree at least 2.

Case A: Suppose G is a plane embedding of a thrackable bipartite graph with n vertices, e edges, and f faces.

Let F be a face of G. By Theorem 1.2.1, the degree of F is at least 6. Let w^* be as defined in Section 3.2.

According to the discharging rules in Section 3.2, $w^*(F) \geq 7$. By Handshaking Lemma ([2]),

$$2e = \sum_F d(F) = \sum_F w(F) = \sum_F w^*(F) \geq 7f, \tag{3.1}$$

so $f \leq \frac{2}{7}e$. Applying Euler's Formula for the plane ([2]), we obtain

$$2 = n + f - e \leq n + \frac{2}{7}e - e = n - \frac{5}{7}e \Rightarrow e \leq \frac{7}{5}(n-2) \Rightarrow e \leq 1.4(n-2).$$

Case B: By Lemma 3.2.3, let G be a even-faced projective plane embedding of a thrackable non-bipartite graph with n vertices, e edges, and f faces. Let F be a face of G.

Thus, $w^*(F) \geq 7$, and by Handshaking Lemma, $f \leq \frac{2}{7}e$. By Euler's Formula for projective plane,

$$1 = n + f - e \leq n + \frac{2}{7}e - e = n - \frac{5}{7}e \Rightarrow e \leq \frac{7}{5}(n-1) \Rightarrow e \leq 1.4(n-1).$$

∎

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