Math 252, Vector Calculus

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Final Examination

Tuesday, April 6th, 1999

Marks

1. An Altarian puffball is a new species of mushroom recently discovered by the scientific team aboard the USS Enterprise. Its stem has a horizontal cross section which looks like a serrated football, whereas the cap looks like a lopsided cook's hat. By a strange freak of nature, the horizontal cross-section of the stem always has an area of exactly 4π m^2. (The cap, on the other hand, varies in diameter from 10 to 20 meters.) The Altarian puffball is oriented along the z-axis with the base of its stem in the xy-plane. A vector field is given by:

$$\vec{\mathbf{F}} = (y^2 + z^2)^{3/2}\hat{\mathbf{i}} + e^{-(z-x)^2}\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Evaluate $\iint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$ where S is the surface of the cap and stem of the Altarian puffball, excluding the base of the stem. Use the following two methods, and make any assumptions which you deem to be reasonable. Your two answers should agree.

(a) Use the Divergence Theorem. [5]

(b) Noting $\vec{\mathbf{F}}$ is solenoidal and has a vector potential ($\vec{\mathbf{G}}$), use a double application of Stokes' theorem. [5]

2. A terrestrial mushroom (M) has a cylindrical stem given by $x^2 + y^2 = 1$, $0 \le z \le 1$, and a spherical cap given by $x^2 + y^2 + (z - (1 + \sqrt{3}))^2 = 4$, $1 < z \le (3 + \sqrt{3})$. A vector field is given by:

$$\vec{\mathbf{F}} = (y - z^2)\hat{\mathbf{i}} - (x + e^{-z^2} - 1)\hat{\mathbf{j}} + \sin(xy^3)\hat{\mathbf{k}}$$

Evaluate $\iint_M (\vec{\nabla} \times \vec{\mathbf{F}}) \bullet d\vec{\mathbf{S}}$ where M is the surface of the cap and stem of the terrestrial mushroom, excluding the base of the stem. [10]

3. A cylinder given by $x^2 + y^2 = 2$ intersects with a sphere given by $x^2 + y^2 + z^2 = 4$, and in so doing creates an enclosed volume V. Let S be the surface which encloses the volume V. (Its inner surface will be cylindrical and its outer surface will be spherical.) A vector field is given by:

$$\vec{\mathbf{F}} = (x + \cosh(yz))\hat{\mathbf{i}} + (y - \sinh(xz))\hat{\mathbf{j}} + (z + \cos x \sin y)\hat{\mathbf{k}}$$

Evaluate $\iint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}}$. [10]

4. The Flux Transport Theorem states:

$$\frac{d\Phi}{dt} = \iint_{S_t} \left(\frac{\partial \vec{\mathbf{F}}}{\partial t} + (\vec{\nabla} \bullet \vec{\mathbf{F}})\vec{\mathbf{v}} \right) \bullet d\vec{\mathbf{S}} + \oint_{C_t} (\vec{\mathbf{F}} \times \vec{\mathbf{v}}) \bullet d\vec{\mathbf{R}}$$

where

$$\Phi(t) = \iint_{S_t} \vec{\mathbf{F}}(\vec{\mathbf{R}}, t) \bullet d\vec{\mathbf{S}}$$

Verify the Flux Transport Theorem for the case that S_t is a unit-square which starts at $0 \le x \le 1$, $0 \le z \le 1$, and is rotated about the z-axis in the counter-clockwise direction (toward the y-axis) with constant angular velocity ω, and $\vec{\mathbf{F}}$ is given by:

$$\vec{\mathbf{F}} = -\sin(\omega t)\hat{\mathbf{i}} + \cos(\omega t)\hat{\mathbf{j}}.$$ [12]

5. One of Maxwell's equations states:

$$\vec{\nabla} \bullet \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_o}$$

where $\vec{\mathbf{E}}$ is the electric field, ρ is the charge density, and ε_o is the electrical permitivity of space. Integrate this and use the Divergence Theorem to obtain Gauss's Law. [5]

6. Use:

$$\vec{\nabla}^2\left(\frac{1}{\left|\vec{\mathbf{R}} - \vec{\mathbf{R}}'\right|}\right) = -4\pi\delta\left(\vec{\mathbf{R}} - \vec{\mathbf{R}}'\right)$$

to show that:

$$\phi(\vec{\mathbf{R}}) = \frac{-1}{4\pi}\iiint_{D'} \frac{\rho(\vec{\mathbf{R}}')}{\left|\vec{\mathbf{R}} - \vec{\mathbf{R}}'\right|} dV'$$

is a solution of Poisson's equation. [5]

7. If $\vec{\mathbf{F}}$ is continuous and smooth, we can write:

$$\vec{\nabla} \bullet \vec{\mathbf{F}} = \lim_{V \to 0} \frac{1}{V} \iiint_V \vec{\nabla} \bullet \vec{\mathbf{F}} dV$$

Use the Divergence Theorem on the right-hand side of this expression and interpret, in words, what this means. [6]

8. If $\vec{\mathbf{F}}$ is continuous and smooth, we can write:

$$(\vec{\nabla} \times \vec{\mathbf{F}}) \bullet \hat{\mathbf{n}} = \lim_{A \to 0} \frac{1}{A} \iint_A (\vec{\nabla} \times \vec{\mathbf{F}}) \bullet \hat{\mathbf{n}} dS$$

Use Stokes' Theorem on the right-hand side of this expression and interpret, in words what this means. [6]

9. Let

$$\vec{\mathbf{F}} = \frac{2x}{x^2 + y^2}\hat{\mathbf{i}} + \frac{2y}{x^2 + y^2}\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}$$

Is this vector field conservative? If so, what is its scalar potential? [6]

10. In orthogonal curvilinear coordinates, the unit vectors are given by:

$$\hat{\mathbf{e}}_i = \frac{\partial \vec{\mathbf{R}}}{\partial u_i} \Bigg/ \left|\frac{\partial \vec{\mathbf{R}}}{\partial u_i}\right| = \frac{\vec{\nabla} u_i}{\left|\vec{\nabla} u_i\right|}$$

and the scale factors are given by:

$$h_i = \left|\frac{\partial \vec{\mathbf{R}}}{\partial u_i}\right| = \frac{1}{\left|\vec{\nabla} u_i\right|}$$

Let:

$$x = u_1^2 - u_2^2$$
$$y = 2u_1u_2$$
$$z = u_3$$

Show that this transformation is a right-handed orthogonal curvilinear coordinate system, and evaluate the scale factors. [10]

11. What is the surface element $d\vec{\mathbf{S}}$, in terms of $dudv$, for a surface given by:

$$x = u^2$$
$$y = \sqrt{2}uv$$
$$z = v^2$$

[7]

12. True or False? [1 pt for each correct, -1 for each wrong, max 11]

(a) A scale factor h_i is one over the rate at which arc length increases on the i^{th} coordinate curve, with respect to coordinate u_i .

(b) The region contained between the outer and inner surfaces of a hollow sphere is simply-connected.

(c) If a vector field $\vec{\mathbf{F}}$ is conservative in a domain D, then $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{R}}$ along a regular curve C in D depends only on the endpoints of the curve.

(d) If $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ throughout a domain D, then $\mathbf{F}$ is conservative.

(e) $\vec{\mathbf{F}} = \dfrac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2 + y^2}$ is conservative.

(f) If the vector potential $\vec{\mathbf{G}}$ of a field $\vec{\mathbf{F}}$ has straight flow lines, then the flow lines of $\vec{\mathbf{F}}$ must curl around those of $\vec{\mathbf{G}}$.

(h) One can calculate the volume of a room by computing the flux of the vector $\vec{\mathbf{R}}$ through the walls, ceiling, and floor.

(i) The volume element in spherical coordinates is given by $r^2 \cos\phi d\phi d\theta dr$.

(j) If $\vec{\nabla}^2\phi > 0$, then ϕ is less than its average value in the immediate vicinity.

(k) Any vector field $\vec{\mathbf{F}}$, which is continuously differentiable in a star-shaped domain D can be decomposed into a gradient and a curl.

(l) Green's Theorem is just a special case of Stokes' Theorem.

Total mark out of [98]