MATHEMATICS 151

Assignment 1, due Wednesday 30 June 1999

Review & Practice Exercises 1 (pg. 14):

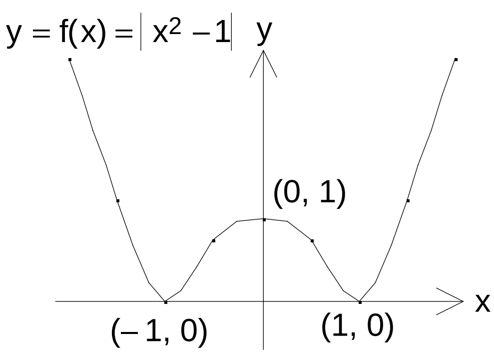
10. $g(x) = \frac{2}{3x-5}$ has domain $(-\infty, 5/3) \cup (5/3, +\infty)$ because we need to have $3x - 5 \neq 0$, and range $(-\infty, 0) \cup (0, +\infty)$ since $y = \frac{2}{3x-5}$ has solution $x = \frac{5y+2}{3y}$ unless $y = 0$.

12. $h(x) = \sqrt[4]{7-3x}$ has domain $(-\infty, 7/3]$ because we need to have $7 - 3x \geq 0$, and range $[0, +\infty)$ since $y = \sqrt[4]{7-3x}$ has solution $x = \frac{7-y^4}{3}$ for $y \geq 0$.

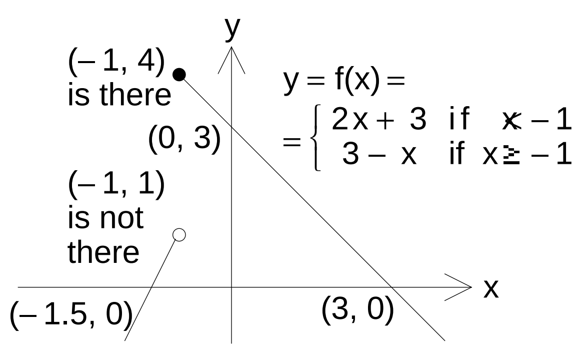
20. $\phi(x) = \sqrt{\frac{x^2-2x}{x-1}} = \sqrt{\frac{x(x-2)}{x-1}}$ has domain $[0, 1) \cup [2, +\infty)$ because we need to have $\frac{x(x-2)}{x-1} \geq 0$.

38. $f(x) = |x^2 - 1|$ has domain $(-\infty, +\infty)$. See graph below.

44. $f(x) = \begin{cases} 2x+3 & \text{if } x < -1 \\ 3-x & \text{if } x \geq -1 \end{cases}$ has domain $(-\infty, +\infty)$. See graph below.



For Exercise 38



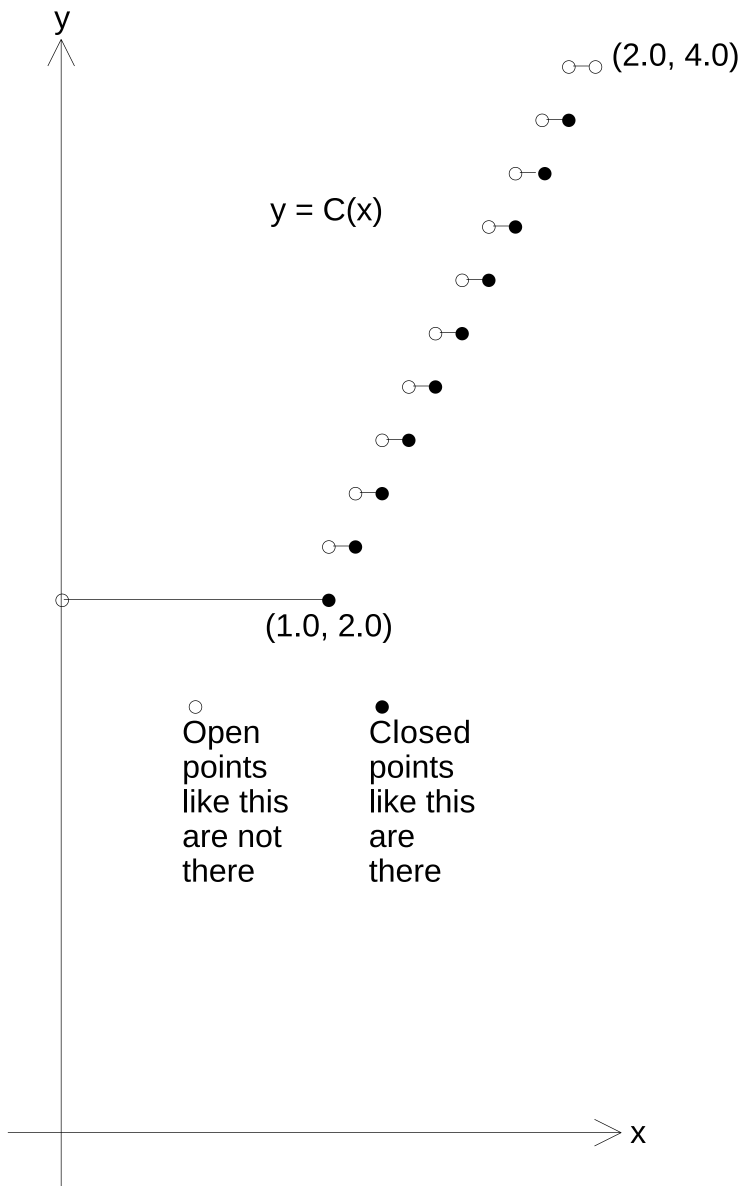
For Exercise 44

52. The curve is not the graph of a function. For any $x \in (-3, 3)$ there are **two** values of y with (x, y) on the graph.

62. If x is the length (in metres) of one side, the adjacent side has length $\frac{16}{x}$ and the perimeter is $P(x) = 2\left(x + \frac{16}{x}\right)$. The domain of $P(x)$ is $(0, +\infty)$.

68. See graph to the right.

$$C(X) = \begin{cases} 2.0 & \text{if } 0 < x \le 1.0 \\ 2.2 & \text{if } 1.0 < x \le 1.1 \\ 2.4 & \text{if } 1.1 < x \le 1.2 \\ 2.6 & \text{if } 1.2 < x \le 1.3 \\ 2.8 & \text{if } 1.3 < x \le 1.4 \\ 3.0 & \text{if } 1.4 < x \le 1.5 \\ 3.2 & \text{if } 1.5 < x \le 1.6 \\ 3.4 & \text{if } 1.6 < x \le 1.7 \\ 3.6 & \text{if } 1.7 < x \le 1.8 \\ 3.8 & \text{if } 1.8 < x \le 1.9 \\ 4.0 & \text{if } 1.9 < x < 2.0 \end{cases}$$



For Exercise 68

90. $f(x) = \frac{1}{x-1}$ has domain $(-\infty, 1) \cup (1, +\infty)$.
$g(x) = \frac{x-1}{x+1}$ has domain $(-\infty, -1) \cup (-1, +\infty)$.
Note $f(x) = -1$ just when $x = 0$,
$f(x) = 1$ just when $x = 2$,
$g(x) = -1$ just when $x = 0$,
and $g(x) = 1$ never.

(a) $(f \circ g)(x) = f(g(x)) = \dfrac{1}{\frac{x-1}{x+1} - 1} = -\dfrac{x+1}{2}$ has domain $(-\infty, -1) \cup (-1, +\infty)$ to keep $g(x)$ defined.

(b) $(g \circ f)(x) = g(f(x)) = \dfrac{\frac{1}{x-1} - 1}{\frac{1}{x-1} + 1} = \dfrac{2-x}{x}$ has domain $(-\infty, 0) \cup (0, 1) \cup (1, +\infty)$ to keep $f(x)$ defined and different from -1 (where $g(f(x))$ would be undefined).

(c) $(f \circ f)(x) = f(f(x)) = \dfrac{1}{\frac{1}{x-1} - 1} = \dfrac{x-1}{2-x}$ has domain $(-\infty, 1) \cup (1, 2) \cup (2, +\infty)$ to keep $f(x)$ defined and different from 1 (where $f(f(x))$ would be undefined).

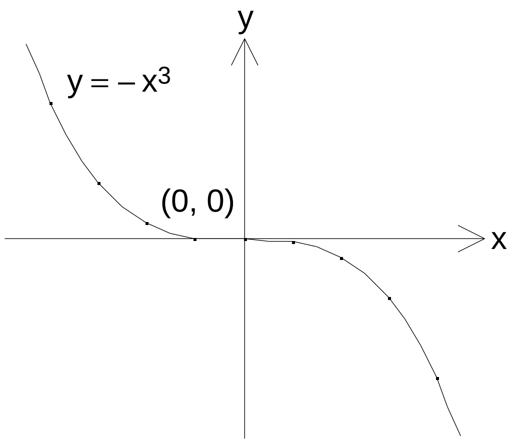
(d) $(g \circ g)(x) = g(g(x)) = \dfrac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = -\dfrac{1}{x}$ has domain $(-\infty, -1) \cup (-1, 0) \cup (0, +\infty)$ to keep $g(x)$ defined and different from -1 (where $g(g(x))$ would be undefined).

100. $F(x) = \sqrt{x} + 1 = (f \circ g)(x)$ where $g(x) = \sqrt{x}$ and $f(x) = x + 1$.
There are other ways to do it.

108. $f(x) = x + 4$; $h(x) = 4x - 1$.
If $g(x) = 4x - 17$ then $(g \circ f)(x) = g(f(x)) = 4(x + 4) - 17 = 4x - 1 = h(x)$.

Review & Practice Exercises 2 (pg. 25):

6.



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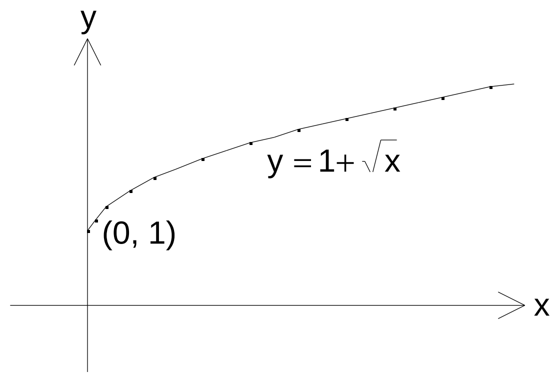
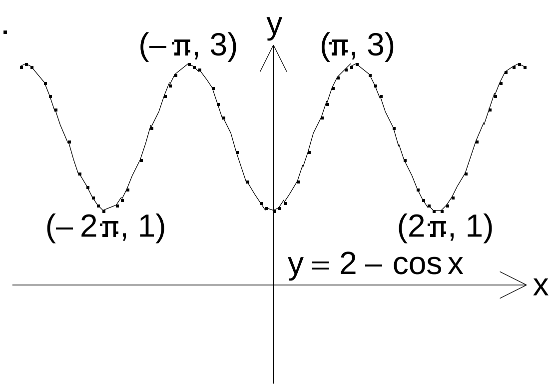


10.



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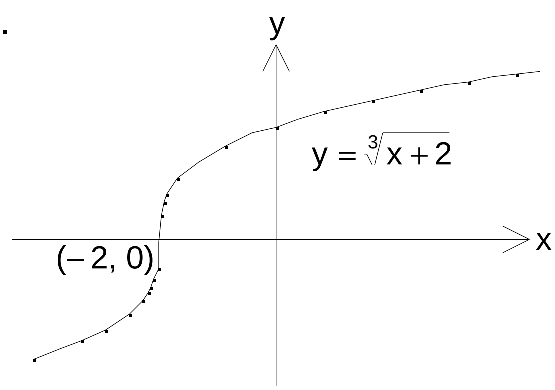

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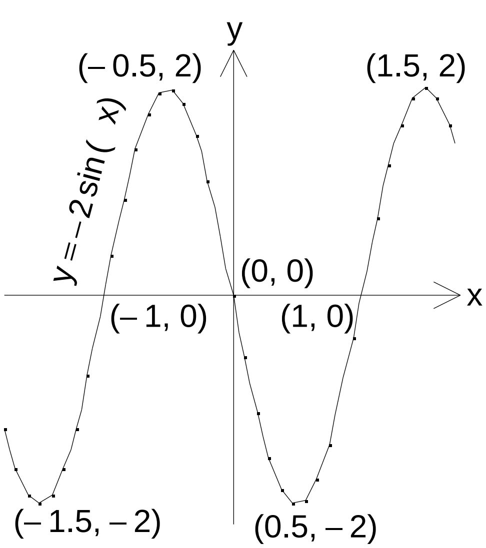

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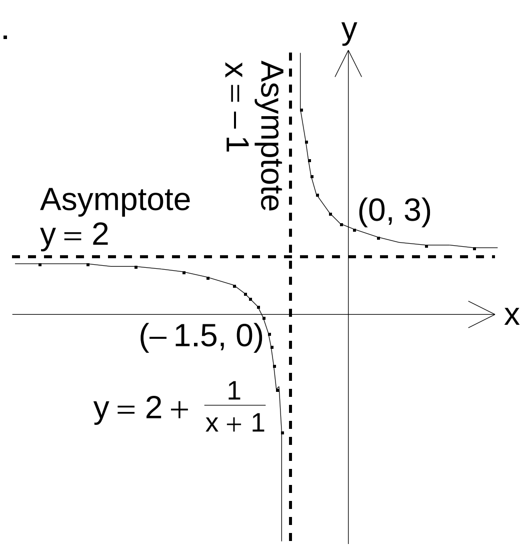

Review & Practice Exercises 4 (pg. 38):

2. The solution to the inequality $|x-1|-|x-3| \geq 5$ is the empty set; in other words, no real values of x make it true. Here are three different arguments showing why this is the case. Perhaps you can come up with other ways to reason it out.

My first argument is an algebraic one. Transpose the term $|x-3|$ to the right hand side of the inequality and solve $|x-1| \geq |x-3|+5$ instead.
To solve that, square both sides of it and obtain $|x-1|^2 \geq |x-3|^2 + 10|x-3| + 25.$
(This is legitimate since both $|x-1|$ and $|x-3|+5$ are nonnegative, and the squaring process preserves order, when applied to nonnegative numbers. This needn't work if at least one of the numbers is negative. For instance, $-5 < 3$ but $(-5)^2 > 3^2$.)
So we now have to solve $(x-1)^2 \geq (x-3)^2 + 10|x-3| + 25$ (since the absolute values can be omitted when numbers get squared). This simplifies to $4x - 33 \geq 10|x-3|$ and to solve that we once again square both sides (dangerous this time since perhaps $4x - 33 < 0$). We obtain $16x^2 - 264x + 1089 \geq 100(x^2 - 6x + 9)$, which simplifies to $84x^2 - 336x - 189 \leq 0$, or $21(4x^2 - 16x - 9) \leq 0$, or $21(2x+1)(2x-9) \leq 0$, and the solution to that inequality is $-0.5 \leq x \leq 4.5$. Now what this argument really shows is that **if** $|x-1|-|x-3| \geq 5$ **then** $-0.5 \leq x \leq 4.5$; in other words, numbers outside that interval cannot possibly be solutions. It does **not** show that any of the numbers inside that interval are solutions, since the step where we squared $4x - 33$ may introduce incorrect "solutions". And that indeed is what happens! The numbers x in the interval $-0.5 \leq x \leq 4.5$ all make $4x - 33$ negative since the largest it gets there is $4 \cdot 4.5 - 33 = -15$, while $10|x-3|$ is of course never negative. So there aren't any solutions.

My second argument is a case-by-case one.

(i) If $-\infty < x \leq 1$ then $|x-1| = 1-x$, $|x-3| = 3-x$, and the inequality becomes $(1-x) - (3-x) \geq 5$, or $-2 \geq 5$, so we get no solutions this way.

(ii) If $1 \leq x \leq 3$ then $|x-1| = x-1$, $|x-3| = 3-x$, and the inequality becomes $(x-1) - (3-x) \geq 5$, or $2x - 4 \geq 5$, which has solution $x \geq 4.5$. However there are no numbers x in the interval $1 \leq x \leq 3$ with $x \geq 4.5$ so we do not get any solutions this way either.

(iii) If $3 \leq x < +\infty$ then $|x-1| = x-1$, $|x-3| = x-3$, and the inequality becomes $(x-1) - (x-3) \geq 5$, or $2 \geq 5$, and our last hope of finding solutions fades away.

My third argument is a geometric one. The inequality $|x-1|-|x-3| \geq 5$ asks x to be at least 5 units farther away from 1 than it is from 3. But that's clearly impossible since 1 and 3 are only 2 units apart. (I like this argument best, but they're all valid.)

8. Here $f_0(x) = \frac{1}{2-x}$, and $f_{n+1}(x) = (f_0 \circ f_n)(x) = f_0(f_n(x))$.
Let's look at a few values of $f_n(3)$.

$f_0(3) = \frac{1}{2-3} = \frac{1}{-1} = \frac{-1}{1}$. $f_1(3) = \frac{1}{2-\frac{-1}{1}} = \frac{1}{\frac{2\cdot 1-(-1)}{1}} = \frac{1}{\frac{3}{1}} = \frac{1}{3}$.

$f_2(3) = \frac{1}{2-\frac{1}{3}} = \frac{1}{\frac{2\cdot 3-1}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$. $f_3(3) = \frac{1}{2-\frac{3}{5}} = \frac{1}{\frac{2\cdot 5-3}{5}} = \frac{1}{\frac{7}{5}} = \frac{5}{7}$.

We now guess that $f_n(3) = \frac{2n-1}{2n+1}$. Note this certainly holds for $n = 0$, 1, 2, and 3.

But $\frac{1}{2-\frac{2n-1}{2n+1}} = \frac{1}{\frac{2\cdot(2n+1)-(2n-1)}{2n+1}} = \frac{1}{\frac{2n+3}{2n+1}} = \frac{2n+1}{2n+3}$,

so once this formula starts to hold it will continue to hold thereafter.
(This is a mathematical induction argument.)

Consequently $f_{100}(3) = \frac{199}{201}$.

12. My TI–36 calculator says that $\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}} = 2$.

$\left(1+\sqrt{3}\right)^2 = 4+2\sqrt{3} = 2\left(2+\sqrt{3}\right)$ so $\frac{1+\sqrt{3}}{\sqrt{2}} = \sqrt{2+\sqrt{3}}$ and $\frac{1+\sqrt{3}}{\sqrt{2+\sqrt{3}}} = \sqrt{2}$.

Multiplying by $\sqrt{2}$, $\frac{\sqrt{2}+\sqrt{6}}{\sqrt{2+\sqrt{3}}} = 2$.

26. The amount of cream in the coffee cup is (almost) exactly the same as the amount of coffee in the cream cup. After the two steps are taken, each cup is just as full as it was before the experiment began, since one teaspoon of liquid has been removed and then replaced, or else added and then removed. The (pure, uncreamed) coffee missing from the coffee cup is now in the cream cup, replacing the cream that is now in the coffee cup. The sizes of the two cups are irrelevant, as long as the coffee cup has enough spare room so that it will not overflow when the cream is first added. We ignore the volume compression taking place. (When two dissimilar liquids are mixed, the total volume can be actually slightly less than the sum of the individual volumes since molecules of the one substance may occupy some of the space between molecules of the other substance.)